Quadratic quantities in acoustics: Scattering cross-section and radiation force

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**Highlights**

- Exact computation of scattering cross-section.
- Exact computation of radiation force.
- All computations done in the near field.

**Abstract**

Time-harmonic acoustic waves interact with a scatterer: this interaction is calculated using linear, first-order theory. However, there are quadratic, second-order quantities that are of interest. These include the scattering cross-section and the steady radiation force; these quantities can be expressed as integrals of products of first-order quantities over a sphere. These integrals are evaluated exactly. The results are infinite series of products of the coefficients in the spherical multipole expansions of the incident and scattered fields; they do not depend on the radius of the spherical integration surface. For a specific scattering problem, the coefficients can be connected by an appropriate $T$-matrix. In most previous work, the spherical surface is moved to infinity so that far-field quantities can be introduced: it is shown that this process is not straightforward and it may introduce spurious difficulties.

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**1. Introduction**

In the context of time-harmonic acoustics, a given incident wave interacts with one or more scatterers surrounded by a lossless inviscid compressible fluid. This process is usually characterised by introducing three "cross-sections" related by

$$
\sigma_{sc} + \sigma_{ab} = \sigma_{ex}.
$$

The scattering cross-section $\sigma_{sc}$ gives a measure of how much energy is radiated away by the scatterers whereas the absorption cross-section $\sigma_{ab}$ gives how much energy is absorbed by the scatterers. Hence the extinction cross-section $\sigma_{ex}$ describes how much energy has been extracted from the incident wave by the scattering process. (Precise definitions will be given later.) When the scatterers are lossless (for example, sound-soft or sound-hard), $\sigma_{ab} = 0$ and then $\sigma_{sc} = \sigma_{ex}$.

Energy is a quadratic quantity: for linear waves of (small) amplitude $A$, energy is proportional to $A^2$. Another quadratic quantity is the acoustic radiation force: for linear time-harmonic waves, it is a steady second-order force, analogous to the Stokes drift force in the context of surface water waves. The study of acoustic radiation forces has a long history, with detailed
computations first made by King [1] for scattering by a small sphere. These forces have attracted more interest in recent times because they can be used to levitate and manipulate small particles. For reviews, see [2,3].

The cross-sections and the radiation force can be expressed as certain integrals of first-order quantities over a sphere of radius \( R \), \( S_R \), enclosing the scatterer; and they are independent of \( R \). The usual strategy is to suppose that \( R \) is large and then far-field estimates can be used, but this is not always straightforward. Here, we adopt a brute-force approach and simply evaluate the integrals over \( S_R \) directly and exactly, having introduced standard expansions of the incident and scattered fields using spherical wavefunctions, together with appropriate Wronskians. The final results are seen to be independent of \( R \). They are infinite series of products of coefficients from the spherical expansions. Some of these series can be expressed, exactly, in terms of far-field quantities. Finally, we connect the coefficients in the wavefunction expansions using the \( T \)-matrix, leading to a viable method for the computation of cross-sections and the radiation force.

The basic equations are introduced briefly in Section 2 and the \( R \)-independent integrals over \( S_R \) are derived in Section 3. These integrals are evaluated in Section 4 after introducing expansions in spherical wavefunctions; most of the details are relegated to an Appendix.

Section 5 is concerned with the use of far-field quantities; various pitfalls and errors are described. It is well known that \( \sigma_{sc} \) can be expressed exactly as an integral of the far-field pattern, and this formula is recovered. However, doing the same for other quadratic quantities is awkward, mainly because they involve the incident field. After some discussion of a paper by Debye from 1909 (Section 5.3), \( T \)-matrix methods are described briefly in Section 6. The paper ends with a summary of the results obtained, covering why they are useful and what they teach us about the computation of quadratic quantities in acoustics.

2. Governing equations

The exact governing equations for a compressible inviscid fluid are

\[
\tilde{\rho} \left( \frac{\partial \tilde{v}}{\partial t} + (\tilde{v} \cdot \nabla) \tilde{v} \right) = -\text{grad} \tilde{p} + \frac{\partial \tilde{\rho}}{\partial t} + \text{div} (\tilde{\rho} \tilde{v}) = 0,
\]

(2)

with \( \tilde{p} = \tilde{p}(\tilde{\rho}) \), where \( \tilde{p}(r, t) \), \( \tilde{v}(r, t) \) and \( \tilde{\rho}(r, t) \) are the pressure, velocity and density, respectively, at position \( r \) and time \( t \). Introduce a small parameter \( \varepsilon \), and expand \( \tilde{p} \), \( \tilde{\rho} \) and \( \tilde{v} \) [2, §III. F]:

\[
\tilde{p} = p_0 + \varepsilon \tilde{p}_1 + \varepsilon^2 \tilde{p}_2 + \cdots, \quad \tilde{\rho} = \rho_0 + \varepsilon \tilde{\rho}_1 + \varepsilon^2 \tilde{\rho}_2 + \cdots, \quad \tilde{v} = \varepsilon \tilde{v}_1 + \varepsilon^2 \tilde{v}_2 + \cdots.
\]

The equation of state gives \( \tilde{p} = \tilde{p}(\tilde{\rho}) = \tilde{p}(\rho_0) + (\tilde{\rho} - \rho_0) \tilde{p}'(\rho_0) + \frac{1}{2}(\tilde{\rho} - \rho_0)^2 \tilde{p}''(\rho_0) + \cdots \). This gives \( \tilde{p}_0 = \tilde{p}(\rho_0), \tilde{p}_1 = c^2 \tilde{\rho}_1 \) and \( \tilde{p}_2 = c^2 \tilde{\rho}_2 + \frac{1}{2}c^2 \tilde{p}'(\rho_0) \) with \( c^2 = \tilde{p}'(\rho_0) \). We assume that \( p_0, \rho_0 \) and \( c^2 \) are constants.

2.1. Linear theory

At first order, \( \varepsilon^1 \), we obtain

\[
\rho_0 \frac{\partial \tilde{\rho}_1}{\partial t} + \text{grad} \tilde{p}_1 = 0, \quad \frac{\partial \tilde{\rho}_1}{\partial t} + \rho_0 c^2 \text{div} \tilde{v}_1 = 0.
\]

(3)

Eliminating \( \tilde{v}_1 \) gives the usual wave equation for \( \tilde{p}_1 \); \( \tilde{p}_1 \) satisfies the same equation.

Taking the scalar product of Eq. (3) with \( \tilde{v}_1 \), making use of Eq. (3)\_2, we obtain the acoustic energy equation,

\[
\frac{\partial \mathcal{E}}{\partial t} + \text{div} (\tilde{p}_1 \tilde{v}_1) = 0
\]

(4)

where \( \mathcal{E} = \frac{1}{2} \rho_0 (\tilde{v}_1 \cdot \tilde{v}_1) + \frac{1}{2} \tilde{p}_1^2 / (\rho_0 c^2) \) is the acoustic energy density [4, §1.9], [5, Eq. (64.5)].

Let \( D \) be a fixed bounded volume with boundary \( S \). Integrating Eq. (4) over \( D \) gives

\[
\frac{d}{dt} \int_D \mathcal{E} \, dV - \int_S \frac{\tilde{p}_1}{\tilde{v}_1} \cdot n \, dS = 0,
\]

(5)

where we have used the divergence theorem and \( n \) is the unit normal on \( S \) pointing into \( D \). This equation has the dimensions of force \( \times \) velocity, that is, power. It says that the rate of increase of energy in \( D \) is balanced by the acoustic power entering through \( S \).

Next, consider time-harmonic motions, and write

\[
\tilde{p}_1(r, t) = \text{Re} \{ p(r) e^{-i\omega t} \} = p_R \cos \omega t + p_I \sin \omega t,
\]

(6)

where \( p_R = \text{Re} p \) and \( p_I = \text{Im} p \) are real. Similarly, \( \tilde{v}_1 = \text{Re} \{ v e^{-i\omega t} \} \) and \( \tilde{\rho}_1 = \text{Re} \{ \rho e^{-i\omega t} \} \).

For quadratic quantities (such as energy) involving products of time-harmonic functions, it is usual to consider time averages over one period. Thus, for any function \( f(r, t) \), we define

\[
\langle f \rangle (r) = \frac{1}{T} \int_0^T f(r, t) \, dt, \quad T = \frac{2\pi}{\omega}.
\]
Simple calculations give \( \langle \partial \varepsilon / \partial t \rangle = 0 \) and \( \langle \bar{p}_1 \bar{v}_1 \rangle = \frac{1}{2} \text{Re} \{ \bar{p} \bar{v} \} \), where the overbar denotes complex conjugation. Hence the time-averaged form of Eq. (4) is \( \text{Re} \{ \text{div} (\bar{p} \bar{v}) \} = 0 \) and the time-averaged form of Eq. (5) is
\[
\text{Re} \int_S \bar{p} \bar{v} \cdot \mathbf{n} \, dS = 0. \tag{7}
\]

For irrotational motion, we can introduce a velocity potential \( \tilde{u}_1 \), with \( \bar{v}_1 = \text{grad} \tilde{u}_1 \) and \( \bar{p}_1 = -\rho_0 \partial \tilde{u}_1 / \partial t \). Furthermore, for time-harmonic motions, we write
\[
\tilde{u}_1 (r, t) = (c/k) \text{Re} \{ u (r) e^{-i\omega t} \} = (c/k) (u_k \cos \omega t + u_t \sin \omega t)
\]
where \( k = \omega / c \), \( u_k = \text{Re} u \) and \( u_t = \text{Im} u \) are real, \( (\nabla^2 + k^2)u = 0 \), and \( u \) is dimensionless.

Some calculation gives \( \langle \varepsilon \rangle = \langle k \rangle + \langle \nu \rangle \) where
\[
\langle \varepsilon \rangle = \frac{\rho_0 c^2}{2k^2} (\text{grad} u) \cdot (\text{grad} \bar{u}),
\]
\[
\langle k \rangle = \frac{\rho_0 c^2}{4k^2} (\text{grad} u) - (\text{grad} \bar{u}),
\]
and \( \langle \nu \rangle = \frac{\rho_0 c^2}{4} u \bar{u} \).

Evidently, \( \langle k \rangle \) is the linear approximation to the average acoustic kinetic energy density and \( \langle \nu \rangle \) is the analogous potential energy density. Also, as \( \text{Re} \{ \bar{p} \bar{v} \} = c^2 (\rho_0 / k) \text{Im} (\bar{u} \text{grad} u) \), Eq. (7) becomes
\[
\text{Im} \int_S \bar{u} \frac{\partial u}{\partial n} \, dS = 0 \tag{9}
\]
where \( \partial u / \partial n = n \cdot \text{grad} u \) is the normal derivative of \( u \) on \( S \). The formula Eq. (9) will be used in Section 3.1.

2.2. Second-order theory

At second order, \( \nu^2 \), Eq. (2), gives
\[
\rho_0 \frac{\partial \bar{v}_1}{\partial t} + \bar{p}_1 \frac{\partial \bar{v}_1}{\partial t} + \rho_0 (\bar{v}_1 \cdot \nabla) \bar{v}_1 = -\text{grad} \bar{p}_2. \tag{10}
\]
Multiply Eq. (3) by \( \bar{v}_1 \) and then add the result to Eq. (10); this gives
\[
\rho_0 \frac{\partial \bar{v}_1^{(2)}}{\partial t} + \frac{\partial (\bar{p}_1 \bar{v}_1^{(1)})}{\partial t} + \rho_0 \frac{\partial (\bar{v}_1^{(1)} \bar{v}_1^{(1)})}{\partial x_j} = -\frac{\partial \bar{p}_2}{\partial x_j}, \quad i = 1, 2, 3. \tag{11}
\]
where \( r \) has components \( x_i \) and we have denoted the components of \( \tilde{v}_h \) by \( \tilde{v}_h^{(i)} \).

It is easy to see that \( \bar{p}_2 \) has the form \( \bar{p}_2 (r, t) = p_2^0 (r) + p_2^1 (r) \cos 2\omega t + p_2^2 (r) \sin 2\omega t \), with similar expressions for \( \bar{p}_2 \) and \( \bar{v}_2 \). We are interested in the steady component of \( \bar{p}_2 \), \( p_2^0 = \langle \bar{p}_2 \rangle \). Extracting the steady component from Eq. (11), we obtain
\[
-\frac{\partial p_2^0}{\partial x_i} = \rho_0 \frac{\partial}{\partial x_j} \left( \bar{v}_1^{(1)} \bar{v}_1^{(1)} \right)
\]
\[
= \rho_0 \frac{c^2}{2k^2} \text{Re} \left( \frac{\partial (\bar{v}_1 \bar{v}_1)}{\partial x_j} \right) \tag{12}
\]
\[
= \rho_0 \frac{c^2}{2k^2} \text{Re} \left( \frac{\partial (\bar{u} \bar{u})}{\partial x_j} \right) \tag{13}
\]
\[
= \rho_0 \frac{c^2}{4k^2} \frac{\partial}{\partial x_i} \left( (\text{grad} u) \cdot (\text{grad} \bar{u}) - k^2 u \bar{u} \right). \tag{14}
\]

We can write Eq. (12) as \( (\partial / \partial x_i) \langle \Pi_y \rangle = 0 \) where \( \Pi_y = p_2^0 \delta_{ij} + \rho_0 \bar{v}_1^{(1)} \bar{v}_1^{(1)} \) [6]. The quantity \( S_y = -\langle \Pi_y \rangle \) is known as the acoustic radiation stress tensor. From Eq. (13), we have
\[
S_y = -p_2^0 \delta_{ij} - \rho_0 \frac{c^2}{2k^2} \text{Re} \left( \frac{\partial u \partial \bar{u}}{\partial x_i} \right). \tag{15}
\]

Integrating Eq. (14) gives (after discarding a constant of integration)
\[
-\rho_0 \frac{c^2}{2k^2} \left( (\text{grad} u) \cdot (\text{grad} \bar{u}) - k^2 u \bar{u} \right) = \langle k \rangle - \langle \nu \rangle,
\]
see Eq. (8), and then
\[
S_y = \rho_0 \frac{c^2}{4k^2} \left( (\text{grad} u) \cdot (\text{grad} \bar{u}) - k^2 u \bar{u} \right) \delta_{ij} - \frac{\partial u \partial \bar{u}}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} - \frac{\partial \bar{u} \partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} \right). \tag{16}
\]

As \( (\partial / \partial x_i) S_{ij} = 0 \), an application of the divergence theorem in the region \( D \) gives
\[
\int_D S_{ij} n_j \, dS = 0.
\]
This formula will be used in Section 3.2.
3. Scattering problems

Suppose that we have a scatterer \( B \) with boundary \( S \). Surround the scatterer by a sphere of radius \( R \), \( S_R \), implying that we have chosen an origin \( O \) in \( B \). We use spherical polar coordinates, \( r, \theta \) and \( \phi \), so that \( S_R \) is \( r = R \).

For scattering problems, we write the total potential as

\[
u = u_{\text{inc}} + u_{\text{sc}},
\]

where \( u_{\text{inc}}(r) \) is the specified incident field, \( u_{\text{sc}}(r) \) is the unknown scattered field, and \( u_{\text{sc}} \) satisfies the Sommerfeld radiation condition. Recall that the wavenumber \( k \) is real: the fluid is lossless.

3.1. Cross-sections

Let \( S = S \cup S_R \) with \( \mathcal{D} \) being the region between \( S \) and \( S_R \). Then Eq. (9) gives

\[
\text{Im} \int_{S_R} \overline{u} \frac{\partial u}{\partial r} \, dS - \text{Im} \int_S \overline{u} \frac{\partial u}{\partial n} \, dS = 0,
\]

(18)

where the normal vector on \( S \) points out of \( B \). Using Eq. (17), we have

\[
\overline{u} \frac{\partial u}{\partial r} = u_{\text{inc}} \frac{\partial u_{\text{inc}}}{\partial r} + u_{\text{sc}} \frac{\partial u_{\text{inc}}}{\partial r} + u_{\text{inc}} \frac{\partial u_{\text{sc}}}{\partial r} + u_{\text{sc}} \frac{\partial u_{\text{sc}}}{\partial r}.
\]

(19)

Substituting in the first integral in Eq. (18), the first term from Eq. (19) gives

\[
\text{Im} \int_{S_R} u_{\text{inc}} \frac{\partial u_{\text{inc}}}{\partial r} \, dS = \frac{1}{2i} \int_{S_R} \left( \overline{u_{\text{inc}}} \frac{\partial u_{\text{inc}}}{\partial r} - u_{\text{inc}} \frac{\partial \overline{u_{\text{inc}}}}{\partial r} \right) \, dS = 0,
\]

by Green’s theorem, assuming that \( u_{\text{inc}} \) is a regular solution of the Helmholtz equation everywhere inside \( S_R \). Substituting the remaining terms from Eq. (19) in Eq. (18) gives

\[
\text{Im} \int_{S_R} u_{\text{inc}} \frac{\partial u_{\text{inc}}}{\partial r} \, dS + \text{Im} \int_{S_R} \left( \overline{u_{\text{inc}}} \frac{\partial u_{\text{inc}}}{\partial r} - u_{\text{inc}} \frac{\partial \overline{u_{\text{inc}}}}{\partial r} \right) \, dS - \text{Im} \int_S \overline{u} \frac{\partial u}{\partial n} \, dS = 0,
\]

(20)

using \( \text{Im} \overline{w} = -\text{Im} w \) for any complex quantity \( w \).

The first term on the left-hand side of Eq. (20) involves the scattered field only; it is closely related to the scattering cross-section. We define

\[
\sigma_{\text{sc}} = k \text{Im} \int_{S_R} \overline{u_{\text{sc}}} \frac{\partial u_{\text{sc}}}{\partial r} \, dS,
\]

(21)

where the factor \( k \) has been inserted so that \( \sigma_{\text{sc}} \) is dimensionless. Physically, \( \sigma_{\text{sc}} \) is proportional to the radiated acoustic power.

The third term on the left-hand side of Eq. (20) involves the total field on \( S \); it is closely related to the absorption cross-section. We define

\[
\sigma_{\text{ab}} = -k \text{Im} \int_S \overline{u} \frac{\partial u}{\partial n} \, dS.
\]

(22)

Physically, \( \sigma_{\text{ab}} \) represents the acoustic power absorbed by the scatterers (if they are lossy).

Combining Eqs. (1) and (20), we define the extinction cross-section by

\[
\sigma_{\text{ex}} = \sigma_{\text{sc}} + \sigma_{\text{ab}} = -k \text{Im} \int_{S_R} \left( \overline{u_{\text{sc}}} \frac{\partial u_{\text{sc}}}{\partial r} - u_{\text{inc}} \frac{\partial \overline{u_{\text{sc}}}}{\partial r} \right) \, dS.
\]

(23)

Sometimes, the scattering cross-section is defined as being proportional to \( \lim_{R \to \infty} \sigma_{\text{sc}} \); see, for example, [7, §2.A.4]. However, as we shall see below, \( \sigma_{\text{sc}} \) does not depend on \( R \). This fact was known to Twersky [8, Eq. (6)] and de Hoop [9, Eq. (4.7)], for example. Indeed, if we write Eq. (21) as

\[
\sigma_{\text{sc}} = \frac{k}{2i} \int_{S_R} \left( \overline{u_{\text{sc}}} \frac{\partial u_{\text{sc}}}{\partial r} - u_{\text{sc}} \frac{\partial \overline{u_{\text{sc}}}}{\partial r} \right) \, dS,
\]

we see that we can move the integration surface onto the scatterer itself,

\[
\sigma_{\text{sc}} = k \text{Im} \int_{S} \overline{u_{\text{sc}}} \frac{\partial u_{\text{sc}}}{\partial n} \, dS.
\]

(24)

The derivation uses Green’s theorem, and so it requires that \( u_{\text{sc}} \) satisfies the Helmholtz equation everywhere between \( S \) and \( S_R \), and that the wavenumber \( k \) is real (which it is, because the exterior medium is assumed to be lossless).
3.2. Radiation force

The steady acoustic radiation force acting on the (fixed) scatterer $S$ is $\mathbf{F}$ with (real) components

$$ F_i = \int_S S_i n_i \, dS, $$

where $S$ is defined by Eq. (15). Moreover, using Eq. (16) with $S = S \cup S_r$, we can move the surface of integration from $S$ to the sphere $S_R$ enclosing $S$.

$$ F_i = \int_{S_R} S_i \frac{x_i}{R} \, dS. $$

The fact that the integration surface can be moved is well known [10–12]. The underlying assumption is that the fluid is lossless. For viscous fluids, the situation is more complicated [13].

Let $A_0 = (\text{grad} \, u) \cdot (\text{grad} \, \overline{u}) - k^2 \overline{u}$. Then, from Eq. (15),

$$ F_z = \frac{\rho_0 c^2}{4k^2} \int_{S_R} \left( A_0 \cos \theta - \frac{\partial u}{\partial z} \frac{\partial \overline{u}}{\partial r} - \frac{\partial \overline{u}}{\partial z} \frac{\partial u}{\partial r} \right) \, dS, $$

$$ F_x \pm iF_y = \frac{\rho_0 c^2}{4k^2} \int_{S_R} \left( A_0 e^{\pm i \phi} \sin \theta - \left( \frac{\partial u}{\partial x} \pm i \frac{\partial u}{\partial y} \right) \frac{\partial \overline{u}}{\partial r} - \left( \frac{\partial \overline{u}}{\partial x} \pm i \frac{\partial \overline{u}}{\partial y} \right) \frac{\partial u}{\partial r} \right) \, dS. $$

In terms of spherical polar coordinates, we have

$$ \frac{\partial u}{\partial z} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}, \quad \text{grad} \, u = \frac{\partial u}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \hat{\phi}, $$

$$ \frac{\partial u}{\partial x} \pm i \frac{\partial u}{\partial y} = e^{\pm i \phi} \left( \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \pm i \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \right). $$

Hence

$$ F_z = \frac{\rho_0 c^2}{4k^2} \int_{S_R} A_z \, dS \quad \text{and} \quad F_x \pm iF_y = \frac{\rho_0 c^2}{4k^2} \int_{S_R} A_{\pm} e^{\pm i \phi} \, dS $$

with

$$ A_z = \left( -\frac{\partial u}{\partial r} \frac{\partial \overline{u}}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial \overline{u}}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \phi} \frac{\partial \overline{u}}{\partial \phi} - k^2 \overline{u} \right) \cos \theta $$

$$ + \frac{\sin \theta}{r} \left( \frac{\partial u}{\partial r} \frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{u}}{\partial r} \frac{\partial u}{\partial \theta} \right), $$

$$ A_{\pm} = \left( -\frac{\partial u}{\partial r} \frac{\partial \overline{u}}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial \overline{u}}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \phi} \frac{\partial \overline{u}}{\partial \phi} - k^2 \overline{u} \right) \sin \theta $$

$$ - \frac{\cos \theta}{r} \left( \frac{\partial u}{\partial r} \frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{u}}{\partial r} \frac{\partial u}{\partial \theta} \right) \mp \frac{i}{r \sin \theta} \left( \frac{\partial u}{\partial \phi} \frac{\partial \overline{u}}{\partial r} + \frac{\partial \overline{u}}{\partial \phi} \frac{\partial u}{\partial r} \right). $$

Formulas of this kind can be found in [6,14,15]. In most of these papers, it is assumed that $S_R$ is a large sphere and then asymptotic estimates based on the assumption that $kR \gg 1$ are used. This is awkward because certain quantities involving $u_{\text{inc}}$ diverge as $kR \to \infty$. For further discussion, see Section 5.1.

4. Use of spherical wavefunctions

In the vicinity of the sphere $S_R$, we can expand $u_{\text{inc}}$ and $u_{\text{sc}}$ as follows,

$$ u_{\text{inc}}(r) = \sum_{n,m} D_{nn}^m (kr) \tilde{Y}_n^m(\hat{r}), $$

$$ u_{\text{sc}}(r) = \sum_{n,m} C_{nn}^m h_n(kr) \tilde{Y}_n^m(\hat{r}), $$

where (using the notation from [16])

$$ \sum_{n,m} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n}, $$

$$ \sum_{m} = \sum_{m=0}^{\infty}. $$
\(j_n\) is a spherical Bessel function, \(h_n \equiv h_n^{(1)}\) is a spherical Hankel function, \(\hat{Y}_m^n\) is a spherical harmonic, \(\hat{r} = r/r\), and \(D_m^n\) and \(C_m^n\) are coefficients. We use unnormalised spherical harmonics

\[
\hat{Y}_m^n(\hat{r}) = P_n^m(\cos \theta) e^{im\phi},
\]

where the associated Legendre functions \(P_n^m\) satisfy the orthogonality relation

\[
\int_{-1}^{1} P_n^m(t) P_v^m(t) \, dt = \frac{\delta_{n,v}}{2n + 1}, \quad \delta_{n,v} = \begin{cases} 1 & \text{if } n = v, \\ 0 & \text{otherwise}. \end{cases}
\]

It is more convenient to use \(\hat{Y}_m^n\) instead of the normalised functions \(Y_m^n\) (with \(Y_0^n = A_n^0 \hat{Y}_0^n\) and \(A_n^0 = (2\pi h_n^0)^{-1/2}\)) because, when calculating \(F\), we shall encounter contiguous spherical harmonics, \(\hat{Y}_{n+1}^{m+1}\).

On \(S_r\), we have \(dS = R^2 \, d\Omega(\hat{r})\), where \(\Omega\) is the unit sphere. The spherical harmonics are orthogonal,

\[
\int_{\Omega} \hat{Y}_m^n \hat{Y}_v^n \, d\Omega = 2\pi h_n^m \delta_{m,v} \delta_{m_t}. \tag{31}
\]

### 4.1 Cross-sections

Starting with Eq. (21), direct calculation (put \(D_n^m = 0\) in Eq. (A.4)) gives

\[
\sigma_{sc} = 2\pi \sum_{n,m} h_n^m |C_n^m|^2. \tag{32}
\]

Similarly, starting with Eq. (23) (put \(D_n^m = E_n^m = F_n^m = D_n^m\) in Eq. (A.3)),

\[
\sigma_{ex} = -2\pi \text{Im} \sum_{n,m} i h_n^m \bar{C}_n^m \bar{D}_n^m = -2\pi \text{Re} \sum_{n,m} h_n^m C_n^m \bar{D}_n^m. \tag{33}
\]

### 4.2 Radiation force: \(F_2\)

The axial component of \(F\), \(F_2\), is given by Eqs. (25) and (26). Expand the total potential as

\[
u = \sum_{n,m} u_n^m(kr) \hat{Y}_m^n(\theta, \phi) \quad \text{with} \quad u_n^m(kr) = D_n^m j_n(kr) + C_n^m h_n(kr). \tag{34}
\]

Then, inspection of Eq. (26) suggests that the simplest term in \(\Lambda_2\) gives the contribution

\[
I_1 = \int_{S_R} \frac{k^2 u \bar{u} \cos \theta \, dS}{2\pi} = 2\pi (kR)^2 \sum_{n,m=0}^{\infty} w_n^m \bar{w}_v^m \int_{-1}^{1} P_n^m(\cos \theta) P_v^m(\cos \theta) \cos \theta \sin \theta \, d\theta
\]

\[
= 2\pi (kR)^2 \sum_{v=1}^{\infty} w_0^0 \bar{w}_0^0 \int_{-1}^{1} t P_v(t) \, dt + 2\pi (kR)^2 \sum_{n=1}^{\infty} \sum_{m=-n}^{n} w_n^m \bar{w}_v^m \int_{-1}^{1} t P_n^m(t) P_v^m(t) \, dt
\]

where \(w_n^m \equiv w_n^m(kR)\) and we have separated off the contribution from \(n = 0\). The first integral is \(\frac{2}{3} \delta_{v,1}\). The second integral can be evaluated using [16, Eq. (A.13)]

\[
(2v + 1) t P_v(t) = (v - m + 1) P_{v+1}(t) + (v + m) P_{v-1}(t)
\]

and the orthogonality relation, Eq. (30):

\[
\int_{-1}^{1} t P_n^m(t) P_v^m(t) \, dt = \frac{h_n^m}{2v + 1} \left\{ (n-m) \delta_{v,n-1} + (n+m+1) \delta_{v,n+1} \right\}. \tag{35}
\]

Note that the term containing \(\delta_{v,n-1}\) is absent when \(|m| = n\) because \(P_v^m = 0\) when \(|m| > v\). Hence

\[
I_1 = \frac{4\pi}{3} (kR)^2 w_0^0 \bar{w}_0^0 + (kR)^2 \sum_{n=1}^{\infty} \sum_{m=-(n-1)}^{n-1} \frac{4\pi (n+m)! \, w_n^m \bar{w}_{n-1}^m}{(2n-1)!(2n+1)(n-1-m)!}
\]

\[
+ (kR)^2 \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{4\pi (n+m+1)! \, w_n^m \bar{w}_{n+1}^m}{(2n+3)(2n+1)(n-m)!}
\]

\[
= 4\pi (kR)^2 \sum_{n,m} \delta_n^m \left\{ w_n^m \bar{w}_{n+1}^m + w_n^m \bar{w}_{n+1}^m \right\}. \tag{36}
\]
Similarly,

\[ I_2 = \int_{S_n} \frac{\partial u \, \partial \tilde{u}}{\partial \theta \, \partial \phi} \cos \theta \, dS = 4\pi (kR)^2 \sum_{n,m} \tilde{z}_n^m \left\{ w_n^{m'} w_{n+1}^{m-m'} + w_n^{m-m'} w_{n+1}^{m'} \right\}. \]  

(37)

The remaining terms in Eq. (26) make the following contributions.

\[ I_3 = \int_{S_n} \frac{1}{r^2} \left( \frac{\partial u \, \partial \tilde{u}}{\partial \theta} \right) \cos \theta \, dS = 4\pi \sum_{n,m} n(n+2) \tilde{z}_n^m \left\{ w_n^m w_{n+1}^{m-m'} + w_n^{m-m'} w_{n+1}^m \right\}, \]

(39)

\[ I_4 = \int_{S_n} \frac{\sin \theta}{r} \left( \frac{\partial u \, \partial \tilde{u}}{\partial \theta \, \partial \phi} + \frac{\partial u \, \partial \tilde{u}}{\partial \phi} \right) \cos \theta \, dS \]

(40)

\[ = 4\pi (kR)^2 \sum_{n,m} \tilde{z}_n^m \left\{ n \left( w_n^m w_{n+1}^{m-m'} + w_n^{m-m'} w_{n+1}^m \right) - (n+2) \left( w_n^m w_{n+1}^{m'} + w_n^{m'} w_{n+1}^m \right) \right\}. \]  

(41)

These integrals are evaluated in the Appendix.

From Eqs. (25) and (26), we have

\[ F_z = \frac{\rho c^2}{4k^2} (-I_1 - I_2 + I_3 + I_4) = -\frac{\pi \rho c^2}{k^2} \sum_{n,m} \tilde{z}_n^m \left( Q_n^m + \bar{Q}_n^m \right) \]  

(42)

where

\[ Q_n^m = s^2 \left( w_n^m w_{n+1}^{m-m'} + w_n^{m-m'} w_{n+1}^m \right) - n(n+2)w_n^m w_{n+1}^{m-m'} - snw_n^m w_{n+1}^{m-m'} + s(n+2)w_n^{m-m'} w_{n+1}^m, \]  

(43)

\[ w_n^m \equiv w_n^m(s) \]  

and \( s = kR. \) In fact, as we know that \( F_z \) does not depend on \( R, \) \( Q_n^m \) cannot depend on \( s; \) this provides a check on subsequent calculations.

Recall that \( w_n^m \) is given by Eq. (34). Letting \( z_n \) denote \( j_n \) or \( h_n, \) we have

\[ (n+2)z_{n+1}(s) = s(z_n(s) - z'_{n+1}(s)), \]

\[ nz_n(s) = s(z'_n(s) + z_{n+1}(s)) \]

whence

\[ (n+2)w_n^{m+1} = s(D_n^{m+1} j_n + c_n^{m+1} h_n - w_n^{m'}), \]

\[ nw_n^m = s(w_n^{m'} + D_n^m j_n + c_n^m h_n). \]  

(44)

Using these in Eq. (43),

\[ s^{-2} Q_n^m = w_n^m w_{n+1}^{m-m'} + w_n^{m-m'} w_{n+1}^m - (w_n^{m'} + D_n^m j_n + c_n^m h_n) \left( D_n^{m+1} j_n + c_n^{m+1} h_n - w_n^{m'} \right) \]

\[ - (w_n^m + D_n^m j_n + c_n^m h_n) w_n^{m+1} - w_n^{m'} \left( D_n^{m+1} j_n + c_n^{m+1} h_n - w_n^{m'} \right) \]

\[ = w_n^{m+1} - (D_n^{m+1} j_n + c_n^{m+1} h_n) \left( D_n^{m+1} j_n + c_n^{m+1} h_n \right) \]

\[ = (D_n^m j_n + c_n^m h_n) \left( D_n^{m+1} j_n + c_n^{m+1} h_n \right) - (D_n^m j_n + c_n^m h_n) \left( D_n^{m+1} j_n + c_n^{m+1} h_n \right) \]

\[ = c_n^m (D_n^{m+1} j_n + c_n^{m+1} h_n) + c_n^{m+1} (D_n^m h_n + h_n + j_n) + c_n^m (c_n^{m+1} (h_n h_n - h_n + j_n) + c_n^{m+1} (h_n h_n - h_n + j_n) \]

\[ = i s^{-2} (D_n^{m+1} j_n + c_n^{m+1} h_n + 2c_n^{m+1} c_n m_{n-1}^m), \]

using \[ [17, \text{10.503}]. \] As expected, \( Q_n^m \) does not depend on \( s. \) In addition, the terms involving products of spherical Bessel functions cancel: there are no radiation forces without scattering.

Substitution in Eq. (42) gives

\[ F_z = \frac{2\pi \rho c^2}{k^2} \sum_{n,m} \tilde{z}_n^m \left( D_n^m c_n^{m+1} + c_n D_n^{m+1} + 2c_n^m c_n^{m-1} \right). \]  

(45)
4.3. Radiation force: $F_x$ and $F_y$

Consider $F_x \pm iF_y$, defined by Eqs. (25), (27) and (34). As $F_x$ and $F_y$ are real, we choose to calculate $F_x - iF_y$ without loss of generality. The following results are found; see the Appendix for details. First,

$$I_5 \equiv \int_{S_R} k^2 u \overline{u} e^{-i\phi} \sin \theta \, dS = 4\pi (kR)^2 \sum_{n,m} \left\{ \chi_n^m w_{n+1}^{m+1} \overline{w}_n^m - \gamma_n^m w_n^m \overline{w}_{n+1}^{m+1} \right\},$$

$$I_6 \equiv \int_{S_R} \frac{\partial u}{\partial r} \frac{\partial \overline{u}}{\partial r} e^{-i\phi} \sin \theta \, dS = 4\pi (kR)^2 \sum_{n,m} \left\{ \chi_n^m w_{n+1}^{m+1} \overline{w}_n^m - \gamma_n^m w_n^m \overline{w}_{n+1}^{m+1} \right\},$$

where

$$\chi_n^m = \frac{(n + m + 2)!}{(2n + 3)(2n + 1)(n - m)!} \quad \text{and} \quad \gamma_n^m = \frac{(n + m)!}{(2n + 3)(2n + 1)(n - m)!}.$$

Two further pieces are as follows:

$$I_7 \equiv \int_{S_R} \frac{1}{r} \left( \frac{\partial u}{\partial \theta} \frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{u}}{\partial \phi} \frac{\partial \overline{u}}{\partial \phi} \right) e^{-i\phi} \sin \theta \, dS = 4\pi \sum_{n,m} n(n+2) \left\{ \chi_n^m w_{n+1}^{m+1} \overline{w}_n^m - \gamma_n^m w_n^m \overline{w}_{n+1}^{m+1} \right\},$$

$$I_8 \equiv \int_{S_R} \left\{ \frac{-\cos \theta}{r^2} \left( \frac{\partial u}{\partial \theta} \frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{u}}{\partial \phi} \frac{\partial \overline{u}}{\partial \phi} \right) + \frac{i}{r \sin \theta} \left( \frac{\partial u}{\partial \overline{u}} \frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{u}}{\partial \theta} \frac{\partial \overline{u}}{\partial \phi} \right) \right\} e^{-i\phi} \, dS = 4\pi kR \sum_{n,m} \chi_n^m \left\{ n \omega_{n+1}^{m+1} \overline{w}_n^m - (n+2) \omega_n^{m+1} \overline{w}_{n+1}^m \right\}$$

$$- 4\pi kR \sum_{n,m} \gamma_n^m \left\{ n \omega_n^{m-1} \overline{w}_{n+1}^m - (n+2) \omega_{n+1}^{m-1} \overline{w}_n^m \right\}. $$

From Eqs. (25) and (27), we have

$$F_x - iF_y = \frac{\rho c^2}{4k^2} (-I_5 - I_6 + I_7 + I_8) = -\frac{\pi \rho c^2}{k^2} \sum_{n,m} \left\{ \chi_n^m \overline{s}_n^m - \gamma_n^m \eta_n^m \right\}$$

where

$$\overline{s}_n^m = s^2 \left( w_{n+1}^{m+1} \overline{w}_n^m + w_n^{m+1} \overline{w}_{n+1}^m \right) - n(n+2) \omega_n^{m+1} \overline{w}_n^m - s \left( n \omega_n^{m+1} \overline{w}_n^m - (n+2) \omega_{n+1}^{m+1} \overline{w}_n^m \right),$$

$$\eta_n^m = s^2 \left( n \omega_n^{m-1} \overline{w}_{n+1}^m + \omega_{n+1}^{m-1} \overline{w}_n^m \right) - n(n+2) \omega_n^{m-1} \overline{w}_{n+1}^m + s \left( n \omega_n^{m-1} \overline{w}_{n+1}^m - n \omega_{n+1}^{m-1} \overline{w}_n^m \right).$$

Making use of Eqs. (34) and (44),

$$s^{-2} \overline{s}_n^m = \frac{w_{n+1}^{m+1} \overline{w}_n^m + w_n^{m+1} \overline{w}_{n+1}^m - \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} w_{n+1}^{m+1} \overline{w}_n^m + \overline{w}_n^m \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} - w_n^{m+1} \overline{w}_{n+1}^m}{w_{n+1}^{m+1} \overline{w}_n^m + \overline{w}_n^m \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} + \overline{w}_n^m \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} - w_n^{m+1} \overline{w}_{n+1}^m}$$

$$= \frac{w_{n+1}^{m+1} \overline{w}_n^m - \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} w_{n+1}^{m+1} \overline{w}_n^m}{w_{n+1}^{m+1} \overline{w}_n^m + \overline{w}_n^m \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} + \overline{w}_n^m \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} - w_n^{m+1} \overline{w}_{n+1}^m}$$

$$= \frac{C_n^{m+1} \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} - \overline{h}_n^{m+1} \overline{h}_n}{C_n^{m+1} \{ D_n^{m+1} j_n + C_n^{m+1} h_n \} + \overline{h}_n^{m+1} \overline{h}_n}$$

Alternatively,

$$\eta_n^m = i \left[ C_n^{m-1} D_n + \overline{D}_n \right] + 2 \overline{C}_n \overline{C}_n^{m-1}. $$

Substitution in Eq. (52) then gives $F_x$ and $F_y$:

$$F_x - iF_y = \frac{\pi \rho c^2}{k^2} \sum_{n,m} \chi_n^m \left\{ C_n^{m+1} D_n + \overline{C}_n D_n + 2 \overline{C}_n C_n^{m+1} \right\}$$

$$+ \frac{\pi \rho c^2}{k^2} \sum_{n,m} \gamma_n^m \left\{ C_n^{m-1} D_n + \overline{C}_n D_n + 2 \overline{C}_n C_n^{m-1} \right\}. $$

(55)
4.4. Radiation force: summary and discussion

Our expressions for the components of the radiation force $\mathbf{F}$ are exact, they are applicable to scatterers of any shape, and for arbitrary incident fields (provided $u_{sc}$ is a regular wavefunction between $S$ and $S_R$). Similar formulas can be found in [18, Eqs. (11)–(13)], [15, p. 28] and [19, p. 666]. Unfortunately, the derivations in these papers contain errors; we return to these in Section 5.2.

5. Use of far-field approximations

Let us introduce the far-field pattern, $f(\hat{r})$, defined by
\[
u_{sc}(r) \sim (ikr)^{-1} e^{ikr} f(\hat{r}) \quad \text{as} \quad r \to \infty. \tag{56}
\]
When $u_{sc}$ is expanded as Eq. (29), we obtain
\[
f(\hat{r}) = \sum_{n,m} (-i)^n C_n^m \tilde{Y}_n^m(\hat{r}). \tag{57}
\]
Formally, this formula is obtained by inserting the asymptotic approximation \[17, 10.52.4],
\[
h_n(w) \sim (iwn)^{-1} e^{iwn} \quad \text{as} \quad w \to \infty, \tag{58}
\]
into Eq. (29). However, although Eq. (57) is correct, we wrote “formally” because the approximation Eq. (58) is not uniform in $n$: it holds for fixed $n$. This difficulty was noted explicitly by Müller [20, p. 241]. For full justification, we follow Colton and Kress [21, Theorem 2.16]. First, as $u_{sc}$ satisfies the Sommerfeld radiation condition, $f(\hat{r})$ is well defined [21, Theorem 2.6] and it has an expansion $f(\hat{r}) = \sum_{n,m} F_n^m \tilde{Y}_n^m(\hat{r})$ for certain coefficients $F_n^m$. We want to show that $F_n^m = (−i)^n C_n^m$. Using Eq. (31), we obtain
\[
2\pi h_n^m F_n^m = \int_{\Omega} f(\hat{r}) \tilde{Y}_n^m(\hat{r}) d\Omega = \int_{\Omega} \lim_{r \to \infty} \left\{ ikr e^{-ikr} u_{sc}(r) \right\} \tilde{Y}_n^m(\hat{r}) d\Omega = \lim_{r \to \infty} \left\{ ikr e^{-ikr} \int_{\Omega} u_{sc}(r) \tilde{Y}_n^m(\hat{r}) d\Omega \right\} = \lim_{r \to \infty} \left\{ ikr e^{-ikr} (2\pi) h_n^m C_n^m h(r) \right\} = 2\pi h_n^m C_n^m,
\]
giving the desired result.

From Eqs. (57) and (32), we obtain a well-known formula for the scattering cross-section,
\[
\int_{\Omega} |f(\hat{r})|^2 d\Omega = 2\pi \sum_{n,m} h_n^m |C_n^m|^2 = \sigma_{sc}. \tag{59}
\]
The formula Eq. (33) for the extinction cross-section, $\sigma_{ex}$, involves the incident field through the coefficients $D_n^m$; we return to this complication in Section 5.1.

Next consider $F_x$, given by Eq. (45). One piece of this formula involves products of the coefficients $C_n^m$; this piece can be expressed in terms of $f$. Specifically, following the calculation of $I_1$, see Eq. (36), we find
\[
\int_{\Omega} |f(\hat{r})|^2 \cos \theta d\Omega = 4\pi \sum_{n,m} z_n^m \left[ i C_n^m \bar{C}_{n+1}^m + i C_n^m \bar{C}_{n+1}^m \right] = -8\pi \sum_{n,m} z_n^m \text{Im} \left( C_n^m \bar{C}_{n+1}^m \right).
\]
Indeed, formally, this result can be obtained by using Eq. (36) with $u = u_{sc}$, together with Eqs. (56) and (58).

Similarly, following the calculation of $I_2$, see Eq. (46),
\[
\int_{\Omega} |f(\hat{r})|^2 e^{-i\phi} \sin \theta d\Omega = -4\pi i \sum_{n,m} \left( \lambda_n^m C_n^m \bar{C}_{n+1}^{m+1} + \lambda_n^m C_n^m \bar{C}_{n+1}^{m-1} \right),
\]
and it is exactly this quantity that appears in the formula for $F_x - iF_y$, Eq. (55).

5.1. Direct calculations avoiding wavefunction expansions

The formula Eq. (59) giving the scattering cross-section in terms of the far-field pattern suggests a direct evaluation, using the definitions Eqs. (21) and (56). The error in Eq. (56) is $O((kr)^{-2})$ as $kr \to \infty$ so that, assuming $kr \gg 1$, Eq. (21) gives
\[
\sigma_{sc} = k \text{Im} \int_{\Omega} \left( \frac{e^{-ikr}}{-ikr} f(\hat{r}) + O((kr)^{-2}) \right) \left( \frac{e^{ikr}}{kr} f(\hat{r}) + O((kr)^{-2}) \right) R^2 d\Omega(\hat{r});
\]
taking the limit $R \to \infty$ gives Eq. (59), as expected. Similar calculations can be made for terms involving products of $u_{\text{inc}}$ with itself in the formulas for the radiation force; these formulas come by substituting $u = u_{\text{inc}} + u_{\text{sc}}$ in $A_2$ and $A_3$, defined by Eqs. (26) and (27), respectively. Clearly, simple calculations succeed because $u_{\text{inc}}$ decays as $R^{-1}$ on $S_R$ whereas $dS = R^2 d\Omega$: the powers of $R$ balance.

The situation is less clear when cross terms are considered, involving products of $u_{\text{inc}}$ and $u_{\text{sc}}$. The simplest example of this occurs with the extinction cross-section, defined by Eq. (23), which we write as

$$\sigma_{\text{ex}} = \lim_{R \to \infty} kR^2 \mathrm{Im} \int_{\Omega} \left( \frac{u_{\text{inc}}}{\hat{r}} - \frac{\partial u_{\text{inc}}}{\partial r} \right) d\Omega. \tag{60}$$

If we insert the far-field approximation for $u_{\text{inc}}$, Eq. (56), we obtain, tentatively,

$$\sigma_{\text{ex}} = \lim_{R \to \infty} kR \int_{\Omega} \left( \frac{1}{i k} \frac{\partial u_{\text{inc}}}{\partial r} - \frac{u_{\text{inc}}}{i k R} \right) e^{ikr} f(\hat{r}) d\Omega(\hat{r}). \tag{61}$$

At first sight, we may worry about the existence of the limit in Eq. (61) for an arbitrary regular incident wave (although we know that $\sigma_{\text{ex}}$ itself does not depend on $R$ so that the limit in Eq. (60) certainly exists), and we may worry that we should include the next term in Eq. (56), the term that decays as $(kr)^{-2}$ (which can be calculated in terms of certain derivatives of $f$, see [22, Corollary 3.8], for example).

It turns out that these anxieties do not cause further difficulties: the limit in Eq. (61) does exist, and the limit can be computed for quite general incident waves; and the omitted term from Eq. (56) leads to a zero contribution in the limit. Similar results can be proved for the computation of the radiation force. For more details, see [23].

### 5.2. Further remarks

The simplifications following from the introduction of the far-field approximation of the scattered field have encouraged some authors to introduce analogous approximations for the incident field. To examine this possibility, suppose that $u_{\text{inc}}$ is a chosen density function, defined on the unit sphere $\hat{\Omega}$.

1. For an incident plane wave, $u_{\text{inc}}(r) = \exp(i k \cdot r)$ where $\mathbf{k}$ is a constant vector with $|\mathbf{k}| = k$. Of course, plane waves do not decay as $r \to \infty$: indeed, $|u_{\text{inc}}(r)| = 1$ everywhere.

2. For an axisymmetric Bessel beam,

$$u_{\text{inc}}(r) = \exp(i k r \cos \theta \cos \beta) J_0(k r \sin \theta \sin \beta),$$

where $\beta$ is a real parameter and $J_0$ is a Bessel function. We see that $u_{\text{inc}}$ decays as $(kr)^{-1/2}$ in all directions $\hat{r} = r/r$ except along the axis ($\theta = 0, \pi$).

3. The incident field could be generated using a plane-wave representation (Herglotz wavefunction, angular spectral representation),

$$u_{\text{inc}}(r) = \int_{\hat{\Omega}} q(\hat{s}) e^{ikr \cdot \hat{s}} d\Omega(\hat{s}), \tag{62}$$

where $q$ is a chosen density function, defined on the unit sphere $\hat{\Omega}$.

When $u_{\text{inc}}$ is defined by Eq. (62), its far-field behaviour depends crucially on the properties of $q$. If $q$ is smooth and bounded, then

$$u_{\text{inc}}(r) \sim 2\pi q(\hat{r}) \frac{\exp(ikr)}{ikr} - 2\pi q(-\hat{r}) \frac{\exp(-ikr)}{ikr}, \text{ as } r \to \infty, \text{ for all directions } \hat{r}. \tag{63}$$

This result is sometimes known as the Jones lemma [24], [25, Appendix XII]. However, if $q$ is not smooth and bounded, then $u_{\text{inc}}$ need not behave as in Eq. (63). For more examples and a detailed study, see [26].

We conclude that Eq. (63) is not always true (despite claims to the contrary [27, p. 154]). When Eq. (63) is true, it shows that $u_{\text{inc}}$ behaves in the far field as the sum of an outgoing spherical wave and an incoming spherical wave.

A plausible strategy for effecting the split in Eq. (63) proceeds by substituting $2j_n = h_n + \hat{h}_n = h_n^{(1)} + h_n^{(2)}$ in the expansion Eq. (28),

$$u_{\text{inc}}(r) = \sum_{n,m} D_{n,m}^{(1)} (h_n^{(1)}(kr) + h_n^{(2)}(kr)) Y_n^m(\hat{r})$$

$$= \sum_{n,m} D_{n,m}^{(1)} h_n^{(1)}(kr) Y_n^m(\hat{r}) + \sum_{n,m} D_{n,m}^{(2)} h_n^{(2)}(kr) Y_n^m(\hat{r}).$$

Although this seems to break $u_{\text{inc}}$ into the sum of an outgoing spherical wave (first sum) and an incoming spherical wave (second sum), the second equality may be false: splitting the sum into two may result in two divergent series. To see this
clearly, consider an incident plane wave,

\[ u_{\text{inc}} = e^{ikz} = \sum_{n=0}^{\infty} (2n+1)i^n j_n(kr)P_n(\cos \theta). \]  

(64)

For fixed \( kr \), the terms in the series decay rapidly because \( (2n+1)j_n(kr) \sim (kr)^n / (2n-1)! \) as \( n \to \infty \) \([17, 10.52.1]\); the Legendre polynomials \( P_n \) satisfy \( P_n(1) = 1 \), \( P_n(-1) = (-1)^n \) and \( P_n(\cos \theta) = O(n^{-1/2}) \) as \( n \to \infty \) for \( \theta \) bounded away from 0 and \( \pi \). On the other hand, both \( h_n^{(1)} \) and \( h_n^{(2)} \) grow rapidly with \( n \) \([17, 10.52.2]\).

The error described above is often compounded by assuming further that \( kr \gg 1 \), so that \( h_n^{(1)}(kr) \) and \( h_n^{(2)}(kr) \) are replaced by their large-\( kr \) approximations (see Eq. (58)) or, equivalently, \( j_n(kr) \) is replaced by \( (kr)^{-1} \sin (kr - \frac{n\pi}{2}) \) \([17, 10.52.3]\); see, for example, \([18, \text{Eq. (4)}]\), \([15, \text{p. 34}]\), \([19, \text{p. 666}]\) and (for an analogous two-dimensional problem) \([28, \text{Eq. (20)}]\). Using this approximation for \( j_n \) in Eq. (64) leads to

\[ u_{\text{inc}} \approx \frac{1}{kr} \sum_{n=0}^{\infty} (2n+1)i^n \sin \left( kr - \frac{n\pi}{2} \right) P_n(\cos \theta); \]  

(65)

apart from the fact that the series is divergent, the factor \( (kr)^{-1} \) leads to the erroneous conclusion that \( u_{\text{inc}} \to 0 \) as \( r \to \infty \) (as stated in \([15, \text{Appendix C}]\), for example).

5.3. Debye (1909)

It is of interest to examine Debye’s famous 1909 paper \([29]\) in which he calculated the electromagnetic radiation force due to a plane wave interacting with a sphere. His calculations are reminiscent of those in Section 4.2; compare \([29, \text{Eq. (57)}]\) with Eq. (42). Then, whereas we used Wronskians to simply further, going from Eq. (58) to Eq. (45), Debye uses the large-argument asymptotic approximations for \( j_n(kr) \) and \( h_n(kr) \), giving \([29, \text{Eq. (58)}]\), which still depends on \( kr \). Perhaps fortunately, this is not the end of the calculation because he has to combine \([29, \text{Eq. (58)}]\) with \([29, \text{Eq. (58')}\); the sum of these two equations does not depend on \( kr \).

6. Use of the \( T \)-matrix

At this stage we have said very little about the scatterer \( S \): its shape, size and constitution have not played an explicit role. However, in order to use the exact formulas presented above, we have to be able to compute the coefficients \( C_m^m \) in the expansion Eq. (29) given the coefficients \( D_m^m \) in the expansion Eq. (28). As the (first-order) scattering problem is linear, the relation between these coefficients is linear. It can be encoded in Waterman’s \( T \)-matrix,

\[ C_m = \sum_{\nu, \mu} \tilde{T}^{\mu \nu} m^{-\mu} D_\nu^{\mu}, \]  

(66)

recall that we are using unnormalised spherical harmonics, which explains the notation \( \tilde{T}^{\mu \nu} \) for the entries in the \( T \)-matrix.

Much is known about the \( T \)-matrix, including how to compute it efficiently for many kinds of scatterers. In particular, for homogeneous spherical scatterers, \( T \) is diagonal, so that we can write \( \tilde{T}^{\mu \nu} m^{-\mu} D_\nu^{\mu} \) as

\[ \sigma_{\text{sc}} = 2\pi \sum_{n,m} h_n^{(1)} D_n^{\mu} \sum_{\nu, \mu} \tilde{T}^{\mu \nu} m^{-\mu} \tilde{T}^{\mu \nu}, \]  

(67)

\[ \sigma_{\text{ex}} = -\pi \sum_{n,m} D_n^{\mu} \left( h_n^{(1)} \tilde{T}^{\mu \nu} m^{-\mu} + h_n^{(2)} \tilde{T}^{\mu \nu} m^{-\mu} \right). \]  

(68)

For lossless scatterers, the identity \( \sigma_{\text{sc}} = \sigma_{\text{ex}} \) combined with the arbitrary nature of the incident field leads to a relation that must be satisfied by the \( T \)-matrix; see \([16, \text{Chapter 7}]\). The far–field pattern can be expressed in terms of the \( T \)-matrix: substitute Eq. (66) in Eq. (57).

Clearly, \( T \)-matrix methods can be used to compute radiation forces by substituting Eq. (66) in Eq. (45), for example. However, although these methods have been used for computing electromagnetic radiation forces \([30]\), we are aware of only one recent study in which \( T \)-matrix methods have been used to compute acoustic radiation forces on non-spherical scatterers \([31]\).
7. Summary and conclusions

It is known that cross-sections and radiation forces can be expressed as integrals over a sphere of radius \( R \), \( S_R \), and that the values of these integrals do not depend on \( R \). It is natural then to let \( R \to \infty \) so that the integrals can be related to far-field quantities such as the far-field pattern, Eq. (56). This familiar strategy works straightforwardly for the scattering cross-section \( \sigma_{sc} \) (see Eqs. (21) and (59)), but this is an exceptional case: if the integrand involves the incident wave \( u_{inc} \), extracting the limiting behaviour as \( R \to \infty \) is difficult, in general; these difficulties are sketched in Section 5, with more details available elsewhere [23,26].

An alternative strategy is to evaluate the integrals over \( S_R \) directly, keeping \( R \) finite. One expands the incident field using regular spherical wavefunctions (Eq. (28) with coefficients \( D_n^{(m)} \)) and the scattered field using outgoing spherical wavefunctions (Eq. (29) with coefficients \( C_n^{(m)} \)), and then integrates with respect to the angular variables. This finite-\( R \) strategy is simple, in principle. The calculations are outlined in Section 4: inevitably, they are complicated but the final results are explicit and exact; they are also seen to be independent of \( R \), which provides a check on the calculations themselves. The resulting formulas for the cross-sections (Eqs. (32) and (33)) and the components of the radiation force (Eqs. (45) and (55)) are useful because they do not assume a specific incident field (we just need the coefficients \( D_n^{(m)} \)) and they do not assume anything about the constitution of the scatterer (such as its shape, boundary conditions or internal properties); one has to be able to find all the \( C_n^{(m)} \) given all the \( D_n^{(m)} \), a task that could be done using a \( T \)-matrix method (Section 6) or by any other convenient numerical method.

From a mathematical point of view, the main merit of the finite-\( R \) strategy adopted here is that it is rigorous: within the derivations, there are no divergent integrals and no divergent series. Given a desire to compute sensible physical quantities such as cross-sections and radiation forces, our goal was to do that using convergent processes: this goal was achieved herein.

Appendix. Some integrals

Suppose that

\[
U(r) = \sum_{n,m} u_n^m(kr) \hat{Y}_n^m(\hat{r}), \quad u_n^m(kr) = D_n^m j_n(kr) + C_n^m h_n(kr), \quad (A.1)
\]

\[
V(r) = \sum_{n,m} v_n^m(kr) \hat{Y}_n^m(\hat{r}), \quad v_n^m(kr) = F_n^m j_n(kr) + E_n^m h_n(kr), \quad (A.2)
\]

in the vicinity of a sphere \( r = R \). Then \[16, Lemma 6.5\]

\[
\int_{r=R} \left( \frac{2V}{dr} - V \frac{2U}{dr} \right) dS = \frac{2\pi i}{k} \sum_{n,m} h_n^m \left( 2C_n^m C_n^m + C_n^m E_n^m + D_n^m E_n^m \right). \quad (A.3)
\]

The proof makes use of the orthogonality of \( \hat{Y}_n^m \), Eq. (31), and the Wronskian for spherical Bessel functions \[17, 10.50.1\]. The special case \( U = V \) gives

\[
k \ln \int_{r=R} \frac{U}{r} \frac{dU}{dr} dS = 2\pi \sum_{n,m} h_n^m \left( |c_n^m|^2 + \text{Re} \{ D_n^m c_n^m \} \right). \quad (A.4)
\]

Next, consider \( I_3 \), defined by Eq. (38). Substituting Eq. (34),

\[
I_3 = 2\pi \sum_{n,m} \sum_{v=0}^{\infty} \left( \int_0^\pi \left( p_n^m(\cos \theta) p_n^m(\cos \theta) \cos \theta \sin^2 \theta \, d\theta \right) p_n^m(\cos \theta) \right) \left( \int_0^\pi \left( m^2 p_n^m(\cos \theta) \cos \theta \cot \theta \, d\theta \right) \right) \quad (A.5)
\]

Note that the terms with \( n = m = 0 \) are absent. The second integral in Eq. (A.5) is

\[
m^2 \int_1^1 \frac{t}{1-t^2} p_n^m(t) p_n^m(t) \, dt \quad (A.6)
\]

whereas the first integral in Eq. (A.5) is

\[
\int_1^1 (1-t^2) p_n^m(t) p_n^m(t) \, dt = \int_1^1 \left( (1-t^2) p_n^m(t) \right) \left( \left( t p_n^m(t) \right) - p_n^m(t) \right) \, dt
\]

\[
= \int_1^1 (1-t^2) p_n^m(t) \left( t p_n^m(t) \right) \, dt - \int_1^1 (1-t^2) p_n^m(t) p_n^m(t) \, dt \quad (A.7)
\]

\[
= - \int_1^1 (1-t^2) p_n^m(t) \, t p_n^m(t) \, dt
\]
\[ -\frac{(n+1)(n+m)}{2n+1} \int_{-1}^{1} p_{n-1}^m(t)p_v^m(t) \, dt + \frac{n(n-m+1)}{2n+1} \int_{-1}^{1} p_{n+1}^m(t)p_v^m(t) \, dt \]
\[ = n(n+1) \int_{-1}^{1} p_n^m(t)p_v^m(t) \, dt - m^2 \int_{-1}^{1} \frac{t}{1-t^2} p_n^m(t)p_v^m(t) \, dt \]
\[ - \frac{2(n+1)(n+m)!}{(2n+1)(2n+3)(n-1-m)!} \delta_{v,n+1} + \frac{2n(n+1+m)!}{(2n+1)(2n+3)(n-1-m)!} \delta_{v,n-1}. \]

where we have used the differential equation satisfied by \( p_v^m(t) \).

\[ (1 - t^2)\nu'(t) + [n(n+1) - m^2/(1-t^2)]\nu(t) = 0, \quad (A.9) \]

and the formula [16, Eq. (A.14)]

\[ (2n+1)(1-t^2)p_n^{m'}(t) = (n+1)(n+m)p_n^{m-1}(t) - n(n-m+1)p_n^{m+1}(t). \quad (A.10) \]

The second integral in Eq. (A.8) cancels with the second integral in Eq. (A.5), which is Eq. (A.6). Then, using Eq. (35), the sum of the two integrals in Eq. (A.5) is

\[ \frac{2n(n+2)(n+m+1)!}{(2n+1)(2n+3)(n-1-m)!} \delta_{v,n+1} + \frac{2(n-1)(n+1)(n+m)!}{(2n+1)(2n-1)(n-1-m)!} \delta_{v,n-1}. \]

Hence Eq. (A.5) reduces to Eq. (39).

From Eq. (40), we obtain

\[ I_4 = -2\pi(kR) \sum_{n,m} \sum_{v=0}^{\infty} (w_n^m w_{n-1}^m + w_{n-1}^m w_n^m) \int_{0}^{\pi} p_n^m(\cos \theta)p_v^m(\cos \theta) \sin^3 \theta \, d\theta. \]

The integral is

\[ \int_{-1}^{1} (1 - t^2)p_n^m(t)p_v^m(t) \, dt = \frac{2(n+1)(n+m)!}{(2n+1)(2n+3)(n-1-m)!} \delta_{v,n+1} - \frac{2n(n+1+m)!}{(2n+1)(2n-1)(n-1-m)!} \delta_{v,n-1}; \]

the same integral appeared in Eq. (A.7). Hence

\[ I_4 = -4\pi(kR) \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left( w_n^m w_{n-1}^m + w_{n-1}^m w_n^m \right) \left( \frac{(n+1)(n+m)!}{(2n+1)(2n-1)(n-1-m)!} \right) \sum_{v=0}^{\infty} \left( w_v^m + w_v^{m+1} \right) \left( \frac{n(n+1+m)!}{(2n+1)(2n+3)(n-1-m)!} \right), \]

and this reduces to Eq. (41).

From the definition Eq. (46)

\[ I_5 = (kR)^2 \sum_{n,m,v} \sum_{n,m,v} \int_{-\pi}^{\pi} e^{icl}\sin d\phi \sum_{n,m} \int_{0}^{\pi} p_n^m(\cos \theta)p_v^m(\cos \theta) \sin^3 \theta \, d\theta \]
\[ = 2\pi(kR)^2 \sum_{v=1}^{\infty} \sum_{n,m} \int_{0}^{\pi} \sqrt{1-t^2} p_v^m(\cos \theta) \sin \theta \, d\theta \]
\[ + 2\pi(kR)^2 \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{v=0}^{\infty} \int_{-1}^{1} \sqrt{1-t^2} p_n^m(t)p_v^{m-1}(t) \, dt \]

after separating off the contribution from \( n = 0 \). The integrals can be evaluated using Eq. (30) and \((2v+1)\sqrt{1-t^2} p_v^{m-1}(t) = p_v^{m-1}(t) - p_v^{m+1}(t)\) [16, Eq. (A.20)]. Thus

\[ \int_{-1}^{1} \sqrt{1-t^2} p_v^m(\cos \theta) \sin \theta \, d\theta = \frac{1}{2v+1} \int_{-1}^{1} (p_{v+1}(t) - p_{v-1}(t)) \, dt = -\frac{2}{3} \delta_{v,1}, \]
\[ \int_{-1}^{1} \sqrt{1-t^2} p_n^m(t)p_v^{m-1}(t) \, dt = \frac{1}{2v+1} \int_{-1}^{1} (p_{v+1}(t) - p_{v-1}(t)) \, dt = \frac{h_n^m}{2v+1} \{ \delta_{n,v+1} - \delta_{n,v-1} \}. \quad (A.11) \]
Hence
\[ I_5 = - \frac{4\pi}{3} (kR)^2 \left( \frac{w_1}{w_0} \right) + (kR)^2 \sum_{m=1}^{\infty} \sum_{n=-m}^{n} \frac{u_n w_{m-1} w_{m-n-1}}{(2n-1)(2n+1)(n-m)!} \]
\[ - (kR)^2 \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{u_n w_{n-m-1}}{(2n+3)(2n+1)(n-m)!} \]

The first term and the second sum can be combined. In the first sum, replace \( n \) by \( n+1 \) and \( m \) by \( m+1 \). Hence we obtain Eq. (46). Similar arguments apply to \( I_6 \), see Eq. (47).

From the definition Eq. (48),
\[ I_7 = \sum_{n,m} \sum_{v,\mu} w_n w_v \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{(m-\mu-1) \phi} \ d\phi \int_{-\pi}^{\pi} p_n^m(\cos \theta) p_v^m(\cos \theta) \sin^4 \theta \ d\theta \]
\[ + \sum_{n,m} \sum_{v,\mu} w_n w_v \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{(m-\mu-1) \phi} \ d\phi \int_{-\pi}^{\pi} p_n^m(\cos \theta) p_v^m(\cos \theta) \sin^4 \theta \ d\theta. \]

The contributions from \( n = m = 0 \) and from \( v = \mu = 0 \) are absent. Hence
\[ I_7 = 2\pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{v=1}^{\infty} \sum_{\mu=1}^{\infty} w_n w_v \int_{-\pi}^{\pi} p_n^m(\cos \theta) p_v^m(\cos \theta) \sin^4 \theta \ d\theta \]
\[ + 2\pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{v=1}^{\infty} \sum_{\mu=1}^{\infty} w_n w_v \int_{-\pi}^{\pi} m(m-1) p_n^m(\cos \theta) p_v^m(\cos \theta) \sin^4 \theta \ d\theta. \]

The second integral is
\[ I_2 \equiv m(m-1) \int_{-1}^{1} p_n^m(t) p_v^m(t) \frac{dt}{\sqrt{1-t^2}} \]
whereas the first integral is
\[ I_1 \equiv \int_{-1}^{1} (1-t^2)^{3/2} p_n^m(t) p_v^{m-1}(t) \ dt = \int_{-1}^{1} \left[ (1-t^2)^{3/2} p_n^m(t) \right] \sqrt{1-t^2} p_v^{m-1}(t) \ dt \]
\[ = \int_{-1}^{1} \left[ (1-t^2)^{3/2} p_n^m(t) \right] \sqrt{1-t^2} p_v^{m-1}(t) \ dt + \int_{-1}^{1} t \sqrt{1-t^2} p_n^m(t) p_v^{m-1}(t) \ dt \]
\[ = - \int_{-1}^{1} \left[ (1-t^2)^{3/2} p_n^m(t) \right] \sqrt{1-t^2} p_v^{m-1}(t) \ dt + \int_{-1}^{1} t (1-t^2)^{3/2} p_n^m(t) p_v^{m-1}(t) \ \frac{dt}{\sqrt{1-t^2}}, \]

using \( \sqrt{1-t^2} p_v^{m-1}(t) = \left( \sqrt{1-t^2} p_v^{m-1}(t) \right) \right) + t(1-t^2)^{-1/2} p_v^{m-1}(t) \right). \]
Now use the differential equation Eq. (A.9) and [16, Eq. (A.12)]
\[ t(1-t^2)^{3/2} p_n^m(t) = -(n-m+1)(n+m)t \sqrt{1-t^2} p_v^{m-1}(t) + m(t^2 - 1)p_n^m(t) + mp_n^m(t). \]

Thus
\[ I_1 = \int_{-1}^{1} \left[ n(n+1) - \frac{m^2}{1-t^2} \right] p_n^m(t) \sqrt{1-t^2} p_v^{m-1}(t) \ dt \]
\[ - \int_{-1}^{1} \left[ (n-m+1)(n+m)t \sqrt{1-t^2} p_v^{m-1}(t) + m \sqrt{1-t^2} p_n^m(t) - \frac{m}{\sqrt{1-t^2}} p_n^m(t) \right] p_v^{m-1}(t) \ dt \]
\[ = \{ n(n+1) - m \} \int_{-1}^{1} \sqrt{1-t^2} p_n^m(t) p_v^{m-1}(t) \ dt \]
\[ - (n+m)(n-m+1) \int_{-1}^{1} t p_n^m(t) p_v^{m-1}(t) \ dt - m(m-1) \int_{-1}^{1} p_n^m(t) p_v^{m-1}(t) \ \frac{dt}{\sqrt{1-t^2}}. \]

The last term cancels with Eq. (A.12). The first integral can be evaluated using Eq. (A.11) and the second integral can be evaluated using Eq. (35). Hence
\[ I = I_1 + I_2 = \{ n(n + 1) - m \} \frac{h_n^m}{2v + 1} \{ \delta_{n,v+1} - \delta_{n,v-1} \}
\]
\[ - \frac{h_n^m}{2v + 1} \{ (v - m + 2)\delta_{n,v+1} + (v + m - 1)\delta_{n,v-1} \}
\]
\[ = \frac{h_n^m}{2v + 1} \{ (n^2 - 1)\delta_{v,n-1} - n(n + 2)\delta_{v,n+1} \}
\]

and

\[ I_2 = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{w_{nm}^m}{w_{nm}^m} (2n - 1)(2n + 1)(n - m)! - \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{w_{nm}^m}{w_{nm}^m} (2n + 3)(2n + 1)(n - m)!
\]

In the first sum, replace \( n \) by \( n + 1 \) and \( m \) by \( m + 1 \). Hence we obtain Eq. (49).

Finally, consider \( I_6 \), defined by Eq. (50). We have

\[ I_6 = kR \sum_{n,m} w_{nm}^m \int_{-\pi}^{\pi} e^{i(m-\mu-1)\phi} \int_{0}^{\pi} P_{nm}(\cos \theta)P_{n}(\cos \theta) \cos \theta \sin^2 \theta \ d\theta
\]
\[ + kR \sum_{n,m} w_{nm}^m \int_{-\pi}^{\pi} e^{i(m-\mu-1)\phi} \int_{0}^{\pi} P_{nm}(\cos \theta)P_{n}(\cos \theta) \cos \theta \sin^2 \theta \ d\theta
\]
\[ - kR \sum_{n,m} w_{nm}^m \int_{-\pi}^{\pi} e^{i(m-\mu-1)\phi} \int_{0}^{\pi} m P_{nm}(\cos \theta)P_{n}(\cos \theta) \ d\theta
\]
\[ + kR \sum_{n,m} w_{nm}^m \int_{-\pi}^{\pi} e^{i(m-\mu-1)\phi} \int_{0}^{\pi} \mu P_{nm}(\cos \theta)P_{n}(\cos \theta) \ d\theta.
\]

(A.14)

The first and third lines combine as

\[ 2\pi kR \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{\nu=0}^{\infty} w_{nm}^m w_{nu}^m J_1
\]

where

\[ J_1 = \int_{1}^{-1} \{ t(1 - t^2)P_{n}^{m-1}(t) - mP_{n}^{m}(t) \} P_{n}^{m-1}(t) \ \frac{dt}{\sqrt{1 - t^2}}
\]
\[ = -(n + m)(n - m + 1) \int_{1}^{-1} tP_{n}^{m-1}(t)P_{n}^{m-1}(t) \ dt - m \int_{1}^{-1} \sqrt{1 - t^2} P_{n}^{m}(t)P_{n}^{m-1}(t) \ dt
\]
\[ = -\frac{h_n^m}{2v + 1} \{ (n + 1)\delta_{v,n-1} + n\delta_{v,n+1} \},
\]

using Eqs. (A.13), (35) and (A.11). Similarly, the second and fourth lines in Eq. (A.14) combine as

\[ 2\pi kR \sum_{n,m} \sum_{\nu=1}^{\infty} w_{nm}^m w_{nu}^m J_2
\]

where

\[ J_2 = \int_{1}^{-1} \{ t(1 - t^2)P_{n}^{m-1}(t) - (m - 1)P_{n}^{m}(t) \} P_{n}^{m}(t) \ \frac{dt}{\sqrt{1 - t^2}}
\]
\[ = \int_{1}^{-1} tP_{n}^{m}(t)P_{n}^{m}(t) \ dt + (m - 1) \int_{1}^{-1} \sqrt{1 - t^2} P_{n}^{m}(t)P_{n}^{m-1}(t) \ dt
\]
\[ = \frac{h_n^m}{2v + 1} \{ (n - 1)\delta_{v,n-1} + (n + 2)\delta_{v,n+1} \},
\]

using [16, Eq. (A.11)]

\[ t(1 - t^2)P_{v}^{m}(t) + \mu P_{v}^{m}(t) = t\sqrt{1 - t^2} P_{v}^{m+1}(t) + \mu(1 - t^2)P_{v}^{m}(t),
\]

with \( \mu = m - 1 \) together with Eqs. (35) and (A.11). Thus
\[ I_8 = -kR \sum_{n=1}^{\infty} \sum_{m=-n}^{n} w_n^m \frac{4\pi(n+1)(n+m)!}{(2n-1)(2n+1)(n-m)!} \frac{w_{n-1}^{m-1}}{2n+1} + kR \sum_{n=1}^{\infty} \sum_{m=-n}^{n} w_n^m \frac{4\pi n(n+m)!}{(2n+3)(2n+1)(n-m)!} \frac{w_{n+1}^{m-1}}{2n+3} \]
\[ + kR \sum_{n,m=1}^{\infty} w_n^m \frac{4\pi(n+2)(n+m)!}{(2n+3)(2n+1)(n-m)!} \frac{w_{n+1}^{m-1}}{2n+3} \].

As before, replace \( n \) by \( n + 1 \) and \( m \) by \( m + 1 \) in the first and third sums, giving Eq. (51).

References