Multiple scattering and scattering cross sections

P. A. Martin

Citation: The Journal of the Acoustical Society of America **143**, 995 (2018); doi: 10.1121/1.5024361
View online: https://doi.org/10.1121/1.5024361
View Table of Contents: http://asa.scitation.org/toc/jas/143/2
Published by the Acoustical Society of America

Multiple scattering and scattering cross sections

P. A. Martin[a)]
Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado 80401, USA

(Received 15 November 2017; revised 29 January 2018; accepted 31 January 2018; published online 15 February 2018)

The scattering cross section for a cluster of scatterers can be calculated using various methods, either exactly or by invoking various approximations. Of special interest are methods in which the scattering properties of individual members of the cluster are used. The underlying question is: Can the contribution to the cluster's cross section from any one member of the cluster be identified? Except for situations in which all effects of multiple scattering are ignored, no such method of identification has been found. https://doi.org/10.1121/1.5024361

[PLM] Pages: 995–1002

I. INTRODUCTION

When a time-harmonic sound wave encounters an obstacle, it is scattered. The scattering cross section σ_{sc} gives a measure of how much energy is scattered by the obstacle. Energy may also be absorbed by the obstacle itself; how much is quantified by the absorption cross section σ_{ab}. Therefore, the extinction cross section σ_{ex}, defined by

$$\sigma_{ex} = \sigma_{sc} + \sigma_{ab}, \qquad (1)$$

quantifies how much energy has been taken from the incident wave by the scattering process.

Much is known about the computation and interpretation of cross sections. This knowledge is often framed in the context of scattering by a single object, although that object can be a cluster of many smaller objects. For cluster problems, multiple scattering may be important, and then one can ask how does any individual object in the cluster contribute to the scattering properties of the cluster as a whole?

A reviewer noted that the desired contribution is defined by subtraction: Solve the N-object cluster problem, remove one object, and then solve the $(N-1)$-object cluster problem. But the main question of interest here is this: Can one extract the contribution of any one object in the cluster from a knowledge of the solution to the N-object problem alone?

No definitive answer has been given. Indeed, no meaningful way to extract the effects of any one scatterer on the global characteristics of the cluster has been found. This accords with arguments advanced by Twersky 30 years ago (see Sec. V below), and is contrary to the views of Mitri and his so-called "intrinsic cross sections" (see Sec. IV G below).

Sections II and III contain information on scattering by one object, even though most of the calculations are valid for a cluster of objects. They include basic definitions, when an integration surface can be moved, far-field patterns, the T-matrix, and the effect of moving a scatterer's location. The last of these gives a useful (and known) result: For incident plane waves, a scatterer can be moved without changing its scattering cross section.

Multiple scattering is considered in Sec. IV, with a focus on the scattering cross section of a cluster, denoted by Σ_{sc}. An explicit exact formula for Σ_{sc} is obtained [see Eq. (46) below], but it is very complicated. However, it can be used to develop approximations. For example, if the effects of multiple scattering are discarded, a classical result is recovered: Σ_{sc} is equal to the sum of the individual scattering cross sections (Sec. IV E). Another option is to adopt Foldy's model, where each scatterer is represented by a monopole source. The consequences of using Foldy's model are investigated in Sec. IV F. Finally, Mitri's cross sections are defined and discussed in Sec. IV G, with concluding remarks in Sec. V.

II. SCATTERING PROBLEMS

A. Preliminaries

Suppose that u is a regular solution of the Helmholtz equation $(\nabla^2 + k^2)u = 0$ everywhere in $\mathcal{D}$, a fixed bounded volume with boundary $\mathcal{S}$. Without loss of generality, assume that u is dimensionless. Assume also that $k = \omega/c$ is real and positive, where the suppressed time dependence is $e^{-i\omega t}$ and c is the constant speed of sound. Thus the fluid in $\mathcal{D}$ is homogeneous, compressible and lossless.

An application of Green's theorem in $\mathcal{D}$ to u and its complex conjugate $\bar{u}$ gives

$$\frac{1}{2i}\int_{\mathcal{S}}\left(\bar{u}\,\frac{\partial u}{\partial n} - u\,\frac{\partial \bar{u}}{\partial n}\right) dS = \mathrm{Im}\int_{\mathcal{S}} \bar{u}\,\frac{\partial u}{\partial n}\, dS = 0, \qquad (2)$$

where $\partial u/\partial n = \boldsymbol{n}\cdot \mathrm{grad}\, u$ is the normal derivative of u on $\mathcal{S}$, and the fact that k^2 is real has been used. Equation (2) will be used repeatedly later.

B. Formulation

Consider a cluster of N scatterers B_j, $j = 1,2,\ldots,N$. Let S_j be the boundary of B_j and put $S = \cup_{j=1}^{N} S_j$. Surround the cluster by a sphere of radius R, S_R, implying that a choice of a primary origin O within the cluster has been made. Spherical polar coordinates, r, θ, and ϕ, are used, so that S_R is $r = R$.

For many purposes, the cluster can be regarded as a single scatterer (with boundary S and interior $B = \cup_{j=1}^{N} B_j$), but

a)Electronic mail: pamartin@mines.edu

the main goal is to elucidate the contributions of each constituent scatterer B_j to the scattering properties of the cluster.

For scattering problems, write the total field as $u = u_{\rm inc} + u_{\rm sc}$, where $u_{\rm inc}(\boldsymbol{r})$ is the specified incident field and $u_{\rm sc}(\boldsymbol{r})$ is the unknown scattered field. The field $u_{\rm sc}$ is a regular solution of the Helmholtz equation everywhere outside S and it satisfies the Sommerfeld radiation condition. For later use, note that $u_{\rm sc}$ is not defined inside S; if an analytic continuation of $u_{\rm sc}$ is made through S into B, singularities will be encountered.

The incident field $u_{\rm inc}$ may have singularities outside S. For example, one could consider incident waves generated by a point source. However, in this paper, it is assumed that $u_{\rm inc}$ is a regular solution of the Helmholtz equation everywhere inside the sphere S_R. For information on the scattering cross section with point-source generated incident fields, see Refs. 1 and 2.

C. Cross sections

Let $\mathcal{S} = S \cup S_R$ with $\mathcal{D}$ being the region between S and S_R. As u is a regular solution of the Helmholtz equation in $\mathcal{D}$, Eq. (2) gives

$$\mathrm{Im}\int_{S_R} \bar{u}\,\frac{\partial u}{\partial r}\,\mathrm{d}S - \mathrm{Im}\int_S \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S = 0, \tag{3}$$

where the normal vector on S_j points out of B_j, $j = 1,2,\ldots,N$. Substituting $u = u_{\rm inc} + u_{\rm sc}$ gives

$$\mathrm{Im}\int_{S_R} \overline{u_{\rm inc}}\,\frac{\partial u_{\rm inc}}{\partial r}\,\mathrm{d}S + \mathrm{Im}\int_{S_R} \overline{u_{\rm sc}}\,\frac{\partial u_{\rm sc}}{\partial r}\,\mathrm{d}S + \mathrm{Im}\int_{S_R}\left(\overline{u_{\rm sc}}\,\frac{\partial u_{\rm inc}}{\partial r} - u_{\rm inc}\,\frac{\partial \overline{u_{\rm sc}}}{\partial r}\right)\mathrm{d}S - \mathrm{Im}\int_S \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S = 0. \tag{4}$$

The first term on the left-hand side of Eq. (4) is

$$\frac{1}{2\mathrm{i}}\int_{S_R}\left(\overline{u_{\rm inc}}\,\frac{\partial u_{\rm inc}}{\partial r} - u_{\rm inc}\,\frac{\partial \overline{u_{\rm inc}}}{\partial r}\right)\mathrm{d}S = 0,$$

because $u_{\rm inc}(\boldsymbol{r})$ is a regular solution of the Helmholtz equation everywhere inside S_R.

The second term on the left-hand side of Eq. (4) involves the scattered field only; it is closely related to the *scattering cross section*. Define

$$\sigma_{\rm sc} = k\,\mathrm{Im}\int_{S_R} \overline{u_{\rm sc}}\,\frac{\partial u_{\rm sc}}{\partial r}\,\mathrm{d}S, \tag{5}$$

where the factor k has been inserted so that $\sigma_{\rm sc}$ is dimensionless. Physically, $\sigma_{\rm sc}$ represents the radiated acoustic power.

The fourth term on the left-hand side of Eq. (4) involves the total field on S; it is closely related to the *absorption cross section*. Define

$$\sigma_{\rm ab} = -k\,\mathrm{Im}\int_S \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S. \tag{6}$$

Physically, $\sigma_{\rm ab}$ represents the acoustic power absorbed by the scatterers (if they are lossy).

Combining Eqs. (1) and (4), define the *extinction cross section* by

$$\sigma_{\rm ex} = -k\,\mathrm{Im}\int_{S_R}\left(\overline{u_{\rm sc}}\,\frac{\partial u_{\rm inc}}{\partial r} - u_{\rm inc}\,\frac{\partial \overline{u_{\rm sc}}}{\partial r}\right)\mathrm{d}S. \tag{7}$$

D. Moving the integration surface

The integration surface in Eqs. (5) and (7) can be moved. For example, considering $\sigma_{\rm sc}$, its value does not depend on R, as is well known [Eq. (6) in Ref. 3, Eq. (4.7) in Ref. 4]. Indeed, if Eq. (5) is written as

$$\sigma_{\rm sc} = \frac{k}{2\mathrm{i}}\int_{S_R}\left(\overline{u_{\rm sc}}\,\frac{\partial u_{\rm sc}}{\partial r} - u_{\rm sc}\,\frac{\partial \overline{u_{\rm sc}}}{\partial r}\right)\mathrm{d}S,$$

it is seen that the integration surface can be moved onto the scatterers themselves,

$$\sigma_{\rm sc} = k\,\mathrm{Im}\int_S \overline{u_{\rm sc}}\,\frac{\partial u_{\rm sc}}{\partial n}\,\mathrm{d}S; \tag{8}$$

recall that the normal vector on S points outward. The derivation uses Green's theorem, and so it requires that $u_{\rm sc}$ satisfies the Helmholtz equation everywhere between S and S_R, and that the wavenumber k is real (which it is, because the exterior medium is assumed to be lossless). Alternatively, letting $R \to \infty$, far-field approximations to $u_{\rm sc}$ can be used in Eq. (5) (see Sec. III B).

E. The scatterers and the absorption cross section

Many kinds of scatterers are of interest. If they are sound-soft ($u = 0$ on S_j) or sound-hard ($\partial u/\partial n = 0$ on S_j), they are lossless and then Eq. (6) gives $\sigma_{\rm ab} = 0$, whence Eq. (1) gives $\sigma_{\rm ex} = \sigma_{\rm sc}$.

One way to introduce loss is to impose an impedance (Robin) condition $\partial u/\partial n + \lambda u = 0$ on S whence $\sigma_{\rm ab}$ will be positive for some choices of λ.

For penetrable scatterers, a transmission problem can be formulated. Suppose that $(\nabla^2 + k_j^2)u_j = 0$ in B_j with transmission conditions $u = u_j$ and $\partial u/\partial n = \varrho_j \partial u_j/\partial n$ on S_j, where k_j can vary with position in B_j and ϱ_j is a dimensionless constant.[5] Using the divergence theorem,

$$\int_{S_j} \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S = \varrho_j\int_{B_j}\left(|\mathrm{grad}\,u_j|^2 - k_j^2|u_j|^2\right)\mathrm{d}V.$$

Then Eq. (6) gives

$$\sigma_{\rm ab} = k\sum_{j=1}^{N}\int_{B_j}\{\mathrm{Im}(\varrho_j k_j^2)|u_j|^2 - \mathrm{Im}(\varrho_j)|\mathrm{grad}\,u_j|^2\}\,\mathrm{d}V, \tag{9}$$

and this will be positive for certain choices of ϱ_j and k_j, $j = 1,2,\ldots,N$; these choices are also sufficient to ensure uniqueness for the transmission problem itself (see p. 309 of Ref. 5).

Some authors prefer to formulate scattering problems as a forced Helmholtz equation, $(\nabla^2 + k^2)u = -k^2\eta(\boldsymbol{r})u$, with η defined in B_j by $k^2(1+\eta) = k_j^2(\boldsymbol{r})$, $j = 1, 2, \ldots, N$, and $\eta = 0$ outside the scatterers. This is equivalent to taking $\varrho_j = 1$, $j = 1,2,\ldots,N$; in this case, Eq. (9) reduces to Eq. (7) in Ref. 6. However, it is known that this simplification does not respect the proper acoustic transmission conditions across interfaces (unless there is no density mismatch across those interfaces).[7]

III. ONE SCATTERER

In this section, known facts concerning scattering by one obstacle are recalled; the obstacle could be a single scatterer ($N = 1$) or a cluster of scatterers ($N > 1$).

A. Spherical wavefunctions

In the vicinity of the sphere S_R, $u_{\rm inc}$ and $u_{\rm sc}$ can be expanded as follows:

$$u_{\rm inc}(\boldsymbol{r}) = \sum_{n,m} d_n^m j_n(kr)\, Y_n^m(\hat{\boldsymbol{r}}), \tag{10}$$

$$u_{\rm sc}(\boldsymbol{r}) = \sum_{n,m} c_n^m h_n(kr)\, Y_n^m(\hat{\boldsymbol{r}}), \tag{11}$$

where j_n is a spherical Bessel function, $h_n \equiv h_n^{(1)}$ is a spherical Hankel function, Y_n^m is a spherical harmonic, $r = |\boldsymbol{r}|$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, d_n^m and c_n^m are coefficients, and

$$\sum_{n,m} \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^{n}.$$

(The notation is as in Ref. 8.) On S_R, $\mathrm{d}S = R^2\,\mathrm{d}\Omega(\hat{\boldsymbol{r}})$, where Ω is the unit sphere. The spherical harmonics are orthonormal,

$$\int_\Omega Y_n^m \overline{Y_\nu^\mu}\,\mathrm{d}\Omega = \delta_{n\nu}\delta_{m\mu}. \tag{12}$$

Starting with Eq. (5), direct calculation [put $A_n^m = 0$ and $B_n^m = c_n^m$ in Eq. (A4)] gives

$$\sigma_{\rm sc} = \sum_{n,m} |c_n^m|^2. \tag{13}$$

Similarly, starting with Eq. (7) [put $A_n^m = D_n^m = 0$, $B_n^m = c_n^m$ and $C_n^m = d_n^m$ in Eq. (A3)],

$$\sigma_{\rm ex} = -\mathrm{Im} \sum_{n,m} \mathrm{i}\, \overline{c_n^m} d_n^m = -\mathrm{Re} \sum_{n,m} c_n^m \overline{d_n^m}. \tag{14}$$

B. The far-field pattern

Introduce the far-field pattern, $f(\hat{\boldsymbol{r}})$, defined by

$$u_{\rm sc}(\boldsymbol{r}) \sim (\mathrm{i}kr)^{-1}\mathrm{e}^{\mathrm{i}kr} f(\hat{\boldsymbol{r}}) \quad \text{as } r \to \infty. \tag{15}$$

Then the following expansion is obtained from Eq. (11),

$$f(\hat{\boldsymbol{r}}) = \sum_{n,m} (-\mathrm{i})^n c_n^m Y_n^m(\hat{\boldsymbol{r}}). \tag{16}$$

(For a proof, see Theorem 2.16 in Ref. 9.) The well-known formula

$$\sigma_{\rm sc} = \int_\Omega |f(\hat{\boldsymbol{r}})|^2 \mathrm{d}\Omega \tag{17}$$

is obtained by substituting Eq. (16) in the integrand followed by use of Eqs. (12) and (13). Alternatively, simply insert Eq. (15) in Eq. (5) followed by letting $R \to \infty$. See Ref. 10 for these and related computations of acoustic radiation forces.

C. The *T*-matrix

Waterman's T-matrix is a convenient way to represent waves scattered by a single obstacle. It connects c_n^m and d_n^m in Eqs. (10) and (11),

$$c_n^m = \sum_{\nu,\mu} T_{n\nu}^{m\mu} d_\nu^\mu. \tag{18}$$

Substituting in Eqs. (13) and (14) gives

$$\sigma_{\rm sc} = \sum_{n,m} \sum_{\nu,\mu} \overline{d_n^m} d_\nu^\mu \sum_{\nu',\mu'} \overline{T_{\nu' n}^{\mu' m}} T_{\nu'\nu}^{\mu'\mu}, \tag{19}$$

$$\sigma_{\rm ex} = -\frac{1}{2} \sum_{n,m} \sum_{\nu,\mu} \overline{d_n^m} d_\nu^\mu \left(T_{n\nu}^{m\mu} + \overline{T_{\nu n}^{\mu m}} \right). \tag{20}$$

For lossless scatterers, the identity $\sigma_{\rm sc} = \sigma_{\rm ex}$ combined with the arbitrary nature of the incident field leads to a relation that must be satisfied by the T-matrix; see Theorem 7.4 in Ref. 8, where the restriction to lossless scatterers was overlooked.

D. Incident plane waves

Up to now, the incident field $u_{\rm inc}$ has not been specified. For an incident plane wave, $u_{\rm inc}(\boldsymbol{r}) = \exp(\mathrm{i}k\boldsymbol{r} \cdot \hat{\boldsymbol{\alpha}})$, where $\hat{\boldsymbol{\alpha}}$ is a unit vector in the direction of propagation. Then the coefficients d_n^m in Eq. (10) are given by Eq. (4.40) in Ref. 8,

$$d_n^m = 4\pi\, \mathrm{i}^n\, \overline{Y_n^m(\hat{\boldsymbol{\alpha}})}. \tag{21}$$

Denoting the corresponding far-field pattern by $f(\hat{\boldsymbol{r}}; \hat{\boldsymbol{\alpha}})$, combining Eqs. (16) and (21) gives

$$f(\hat{\boldsymbol{\alpha}}; \hat{\boldsymbol{\alpha}}) = \frac{1}{4\pi} \sum_{n,m} c_n^m \overline{d_n^m}.$$

Then, for lossless scatterers, Eqs. (14) and (17), and $\sigma_{\rm sc}(\hat{\boldsymbol{\alpha}}) = \sigma_{\rm ex}(\hat{\boldsymbol{\alpha}})$ give

$$\sigma_{\rm sc}(\hat{\boldsymbol{\alpha}}) = \int_\Omega |f(\hat{\boldsymbol{r}}; \hat{\boldsymbol{\alpha}})|^2 \mathrm{d}\Omega = -\mathrm{Re}\,\{4\pi f(\hat{\boldsymbol{\alpha}}; \hat{\boldsymbol{\alpha}})\}. \tag{22}$$

This is the famous *optical theorem*; its early history was reviewed by Newton.[11]

E. Moving the scatterer

It is interesting to see what happens to the far-field pattern when the scatterer is moved (see Sec. 4.6.1 in Ref. 8). Start by assuming an incident plane wave. Without loss of generality, take the origin O at some point inside the scatterer and suppose that the same point is moved to O_1 when the scatterer is moved (translated). Let $\boldsymbol{b}_1$ be the position vector of O_1 with respect to O and let $f_1(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}})$ be the new far-field pattern. Let $\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{b}_1$ so that $u_{\text{inc}} = \exp(\mathrm{i}k\boldsymbol{b}_1 \cdot \hat{\boldsymbol{\alpha}})\exp(\mathrm{i}k\boldsymbol{r}_1 \cdot \hat{\boldsymbol{\alpha}})$. In the far field, the scattered field behaves as

$$\exp(\mathrm{i}k\boldsymbol{b}_1 \cdot \hat{\boldsymbol{\alpha}})(\mathrm{i}kr_1)^{-1}\mathrm{e}^{\mathrm{i}kr_1}f(\hat{\boldsymbol{r}}_1;\hat{\boldsymbol{\alpha}}) \quad \text{as } r_1 = |\boldsymbol{r}_1| \to \infty, \tag{23}$$

where $\hat{\boldsymbol{r}}_1 = \boldsymbol{r}_1/r_1$. But $r_1 \sim r - \boldsymbol{b}_1 \cdot \hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{r}}_1 \sim \hat{\boldsymbol{r}}$ as $r_1 \to \infty$ for fixed $\boldsymbol{b}_1$. Hence

$$f_1(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}}) = \exp\{\mathrm{i}k(\hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{r}}) \cdot \boldsymbol{b}_1\}f(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}}).$$

This shows that, for incident plane waves, moving the scatterer changes the phase of the far-field pattern [that is, changes the argument of the complex number $f(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}})$] but does not change $|f(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}})|$. Thus, from Eq. (22), moving the scatterer does not change the value of σ_{sc}.

These results explain why the inverse problem of determining the shape of a scatterer from $|f(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}})|$ suffers from non-uniqueness. Uniqueness may be restored by changing the incident fields; for example, Klibanov[12] uses waves generated by point sources.

Having seen what happens to f when the incident field is a plane wave, consider now an arbitrary u_{inc}. There are expansions of u_{inc} and u_{sc} around O_1 analogous to Eqs. (10) and (11),

$$u_{\text{inc}} = \sum_{n,m} \tilde{d}_n^m j_n(kr_1)\, Y_n^m(\hat{\boldsymbol{r}}_1), \tag{24}$$

$$u_{\text{sc}} = \sum_{n,m} \tilde{c}_n^m h_n(kr_1)\, Y_n^m(\hat{\boldsymbol{r}}_1), \tag{25}$$

with coefficients $\tilde{d}_n^m$ and $\tilde{c}_n^m$. As $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{b}_1$, an addition theorem (Theorem 3.26 in Ref. 8) can be used to write Eq. (10) as

$$u_{\text{inc}} = \sum_{n,m} d_n^m \sum_{\nu,\mu} \hat{S}_{n\nu}^{m\mu}(\boldsymbol{b}_1) j_\nu(kr_1)\, Y_\nu^\mu(\hat{\boldsymbol{r}}_1),$$

where $\hat{S}_{n\nu}^{m\mu}$ is known as a *separation matrix*. Comparison with Eq. (24) gives

$$\tilde{d}_n^m = \sum_{\nu,\mu} d_\nu^\mu \hat{S}_{\nu n}^{\mu m}(\boldsymbol{b}_1). \tag{26}$$

Having computed $\tilde{d}_n^m$, giving the incident field around O_1, scatter that field

$$\tilde{c}_n^m = \sum_{\nu,\mu} T_{n\nu}^{m\mu} \tilde{d}_\nu^\mu. \tag{27}$$

Then re-expand the scattered field, Eq. (25), about O using $\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{b}_1$ and another addition theorem (Theorem 3.27 in Ref. 8)

$$u_{\text{sc}} = \sum_{n,m} \tilde{c}_n^m \sum_{\nu,\mu} \hat{S}_{n\nu}^{m\mu}(-\boldsymbol{b}_1) h_\nu(kr)\, Y_\nu^\mu(\hat{\boldsymbol{r}}),$$

valid for $r > b_1$. Comparison with Eq. (11) gives

$$c_n^m = \sum_{\nu,\mu} \tilde{c}_\nu^\mu \hat{S}_{\nu n}^{\mu m}(-\boldsymbol{b}_1).$$

For the far-field pattern, Eq. (16) can be used together with a formula for $\hat{S}_{n\nu}^{m\mu}$ [Eq. (3.79) in Ref. 8],

$$\hat{S}_{n\nu}^{m\mu}(\boldsymbol{b}) = \mathrm{i}^{\nu-n} \int_\Omega \exp(\mathrm{i}k\boldsymbol{b} \cdot \hat{\boldsymbol{r}})\, Y_n^m(\hat{\boldsymbol{r}})\, \overline{Y_\nu^\mu(\hat{\boldsymbol{r}})}\, \mathrm{d}\Omega, \tag{28}$$

and the expansion formula

$$f(\hat{\boldsymbol{r}}) = \sum_{n,m} Y_n^m(\hat{\boldsymbol{r}}) \int_\Omega f(\hat{\boldsymbol{r}}')\, \overline{Y_n^m(\hat{\boldsymbol{r}}')}\, \mathrm{d}\Omega(\hat{\boldsymbol{r}}'). \tag{29}$$

The result is

$$f_1(\hat{\boldsymbol{r}}) = \exp(-\mathrm{i}k\hat{\boldsymbol{r}} \cdot \boldsymbol{b}_1) \sum_{n,m} (-\mathrm{i})^n \tilde{c}_n^m Y_n^m(\hat{\boldsymbol{r}}).$$

The same result can also be obtained directly from Eq. (25), using Eq. 10.52.4 in Ref. 13 and the estimates below Eq. (23).

The corresponding scattering cross section is

$$\begin{aligned}\tilde{\sigma}_{\text{sc}} &= \int_\Omega |f_1(\hat{\boldsymbol{r}})|^2\, \mathrm{d}\Omega = \sum_{n,m} |\tilde{c}_n^m|^2 \\ &= \sum_{n,m} \sum_{\nu,\mu} \overline{\tilde{d}_n^m}\, \tilde{d}_\nu^\mu \sum_{\nu',\mu'} \overline{T_{\nu' n}^{\mu' m}}\, T_{\nu'\nu}^{\mu'\mu},\end{aligned}$$

where $\tilde{d}_n^m$ is given by Eq. (26) in terms of d_n^m. This formula for $\tilde{\sigma}_{\text{sc}}$ should be compared with Eq. (19) for σ_{sc}; in general, they are not equal.

For a plane wave, d_n^m is defined by Eq. (21) whence Eqs. (26), (28), and (29) give $\tilde{d}_n^m = d_n^m \exp(\mathrm{i}k\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{b}_1)$. Thus, $\overline{\tilde{d}_n^m}\, \tilde{d}_\nu^\mu = \overline{d_n^m}\, d_\nu^\mu$ and $\tilde{\sigma}_{\text{sc}} = \sigma_{\text{sc}}$, as was already seen above for this special case.

IV. MULTIPLE SCATTERING

As noted earlier, the calculations above are concerned with scattering by one obstacle, but that obstacle could be a cluster of N disjoint scatterers, each with its own scattering properties. Denote the scattering cross section of the cluster by Σ_{sc}; it is defined by Eq. (5) as

$$\Sigma_{\text{sc}} = k\, \mathrm{Im} \int_{S_R} \overline{u_{\text{sc}}}\, \frac{\partial u_{\text{sc}}}{\partial r}\, \mathrm{d}S, \tag{30}$$

where S_R is a sphere enclosing the cluster.

As in the derivation of Eq. (8), the integration surface can be moved onto the scatterers themselves. Hence,

$$\Sigma_{\text{sc}} = \sum_{j=1}^N \Sigma_{\text{sc}}^j \tag{31}$$

with

$$\Sigma_{\rm sc}^{j} = k\,{\rm Im}\int_{S_j} \overline{u_{\rm sc}}\,\frac{\partial u_{\rm sc}}{\partial n}\,{\rm d}S. \tag{32}$$

This decomposes $\Sigma_{\rm sc}$ into an integration over each scatterer, but the integrand involves $u_{\rm sc}$, which is the solution of the full N-body multiple scattering problem.

For each $j = 1,2,\ldots,N$, choose an origin O_j inside (or near) S_j and let S_j^+ be the escribed sphere to S_j. (Thus, S_j^+ is the smallest sphere, centered at O_j, that encloses S_j.) Then in the formula for $\Sigma_{\rm sc}^j$, Eq. (32), S_j can be replaced by S_j^+, giving

$$\Sigma_{\rm sc}^{j} = k\,{\rm Im}\int_{S_j^+} \overline{u_{\rm sc}}\,\frac{\partial u_{\rm sc}}{\partial n}\,{\rm d}S. \tag{33}$$

This is convenient if one wants to represent the scattered field using outgoing spherical wavefunctions centered at O_j; see Eq. (35) below.

A. Use of spherical wavefunctions

Let a typical point P have position vectors $\boldsymbol{r}$ with respect to the primary origin O, and $\boldsymbol{r}_j$ with respect to the (local) origin O_j, $j = 1,2,\ldots,N$. Then the scattered field can be written as

$$u_{\rm sc}(P) = \sum_{n,m} c_n^m h_n(kr) Y_n^m(\hat{\boldsymbol{r}}), \quad P \text{ outside } S_R \tag{34}$$

or as

$$u_{\rm sc}(P) = \sum_{j=1}^{N}\sum_{n,m} c_{nj}^m h_n(kr_j) Y_n^m(\hat{\boldsymbol{r}}_j), \quad P \text{ outside } \bigcup_{j=1}^{N} S_j^+. \tag{35}$$

Using an appropriate addition theorem, write Eq. (35) as

$$u_{\rm sc}(P) = \sum_{n,m}\{c_{nj}^m h_n(kr_j) + a_{nj}^m j_n(kr_j)\} Y_n^m(\hat{\boldsymbol{r}}_j) \tag{36}$$

for P in the vicinity of S_j^+, where the coefficients a_{nj}^m can be calculated [see Eq. (41) below]. Substituting in Eq. (33) and integrating [using Eq. (A4) with $A_n^m = a_{nj}^m$ and $B_n^m = c_{nj}^m$],

$$\Sigma_{\rm sc}^{j} = \sum_{n,m}\left(|c_{nj}^m|^2 + {\rm Re}\{a_{nj}^m \overline{c_{nj}^m}\}\right). \tag{37}$$

This formula is exact but the coefficients c_{nj}^m and a_{nj}^m have not been specified; of course, they will depend on the nature of the scatterers and on the geometrical configuration of the cluster.

B. Solving the scattering problem exactly

The N-body scattering problem can be solved in various ways, as explained in Ref. 8. Here, T-matrix methods are used because they separate out the behavior of individual scatterers and enable detailed comparisons with previous work, but the main conclusions do not depend on the use of T-matrix methods. From Eq. (35),

$$u_{\rm sc} = \sum_{n,m} c_{nj}^m h_n(kr_j) Y_n^m(\hat{\boldsymbol{r}}_j) + u_{\rm exc}^j, \tag{38}$$

where $u_{\rm exc}^j$ is the "exciting field" on S_j,

$$u_{\rm exc}^j = \sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{n,m} c_{nl}^m h_n(kr_l) Y_n^m(\hat{\boldsymbol{r}}_l). \tag{39}$$

In the vicinity of S_j, Eq. (38) reduces to Eq. (36) after using the addition theorem (Theorem 3.27 in Ref. 8)

$$h_n(kr_l) Y_n^m(\hat{\boldsymbol{r}}_l) = \sum_{\nu,\mu} S_{n\nu}^{m\mu}(\boldsymbol{b}_{jl}) j_\nu(kr_j) Y_\nu^\mu(\hat{\boldsymbol{r}}_j), \tag{40}$$

where $S_{n\nu}^{m\mu}$ is another separation matrix and $\boldsymbol{b}_{jl} = \boldsymbol{b}_j - \boldsymbol{b}_l$ is the position vector of O_j with respect to O_l. Hence, the coefficients a_{nj}^m in Eq. (36) are given by

$$a_{nj}^m = \sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{\nu,\mu} c_{\nu l}^\mu S_{\nu n}^{\mu m}(\boldsymbol{b}_{jl}). \tag{41}$$

The derivation here assumes that O_l is outside S_j^+ for $l = 1,2,\ldots,N$, $l \neq j$.

Expand the incident field in the vicinity of S_j^+,

$$u_{\rm inc} = \sum_{n,m} d_n^m j_n(kr)\, Y_n^m(\hat{\boldsymbol{r}}) = \sum_{n,m} d_{nj}^m j_n(kr_j)\, Y_n^m(\hat{\boldsymbol{r}}_j) \tag{42}$$

where

$$d_{nj}^m = \sum_{\nu,\mu} d_\nu^\mu \hat{S}_{\nu n}^{\mu m}(\boldsymbol{b}_j)$$

and $\boldsymbol{b}_j$ is the position vector of O_j with respect to O. This gives the local coefficients d_{nj}^m in terms of the global coefficients d_n^m.

Thus the field exciting the jth scatterer has the form

$$\sum_{n,m} D_{nj}^m j_n(kr_j) Y_n^m(\hat{\boldsymbol{r}}_j) \quad \text{with } D_{nj}^m = d_{nj}^m + a_{nj}^m. \tag{43}$$

Self-consistency then gives

$$c_{nj}^m = \sum_{\nu,\mu} T_{n\nu}^{m\mu}(j)\, D_{\nu j}^\mu \tag{44}$$

where $T_{n\nu}^{m\mu}(j)$ denotes the entries in the T-matrix for the jth scatterer. Substitution for D_{nj}^m then gives a system of algebraic equations to solve for c_{nj}^m.

C. Scattering cross section of the cluster, $\Sigma_{\rm sc}$

Combining Eqs. (31), (32), (37), and (41) gives

$$\begin{aligned}\Sigma_{\rm sc} &= \sum_{j=1}^{N}\sum_{n,m}\left(|c_{nj}^m|^2 + {\rm Re}\{a_{nj}^m \overline{c_{nj}^m}\}\right)\\ &= \sum_{j=1}^{N}\sum_{n,m}|c_{nj}^m|^2 + \Sigma_{\rm sc}^{\rm c},\end{aligned} \tag{45}$$

where

$$\Sigma_{\rm sc}^{\rm c} = \mathrm{Re}\sum_{j=1}^{N}\sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{n,m}\sum_{\nu,\mu}\overline{c_{nj}^{m}}c_{\nu l}^{\mu}S_{\nu n}^{\mu m}(\boldsymbol{b}_{jl})$$
$$= \frac{1}{2}\sum_{j=1}^{N}\sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{n,m}\sum_{\nu,\mu}\overline{c_{nj}^{m}}c_{\nu l}^{\mu}S_{\nu n}^{\mu m}(\boldsymbol{b}_{jl})$$
$$+\frac{1}{2}\sum_{j=1}^{N}\sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{n,m}\sum_{\nu,\mu}c_{nj}^{m}\overline{c_{\nu l}^{\mu}}\,\overline{S_{\nu n}^{\mu m}(\boldsymbol{b}_{jl})}$$
$$= \frac{1}{2}\sum_{j=1}^{N}\sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{n,m}\sum_{\nu,\mu}\overline{c_{nj}^{m}}c_{\nu l}^{\mu}\left(S_{\nu n}^{\mu m}(\boldsymbol{b}_{jl}) + \overline{S_{n\nu}^{m\mu}(\boldsymbol{b}_{lj})}\right).$$

As $\boldsymbol{b}_{lj} = -\boldsymbol{b}_{jl}$, Lemma 3.29 from Ref. 8 can be used to simplify $\Sigma_{\rm sc}^{c}$ further, whence

$$\Sigma_{\rm sc} = \sum_{j=1}^{N}\sum_{n,m}|c_{nj}^{m}|^2 + \sum_{j=1}^{N}\sum_{\substack{l=1\\ l\neq j}}^{N}\sum_{n,m}\sum_{\nu,\mu}\overline{c_{nj}^{m}}c_{\nu l}^{\mu}\hat{S}_{\nu n}^{\mu m}(\boldsymbol{b}_{jl}). \quad (46)$$

[Despite appearances, the last term is real; to see this, use Eq. (3.95) in Ref. 8.] Equation (46) is exact. It was derived previously as Eq. (33) in Ref. 14; it is also similar to Eq. (26) in Ref. 15, where a study of electromagnetic scattering by N spheres is presented.

Equation (46) contains all the coefficients c_{nj}^{m} in the multipole expansion Eq. (35). However, it does not reveal how an individual scatterer S_j contributes to the properties of the cluster.

D. Extinction cross section of the cluster, $\Sigma_{\rm ex}$

The extinction cross section $\Sigma_{\rm ex}$ is defined by Eq. (7). Proceeding as with $\Sigma_{\rm sc}$ gives $\Sigma_{\rm ex} = \sum_{j=1}^{N}\Sigma_{\rm ex}^{j}$ with

$$\Sigma_{\rm ex}^{j} = -k\,\mathrm{Im}\int_{S_j^+}\left(\overline{u_{\rm sc}}\frac{\partial u_{\rm inc}}{\partial r} - u_{\rm inc}\frac{\partial\,\overline{u_{\rm sc}}}{\partial r}\right)\mathrm{d}S$$
$$= -\mathrm{Re}\sum_{n,m}c_{nj}^{m}\overline{d_{nj}^{m}}, \quad (47)$$

which is very similar to Eq. (14). Here, Eqs. (36) and (42) have been used; the terms involving a_{nj}^{m} cancel.

E. Single scattering

The entries in the separation matrix $S_{n\nu}^{m\mu}(\boldsymbol{b})$ decay to zero as $kb \to \infty$ (Sec. 3.13.2 in Ref. 8). This corresponds to widely spaced scatterers. In this limit, Eq. (41) gives $a_{nj}^{m} = 0$ whence Eq. (43) gives $D_{nj}^{m} = d_{nj}^{m}$. This uncouples the N-scatterer problem into N separate problems: There is no multiple scattering in this limit. The coefficients $c_{nj}^{m}\ (=\tilde{c}_{n}^{m})$ are related to $d_{nj}^{m}\ (=\tilde{d}_{n}^{m})$ by the jth T-matrix, taking into account that the origin has been moved from O to O_j. [Compare Eqs. (27) and (44) with $D_{\nu j}^{\mu} = d_{\nu j}^{\mu}$.] Therefore, in the wide-spacing limit, Eq. (37) gives $\Sigma_{\rm sc}^{j} \simeq \sigma_{\rm sc}^{j}(O_j)$, where $\sigma_{\rm sc}^{j}(O_j)$ is the scattering cross section for the jth scatterer (which is located at O_j) in the absence of all the other scatterers. Finally, for the special case of an incident plane wave, it is known (Sec. III E) that location is irrelevant, and so the scatterer can be moved from O_j to O, giving

$$\Sigma_{\rm sc} \simeq \sum_{j=1}^{N}\sigma_{\rm sc}^{j} \quad \text{with} \quad \sigma_{\rm sc}^{j} \equiv \sigma_{\rm sc}^{j}(O). \quad (48)$$

The additive property Eq. (48) is noted by Newton [Eq. (4) in Ref. 11 and Eq. (1.90) in Ref. 16] and by van de Hulst (Sec. 4.22 in Ref. 17). Their arguments lack details, but they do assume that the scatterers are distributed randomly and invoke the single-scattering approximation. For a derivation along these lines, for electromagnetic waves, see Sec. 10 in Ref. 18. The fact that randomness plays no part in the derivation of Eq. (48) is noted explicitly on p. 1516 of Ref. 19. Randomness can be used to achieve additivity of other far-field quantities,[20] but it is not needed for Eq. (48).

F. Foldy's method: Small scatterers

Foldy's method[21] is an approximate method for multiple scattering problems. It starts by assuming that the total field can be written as

$$u(\boldsymbol{r}) = u_{\rm inc}(\boldsymbol{r}) + \sum_{j=1}^{N}A_jG(\boldsymbol{r}-\boldsymbol{b}_j) \quad (49)$$

with $G(\boldsymbol{r}) = h_0(kr) = \mathrm{e}^{\mathrm{i}kr}/(\mathrm{i}kr)$. Thus, the scattered field is represented by a point source at the center of each scatterer. This is a good approximation for small sound-soft (Dirichlet condition, lossless) scatterers. The coefficients are determined by solving

$$g_n^{-1}A_n = u_{\rm inc}(\boldsymbol{b}_n) + \sum_{\substack{j=1\\ j\neq n}}^{N}A_jG(\boldsymbol{b}_{nj}), \quad n = 1,2,\ldots,N. \quad (50)$$

The "scattering coefficient" g_j is defined as follows. When an incident field $u_{\rm inc}$ is scattered by the jth obstacle in isolation, the total field is

$$u(\boldsymbol{r}) = u_{\rm inc}(\boldsymbol{r}) + g_ju_{\rm inc}(\boldsymbol{b}_j)\,G(\boldsymbol{r}-\boldsymbol{b}_j).$$

This identifies the far-field pattern and hence gives the approximation

$$\sigma_{\rm sc}^{j}(O_j) = 4\pi\,|g_j|^2|u_{\rm inc}(\boldsymbol{b}_j)|^2. \quad (51)$$

For more details on Foldy's method, including the choice of g_j, see Sec. 8.3 of Ref. 8. For applications, see Refs. 22–24, and references therein.

From Eq. (49), the far-field pattern of the cluster is

$$F(\hat{\boldsymbol{r}}) = \sum_{j=1}^{N} A_j \exp(-\mathrm{i}k\boldsymbol{b}_j \cdot \hat{\boldsymbol{r}}). \qquad (52)$$

Hence, from Eqs. (17) and (52),

$$\Sigma_{\mathrm{sc}} = \int_{\Omega} |F(\hat{\boldsymbol{r}})|^2 \, \mathrm{d}\Omega = 4\pi \sum_{j=1}^{N} \sum_{l=1}^{N} A_j \overline{A_l} \, j_0(k|\boldsymbol{b}_{lj}|); \qquad (53)$$

the integration is standard [Eq. (3.44) in Ref. 8].

Equation (53) agrees with Eq. (16) in Ref. 25. In that paper, the authors try to maximize Σ_{sc} with respect to the locations of the N scatterers.

For wide spacings, the system Eq. (50) reduces to $g_n^{-1} A_n = u_{\mathrm{inc}}(\boldsymbol{b}_n)$, whence Eq. (51) becomes $\sigma_{\mathrm{sc}}^j(O_j) = 4\pi|A_j|^2$ and [using $j_0(x) = x^{-1} \sin x$] Eq. (53) reduces to

$$\Sigma_{\mathrm{sc}} \simeq 4\pi \sum_{j=1}^{N} |A_j|^2 \simeq \sum_{j=1}^{N} \sigma_{\mathrm{sc}}^j(O_j).$$

Thus the classical additive result Eq. (48) is recovered when there are incident plane waves.

G. Mitri's cross sections

Mitri has written several papers[26–28] on what he calls *extrinsic* and *intrinsic* cross sections. He is mainly concerned with multiple scattering by two cylinders.

Mitri refers to Σ_{sc} as the extrinsic scattering cross section; it is the scattering cross section for the cluster. He obtains a formula for Σ_{sc} [Eq. (26) in Ref. 26] that is reminiscent of Eq. (46) (in two dimensions with $N=2$), but the derivation is suspect because it is derived by going into the far field leading to divergent series [such as Eq. (20) in Ref. 26].

Next, consider Mitri's intrinsic scattering cross sections, which are denoted here by σ_{M}^j, $j = 1, 2, \ldots, N$. They are defined as follows. Returning to Eq. (35), write $u_{\mathrm{sc}}(P) = \sum_{j=1}^{N} u_{\mathrm{sc}}^j(P)$ with

$$u_{\mathrm{sc}}^j(P) = \sum_{n,m} c_{nj}^m h_n(kr_j) Y_n^m(\hat{\boldsymbol{r}}_j), \quad P \text{ outside } S_j^+.$$

Then [see Eq. (15) in Ref. 27, and compare with Eq. (5)]

$$\sigma_{\mathrm{M}}^j = k \lim_{R\to\infty} \mathrm{Im} \int_{S_R} \overline{u_{\mathrm{sc}}^j} \frac{\partial u_{\mathrm{sc}}^j}{\partial r} \, \mathrm{d}S. \qquad (54)$$

(As noted in Sec. II D, the limiting operation is redundant.) A straightforward calculation gives [see Eq. (13)]

$$\sigma_{\mathrm{M}}^j = \sum_{n,m} |c_{nj}^m|^2, \qquad (55)$$

in agreement with Eqs. (22) and (25) in Ref. 27. Thus, from Eq. (45), $\Sigma_{\mathrm{sc}} = \sum_{j=1}^{N} \sigma_{\mathrm{M}}^j + \Sigma_{\mathrm{sc}}^{\mathrm{c}}$. Mitri writes (p. 5 of Ref. 27):

> The intrinsic scattering cross section [σ_{M}^j] provides quantitative information on the scattering properties of the probed object [S_j] exclusively (but in the presence of the [other objects] without multiple interference effects), as shown in equation (55), where the dependence of σ_{M}^j is only on the scattering coefficient c_{nj}^m, keeping in mind that those coefficients are a function of [all N] objects.

Certainly, σ_{M}^j can be computed (see Sec. 5 in Ref. 27), but there is no justification to the claim that σ_{M}^j provides useful information that is *intrinsic* to the jth scatterer: The word "intrinsic" is a misnomer when applied to σ_{M}^j. (Note that all "multiple interference effects" *are* included in the calculation of c_{nj}^m.)

It is of interest to compare σ_{M}^j with Σ_{sc}^j. As u_{sc}^j satisfies the Helmholtz equation everywhere between S_j and S_R, the integration surface in Eq. (54) can be moved,

$$\sigma_{\mathrm{M}}^j = k \, \mathrm{Im} \int_{S_j} \overline{u_{\mathrm{sc}}^j} \frac{\partial u_{\mathrm{sc}}^j}{\partial n} \, \mathrm{d}S. \qquad (56)$$

Now recall the formula Eq. (31) in which

$$\Sigma_{\mathrm{sc}}^j = k \, \mathrm{Im} \int_{S_j} \overline{u_{\mathrm{sc}}} \frac{\partial u_{\mathrm{sc}}}{\partial n} \, \mathrm{d}S \qquad (57)$$

and u_{sc} is the N-body scattered field. [Note that S_j cannot be replaced by S_R in Eq. (57) because u_{sc} does not satisfy the Helmholtz equation everywhere outside S_j, in particular, not inside S_l with $l \neq j$.] Equation (31) gives the exact decomposition $\Sigma_{\mathrm{sc}} = \sum_{j=1}^{N} \Sigma_{\mathrm{sc}}^j$, but it cannot be claimed that Σ_{sc}^j gives information intrinsic to S_j [because of the presence of u_{sc} in Eq. (57)]. In more detail, using Eqs. (33) and (35),

$$\Sigma_{\mathrm{sc}}^j = k \, \mathrm{Im} \sum_{\ell=1}^{N} \sum_{m=1}^{M} \int_{S_j^+} \overline{u_{\mathrm{sc}}^\ell} \frac{\partial u_{\mathrm{sc}}^m}{\partial r_j} \, \mathrm{d}S.$$

There are N^2 integrals. The one with $\ell = m = j$ gives σ_{M}^j. Those with neither ℓ nor m equal to j are zero (because u_{sc}^n is a regular solution of the Helmholtz equation inside S_j^+ when $n \neq j$). The remaining $2N - 2$ integrals can be expressed using the exciting field u_{exc}^j, Eq. (39),

$$\sigma_{\mathrm{M}}^j - \Sigma_{\mathrm{sc}}^j = -k \, \mathrm{Im} \int_{S_j} \left(\overline{u_{\mathrm{sc}}^j} \frac{\partial u_{\mathrm{exc}}^j}{\partial n} - u_{\mathrm{exc}}^j \frac{\partial \overline{u_{\mathrm{sc}}^j}}{\partial n} \right) \mathrm{d}S. \qquad (58)$$

This quantity may be interpreted as extinction by the jth scatterer, with u_{exc}^j being the incident field; see Eq. (7). However, this interpretation is not straightforward because both u_{sc}^j and u_{exc}^j come from solving the full N-body scattering problem. Perhaps the best one can say about σ_{M}^j is that it gives one piece of Σ_{sc}^j; see Eqs. (37) and (55).

V. DISCUSSION

One cannot think about multiple scattering without thinking about Victor Twersky (1923–1998). Near the end of his life, he used the Foldy model Eq. (49) to investigate scattering by arrays of N identical small scatterers, equally spaced around a circle; the far-field pattern F and the scattering cross section Σ_{sc} are given by Eqs. (52) and (53), respectively, in terms of the strengths A_j, $j=1,2,\ldots,N$.[22,23] Twersky (p. 25 in Ref. 22) pointed out that no individual A_j is observable, and that numerical results for an individual A_j "lead to incorrect notions of the behavior of the observable multiple scattering amplitude [F] for the array" (Appendix B in Ref. 22).

Reiterating, just because a quantity (such as A_j or σ_{M}^j) is computable does not mean that meaningful information can be extracted about the physical problem from that quantity. Further analysis is required.

ACKNOWLEDGMENTS

The author is grateful to the editor and the reviewers for their encouragement and constructive comments.

APPENDIX: SOME INTEGRALS

Suppose that, in the vicinity of a sphere $r=R$,

$$U(\boldsymbol{r}) = \sum_{n,m} \{A_n^m j_n(kr) + B_n^m h_n(kr)\} Y_n^m(\hat{\boldsymbol{r}}), \tag{A1}$$

$$V(\boldsymbol{r}) = \sum_{n,m} \{C_n^m j_n(kr) + D_n^m h_n(kr)\} Y_n^m(\hat{\boldsymbol{r}}). \tag{A2}$$

Then (Lemma 6.5 in Ref. 8)

$$\int_{r=R} \left(\bar{U} \frac{\partial V}{\partial r} - V \frac{\partial \bar{U}}{\partial r} \right) \mathrm{d}S = \frac{\mathrm{i}}{k} \sum_{n,m} \left(2\overline{B_n^m} D_n^m + \overline{A_n^m} D_n^m + \overline{B_n^m} C_n^m \right). \tag{A3}$$

The proof makes use of the orthogonality of Y_n^m, Eq. (12), and Wronskians for spherical Bessel and Hankel functions (Eqs. 10.50.1 in Ref. 13). The special case $U=V$ gives

$$k \,\mathrm{Im} \int_{r=R} \bar{U} \frac{\partial U}{\partial r} \,\mathrm{d}S = \sum_{n,m} \left(|B_n^m|^2 + \mathrm{Re}\left\{ \overline{A_n^m} B_n^m \right\} \right). \tag{A4}$$

[1] D. S. Jones, "On the scattering cross section of an obstacle," Philos. Mag. **46**, 957–962 (1955).

[2] C. Athanasiadis, P. A. Martin, A. Spyropoulos, and I. G. Stratis, "Scattering relations for point sources: Acoustic and electromagnetic waves," J. Math. Phys. **43**, 5683–5697 (2002).

[3] V. Twersky, "Certain transmission and reflection theorems," J. Appl. Phys. **25**, 859–862 (1954).

[4] A. T. de Hoop, "On the plane-wave extinction cross-section of an obstacle," Appl. Sci. Res., Sec. B **7**, 463–469 (1959).

[5] R. E. Kleinman and P. A. Martin, "On single integral equations for the transmission problem of acoustics," SIAM J. Appl. Math. **48**, 307–325 (1988).

[6] P. S. Carney, J. C. Schotland, and E. Wolf, "Generalized optical theorem for reflection, transmission, and extinction of power for scalar fields," Phys. Rev. E **70**, 036611 (2004).

[7] P. A. Martin, "Acoustic scattering by inhomogeneous obstacles," SIAM J. Appl. Math. **64**, 297–308 (2003).

[8] P. A. Martin, *Multiple Scattering* (Cambridge University Press, Cambridge, UK, 2006).

[9] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd ed. (Springer, New York, 2013).

[10] P. A. Martin, "On the far-field computation of acoustic radiation forces," J. Acoust. Soc. Am. **142**, 2094–2100 (2017).

[11] R. G. Newton, "Optical theorem and beyond," Am. J. Phys. **44**, 639–642 (1976).

[12] M. V. Klibanov, "Phaseless inverse scattering problems in three dimensions," SIAM J. Appl. Math. **74**, 392–410 (2014).

[13] *Digital Library of Mathematical Functions*, available at www.dlmf.nist.gov (Last viewed 27 January 2018).

[14] R. Lim and R. H. Hackman, "A formulation of multiple scattering by many bounded obstacles using a multicentered, *T* supermatrix," J. Acoust. Soc. Am. **91**, 613–638 (1992).

[15] F. Borghese, P. Denti, and R. Saija, "Superposition through phases of the far fields scattered by the spheres of an aggregate," J. Quant. Spectrosc. Radiat. Transfer **129**, 69–78 (2013).

[16] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Springer, New York, 1982).

[17] H. C. van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981).

[18] M. I. Mishchenko, J. M. Dlugach, M. A. Yurkin, L. Bi, B. Cairns, L. Liu, R. L. Panetta, L. D. Travis, P. Yang, and N. T. Zakharova, "First-principles modeling of electromagnetic scattering by discrete and discretely heterogeneous random media," Phys. Rep. **632**, 1–75 (2016).

[19] M. J. Berg, C. M. Sorensen, and A. Chakrabarti, "Extinction and the optical theorem. Part II. Multiple particles," J. Opt. Soc. Am. A **25**, 1514–1520 (2008).

[20] A. E. Moskalensky, D. I. Strokotov, A. V. Chernyshev, V. P. Maltsev, and M. A. Yurkin, "Additivity of light-scattering patterns of aggregated biological particles," J. Biomed. Opt. **19**, 085004 (2014).

[21] L. L. Foldy, "The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers," Phys. Rev. **67**, 107–119 (1945).

[22] V. Twersky, "Multiple scattering by finite regular arrays of resonators," J. Acoust. Soc. Am. **87**, 25–41 (1990).

[23] V. Twersky, "Regular polygonal arrays of resonant scatterers," J. Acoust. Soc. Am. **89**, 991–998 (1991).

[24] P. A. Martin, "On acoustic and electric Faraday cages," Proc. R. Soc. A **470**, 20140344 (2014).

[25] F. Schäfer, F. Eckert, and T. Wellens, "Cooperative scattering of scalar waves by optimized configurations of point scatterers," J. Phys. B: At. Mol. Opt. Phys. **50**, 235502 (2017).

[26] F. G. Mitri, "Extrinsic extinction cross-section in the multiple acoustic scattering by fluid particles," J. Appl. Phys. **121**, 144904 (2017).

[27] F. G. Mitri, "Intrinsic acoustical cross sections in the multiple scattering by a pair of rigid cylindrical particles in 2D," J. Phys. D: Appl. Phys. **50**, 325601 (2017).

[28] F. G. Mitri, "Extrinsic and intrinsic acoustical cross-sections for a viscous fluid particle near a planar rigid boundary: Circular cylinder example," J. Phys. Commun. **1**, 055015 (2017).