# TEMPORALLY MANIPULATED PLASMONS ON GRAPHENE* 

JOSH WILSON ${ }^{\dagger}$, FADIL SANTOSA ${ }^{\dagger}$, AND P. A. MARTIN ${ }^{\ddagger}$


#### Abstract

This paper studies the propagation of plasmons on graphene when the Drude weight is varied in time. The phenomenon of plasmon propagation is modeled by considering the graphene as a conductive sheet. Under the assumption that the field is oscillatory in the direction parallel to the sheet, it can be shown that the coupled electromagnetic field can be reduced to a single timedependent equation describing the current density on the sheet. The current density depends on the wave number $\xi$ and is shown to satisfy an integro-differential equation in time. Well-posedness for this equation is established. A numerical scheme to solve the current equations based on convolution quadrature is developed. An approximate equation, based on large $\xi$ with the physical interpretation of a quasi-static approximation, is derived and its accuracy assessed. The phenomena of wave reversal and parametric amplification are studied. Numerical calculations are conducted to address several theoretical issues as well as to demonstrate the main ideas.


Key words. plasmons, graphene, wave propagation, Maxwell's equations, time reversal, amplification

AMS subject classifications. $78 \mathrm{~A} 40,78 \mathrm{M} 25,35 \mathrm{Q} 60,45 \mathrm{~A} 05$
DOI. 10.1137/18M1226889

1. Introduction. In this work, we study the phenomenon of plasmon propagation along a graphene sheet. Plasmons are electromagnetic waves which are concentrated near the interface between a dielectric, such as air, and a highly conductive material. By "concentrated" we mean that the field decays exponentially away from the interface and propagation is mainly confined along the interface. Classically, plasmons were studied at metal-dielectric interfaces. However, there are several drawbacks to using metals for plasmonics. First, plasmons occur in the near infrared, which is not suitable for all applications. Second, metals strongly damp electromagnetic waves, leading to short plasmon lifetimes. Finally, carrier concentration in metals is not tunable, which means that plasmonic devices built from metals can only operate around a specific wavelength of light.

Recently, two-dimensional materials have emerged as an alternative platform for plasmonics that overcome the shortcomings of metals. Graphene in particular can accommodate tunable and highly confined low-loss plasmons $[9,13,22,25,16,26,3]$. Furthermore, the plasmons occur in the highly sought after terahertz to mid-infrared regime, with applications in optoelectronics [15, 21], optical modulators [20, 38, 39], beamforming [7], and the detection and fingerprinting of biomolecules [34, 19]. Enabling these applications is the ease in tuning of graphene plasmon resonances through modulation of its electronic doping. Although this modulation can be achieved in real time in a practical setup, its consequences on graphene plasmon dynamics are not as well understood. On the other hand, temporal modulation of waves has been studied

[^0]in many physical contexts, revealing interesting phenomena, including time-reversed acoustic [14], elastic [11], electromagnetic [24], and water waves [2, 33, 8] and the modulation of refractive index in optics [10, 35, 32].

In this work, we investigate the effects of modulating the electronic doping of graphene in time on plasmon dynamics. This introduction is followed by a quick review of the model for the graphene plasmons in section 2 . In section 3 we show that when the Drude weight is a constant, we can write an explicit solution to the plasmon state. The constant Drude weight case is important as it represents the background or initial state for the fields in the case of the time-varying Drude weight. We next derive an integro-differential that describes the amplitude of the current on the graphene in section 4 . Knowing the current allows us to find the electromagnetic field away from the graphene, and therefore we use the current as the dependent variable for the system. In section 5 we provide an analysis of the resulting current equation, establishing existence, uniqueness, and regularity for solutions, as well as an a priori estimate on the energy of the system. A numerical scheme for solving the integro-differential equation based on convolution quadrature is proposed in section 6 . A quasi-static approximation is introduced in section 7 and its accuracy assessed. It is shown that in this regime, the dynamics of graphene plasmons resemble the dynamics of a certain harmonic oscillator. We will use this model to study properties of plasmon dynamics, such as temporal time-reversal and parametric amplification. Our findings are illustrated with numerical experiments.
2. Model. We start with the TE-mode of electromagnetic wave propagation wherein the magnetic field is $\mathbf{H}=\left(0,0, H_{z}\right)$, the electric field is $\mathbf{E}=\left(E_{x}, E_{y}, 0\right)$, and both are invariant in $z$. For convenience, we place the graphene sheet on the $x z$ plane $(y=0)$. Away from the sheet, the electromagnetic fields satisfy Maxwell's equations,

$$
\begin{align*}
\mu \frac{\partial H_{z}}{\partial t} & =\frac{\partial E_{x}}{\partial y}-\frac{\partial E_{y}}{\partial x}  \tag{2.1}\\
\epsilon \frac{\partial E_{x}}{\partial t} & =\frac{\partial H_{z}}{\partial y}  \tag{2.2}\\
\epsilon \frac{\partial E_{y}}{\partial t} & =-\frac{\partial H_{z}}{\partial x} \tag{2.3}
\end{align*}
$$

Following [36, 29], the presence of the graphene is modeled as a boundary condition. Implicit in this model is the assumption that the graphene sheet can be homogenized in both $x$ and $z$ directions in the wavelength regime under consideration. The current density on the graphene, $j(x, t)$, is related to the discontinuity in magnetic field across the sheet at $y=0$,

$$
\begin{equation*}
\llbracket H_{z} \rrbracket=j(x, t) . \tag{2.4}
\end{equation*}
$$

The tangential electric field is continuous across the boundary,

$$
\begin{equation*}
\llbracket E_{x} \rrbracket=0 \tag{2.5}
\end{equation*}
$$

The model is completed by Drude's law for the current density $j(x, t)$,

$$
\begin{equation*}
\frac{\partial j}{\partial t}=-\frac{1}{\tau} j+\mathcal{D}(t) E_{x}(x, 0, t) \tag{2.6}
\end{equation*}
$$

Here $\tau$ is the damping factor and $\mathcal{D}(t)$ is the time-dependent Drude weight.

Combining (2.2) and (2.5) shows that $\llbracket \partial H_{z} / \partial y \rrbracket=0$. On the other hand, $\llbracket E_{y} \rrbracket$ and $\llbracket \partial E_{x} / \partial y \rrbracket$ are nonzero in general.

We will solve an initial value problem for (2.1)-(2.3) and (2.6) with boundary conditions (2.4)-(2.5). For initial conditions, we use the fields and current density associated with $\mathcal{D}(0)=\mathcal{D}_{0}$. These fields are calculated next.
3. Constant Drude weight. When $\mathcal{D}$ is a constant, $\mathcal{D}_{0}$, we can assume that all fields are time-harmonic and make an ansatz of the form

$$
\begin{equation*}
H_{z}(x, y, t)=\operatorname{sgn}(y) e^{i \xi x-\gamma_{0}|y|-s_{0} t} \tag{3.1}
\end{equation*}
$$

Here $\xi$ is real and fixed, and it represents the plasmon wave number. The parameters $\gamma_{0}$ and $s_{0}$ are in general complex and are to be determined by the governing equations. We can interpret $\operatorname{Im} s_{0}$ as the frequency of light and $\operatorname{Re} s_{0}$ as the damping in the system. Hence we require $\operatorname{Im} s_{0} \neq 0$ and $\operatorname{Re} s_{0} \geq 0$ so that the plasmon propagates and does not grow in time.

Remark 3.1. In contrast to the discussion here, in some applications one specifies $\operatorname{Im} s_{0}$ and then determines $\xi$ from the governing equations.

Substituting the ansatz (3.1) in (2.2), (2.3), and (2.4) gives us

$$
\begin{align*}
E_{x}(x, y, t) & =\frac{\gamma_{0}}{\epsilon s_{0}} e^{i \xi x-\gamma_{0}|y|-s_{0} t}, \quad E_{y}(x, y, t)=\frac{i \xi}{\epsilon s_{0}} \operatorname{sgn}(y) e^{i \xi x-\gamma_{0}|y|-s_{0} t}  \tag{3.2}\\
j(x, t) & =2 e^{i \xi x-s_{0} t} \tag{3.3}
\end{align*}
$$

Now from (2.1) we get the dispersion relation

$$
\begin{equation*}
\gamma_{0}^{2}=\mu \epsilon s_{0}^{2}+\xi^{2} \tag{3.4}
\end{equation*}
$$

and from (2.6) we get the relation

$$
\begin{equation*}
j(x, t)=\sigma\left(s_{0}\right) E_{x}(x, 0, t) \quad \text { with } \quad \sigma\left(s_{0}\right)=\frac{\mathcal{D}_{0}}{1 / \tau-s_{0}} \tag{3.5}
\end{equation*}
$$

We can interpret $\sigma\left(s_{0}\right)$ as the conductivity of the graphene sheet; evidently, it is frequency dependent. Substituting the expressions for $j$ and $E_{x}$ in (3.5) gives

$$
\begin{equation*}
\gamma_{0}=\frac{2 \epsilon s_{0}}{\sigma\left(s_{0}\right)} \tag{3.6}
\end{equation*}
$$

Upon inserting this relation in (3.4) we obtain

$$
\begin{equation*}
\xi^{2}=\frac{s_{0}^{2}}{c^{2}}\left(\frac{4}{\sigma^{2}\left(s_{0}\right) \eta^{2}}-1\right) \tag{3.7}
\end{equation*}
$$

where $\eta=\sqrt{\mu / \epsilon}$ and $c^{-2}=\mu \epsilon$. Equation (3.7) is the dispersion relation for a plasmon [1]. We can rewrite it as

$$
\begin{equation*}
s_{0}^{4}-\frac{2}{\tau} s_{0}^{3}+\left(\frac{1}{\tau^{2}}-\frac{\eta^{2} \mathcal{D}_{0}^{2}}{4}\right) s_{0}^{2}-\frac{\xi^{2} \mathcal{D}_{0}^{2}}{4 \epsilon^{2}}=0 \tag{3.8}
\end{equation*}
$$

This quartic equation has two real roots and one complex-conjugate pair of roots. The real roots are of no interest because we want $\operatorname{Im} s_{0} \neq 0$ so as to have propagating


Fig. 1. Visualization of the $H_{z}$ and $E_{x}$ fields corresponding to a plasmon. Shown are the real parts of $H_{z}(x, y, t)$ and $E_{x}(x, y, t)$ for some $t$. Note the exponential decay of both fields and the jump in $H_{z}$ across the graphene, which is lying along the $x$-axis.
waves. The complex-conjugate pair of the form $e^{s_{0} t+i \xi x}$ gives the physical propagating solutions, as described below (3.1).

Given a wave number $\xi$, we solve (3.8) for the two physically meaningful roots. With those in hand, we can visualize the electromagnetic fields corresponding to the plasmons by inserting the obtained $s_{0}$ and $\gamma_{0}$ in (3.1)-(3.2). We display sample $H_{z}(x, y, t)$ and $E_{x}(x, y, t)$ fields in Figure 1 at a fixed $t$. Note the jump in the $H_{z}$ field across the graphene sheet at $y=0$. On the other hand, $E_{x}$ is continuous across the sheet. Both fields decay exponentially away from the $x$-axis. The constants chosen in this visualization are $\xi=100, \mathcal{D}_{0}=0.5, \tau=2$, and $\mu=\epsilon=1$.

Finally, we note that for large $\xi$, the two physically relevant roots are given approximately by $s_{0}$ and its complex conjugate, $\overline{s_{0}}$, where

$$
\begin{equation*}
s_{0}=i \sqrt{\frac{\xi \mathcal{D}_{0}}{2 \epsilon}}+\frac{1}{2 \tau}+O\left(\xi^{-1 / 2}\right) \tag{3.9}
\end{equation*}
$$

Therefore, the plasmon frequency is approximately $\sqrt{\xi \mathcal{D}_{0} /(2 \epsilon)}$. We also have

$$
\begin{equation*}
\gamma_{0}=\xi+O(1) \quad \text { as } \xi \rightarrow \infty \tag{3.10}
\end{equation*}
$$

4. Current density evolution. In the case where the Drude weight is not constant, we will derive a single integro-differential equation in $t$ for the amplitude of the current density. We assume the fields are of the form

$$
H_{z}(x, y, t)=\widetilde{H_{z}}(y, t) e^{i \xi x}, \quad E_{x}(x, y, t)=\widetilde{E_{x}}(y, t) e^{i \xi x}, \quad E_{y}(x, y, t)=\widetilde{E_{y}}(y, t) e^{i \xi x}
$$

Similarly, we assume $j(x, t)=\tilde{\jmath}(t) e^{i \xi x}$. Henceforth, we will deal with $\widetilde{H_{z}}(y, t), \widetilde{E_{x}}(y, t)$, $\widetilde{E_{y}}(y, t)$, and $\tilde{j}(t)$ and drop the tildes for convenience. Maxwell's equations, (2.1)-
(2.3), reduce to

$$
\mu \frac{\partial H_{z}}{\partial t}=\frac{\partial E_{x}}{\partial y}-i \xi E_{y}, \quad \epsilon \frac{\partial E_{x}}{\partial t}=\frac{\partial H_{z}}{\partial y}, \quad \epsilon \frac{\partial E_{y}}{\partial t}=-i \xi H_{z}
$$

Next we take a Laplace transform with respect to $t$ and obtain

$$
\begin{align*}
& \mu\left\{s \widehat{H_{z}}-H_{z}(y, 0)\right\}=\frac{\partial \widehat{E_{x}}}{\partial y}-i \xi \widehat{E_{y}}  \tag{4.1}\\
& \epsilon\left\{s \widehat{E_{x}}-E_{x}(y, 0)\right\}=\frac{\partial \widehat{H_{z}}}{\partial y}  \tag{4.2}\\
& \epsilon\left\{s \widehat{E_{y}}-E_{y}(y, 0)\right\}=-i \xi \widehat{H_{z}} \tag{4.3}
\end{align*}
$$

The transformed variables, decorated with hats, now depend on $s$ and $y$.
We can obtain an equation for $\widehat{H_{z}}$ from (4.1)-(4.3),

$$
\begin{equation*}
\frac{\partial^{2} \widehat{H_{z}}}{\partial y^{2}}-\gamma^{2} \widehat{H_{z}}=f(y, s) \tag{4.4}
\end{equation*}
$$

where the forcing term is given by

$$
\begin{equation*}
f(y, s)=-\mu \epsilon s H_{z}(y, 0)-\epsilon \frac{\partial E_{x}}{\partial y}(y, 0)+i \xi \epsilon E_{y}(y, 0) \tag{4.5}
\end{equation*}
$$

and $\gamma=\sqrt{\mu \epsilon s^{2}+\xi^{2}}$. We choose the principal branch of the square root, meaning that $\operatorname{Re} \gamma \geq 0$. We could solve (4.4) using a one-dimensional Green function $G\left(y, y_{0}, s\right)=$ $-e^{-\gamma\left|y-y_{0}\right|} /(2 \gamma)$, but, as we are mainly interested in the behavior at $y=0$, we can proceed more directly. Multiply (4.4) by $e^{-\gamma y}$ and integrate from $y=0$ to $\infty$ to obtain

$$
\int_{0}^{\infty} f(y, s) e^{-\gamma y} d y=-\gamma \widehat{H_{z}}(0+, s)-\frac{\partial \widehat{H_{z}}}{\partial y}(0+, s)
$$

Similarly,

$$
\int_{-\infty}^{0} f(y, s) e^{\gamma y} d y=-\gamma \widehat{H_{z}}(0-, s)+\frac{\partial \widehat{H_{z}}}{\partial y}(0-, s)
$$

Subtracting, using the Laplace transform version of (2.4) (after removing the $e^{i \xi x}$ factor), we obtain

$$
-\gamma \hat{\jmath}(s)-2 \frac{\partial \widehat{H_{z}}}{\partial y}(0, s)=\int_{-\infty}^{\infty} \operatorname{sgn}(y) e^{-\gamma|y|} f(y, s) d y
$$

using $\llbracket \partial \widehat{H_{z}} / \partial y \rrbracket=0$. Then from (4.2),

$$
\begin{align*}
\widehat{E_{x}}(0, s) & =\frac{1}{s} E_{x}(0,0)+\frac{1}{\epsilon s} \frac{\partial \widehat{H_{z}}}{\partial y}(0, s)  \tag{4.6}\\
& =\frac{1}{s} E_{x}(0,0)-\frac{\gamma}{\epsilon s} \hat{\jmath}(s)-\frac{1}{2 \epsilon s} \int_{-\infty}^{\infty} \operatorname{sgn}(y) e^{-\gamma|y|} f(y, s) d y .
\end{align*}
$$

We assume that $\mathcal{D}(t \leq 0)=\mathcal{D}_{0}$ and use (3.1)-(3.3) as initial conditions. To avoid confusion, we denote the fields for $t \leq 0$ by $H_{z 0}(y, t), E_{x 0}(y, t), E_{y 0}(y, t)$, and $j_{0}(t)$, suppressing the (known) $x$-dependence. The forcing term (4.5) reduces to

$$
f(y, s)=-\operatorname{sgn}(y) \mu \epsilon e^{-\gamma_{0}|y|}\left(s-s_{0}\right)
$$

Therefore, the integral on the right-hand side of (4.6) becomes

$$
\mu \epsilon\left(s_{0}-s\right) \int_{-\infty}^{\infty} e^{-\left(\gamma+\gamma_{0}\right)|y|} d y=\frac{2 \mu \epsilon\left(s_{0}-s\right)}{\gamma+\gamma_{0}}=\frac{2\left(\gamma_{0}-\gamma\right)}{s+s_{0}}
$$

after multiplying the numerator and denominator by $\left(\gamma-\gamma_{0}\right)$ and using the definitions of $\gamma$ and $\gamma_{0}$. So, from (4.6) we have the equation

$$
\begin{equation*}
\widehat{E_{x}}(0, s)=-\frac{\gamma}{2 \epsilon s} \hat{\jmath}(s)+\frac{1}{s} E_{x}(0,0)+\frac{\gamma-\gamma_{0}}{\epsilon s\left(s+s_{0}\right)} \tag{4.7}
\end{equation*}
$$

Next we will perform the inverse Laplace transform of this equation. Recall that from (3.3), $j_{0}(t)=2 e^{-s_{0} t}$ which gives $\hat{\jmath_{0}}(s)=\mathcal{L}\left\{j_{0}\right\}=2 /\left(s+s_{0}\right)$. Then rewrite (4.7) as

$$
\mathcal{L}\left\{E_{x}(0, t)-E_{x}(0,0)\right\}=-\frac{\gamma}{2 \epsilon s}\left(\hat{\jmath}(s)-\hat{\jmath}_{0}(s)\right)-\frac{\gamma_{0}}{\epsilon s\left(s+s_{0}\right)}
$$

the last term is $-\left(\gamma_{0} /\left[\epsilon s_{0}\right]\right) \mathcal{L}\left\{1-e^{-s_{0} t}\right\}$, which we identify as $\mathcal{L}\left\{E_{x 0}(0, t)-E_{x 0}(0,0)\right\}$. For the term involving $\gamma / s$, we observe that

$$
\begin{align*}
\frac{\gamma}{s} & =\frac{\sqrt{s^{2}+a^{2}}}{c s}=\frac{1}{c s}\left(\sqrt{s^{2}+a^{2}}-s\right)+\frac{1}{c}  \tag{4.8}\\
& =\frac{a}{c s} \mathcal{L}\left\{t^{-1} J_{1}(a t)\right\}+\frac{1}{c}=\frac{1}{c} \mathcal{L}\left\{k_{1}(t)\right\}+\frac{1}{c}
\end{align*}
$$

where $a=c \xi, c=1 / \sqrt{\mu \epsilon}, J_{1}$ is a Bessel function, we have used [17, eq. 6.623.3], and

$$
\begin{equation*}
k_{1}(t)=\int_{0}^{t} \frac{a}{t^{\prime}} J_{1}\left(a t^{\prime}\right) d t^{\prime} \tag{4.9}
\end{equation*}
$$

Hence using the convolution theorem, we can invert (4.7) to obtain

$$
\begin{equation*}
E_{x}(0, t)-E_{x 0}(0, t)=-(\eta / 2)\left[\left(j-j_{0}\right)+k_{1} *\left(j-j_{0}\right)\right] \tag{4.10}
\end{equation*}
$$

noting that $\eta=\sqrt{\mu / \epsilon}$ and $E_{x 0}(0,0)=E_{x}(0,0)$. Finally, eliminating $E_{x}(0, t)$ between (2.6) and (4.10) gives an integro-differential equation for $j(t)$,

$$
\begin{equation*}
\frac{d j}{d t}+\frac{1}{\tau} j+\frac{\eta \mathcal{D}}{2}\left(j+k_{1} * j\right)=\frac{\eta \mathcal{D}}{2}\left(j_{0}+k_{1} * j_{0}\right)+\mathcal{D}(t) E_{x 0}(0, t) \tag{4.11}
\end{equation*}
$$

Next we derive several useful variants of (4.11). Define

$$
\mathcal{J}(t)=j(t)-j_{0}(t)=j(t)-2 e^{-s_{0} t}
$$

which satisfies $\mathcal{J}(0)=0$. Take (2.6) for both $j$ and $j_{0}$, and subtract to get

$$
\begin{equation*}
\frac{d \mathcal{J}}{d t}+\frac{1}{\tau} \mathcal{J}=\mathcal{D}(t) E_{x}(0, t)-\mathcal{D}_{0} E_{x 0}(0, t) \tag{4.12}
\end{equation*}
$$

Eliminating $E_{x}(0, t)$ using (4.10) gives

$$
\begin{equation*}
\frac{d \mathcal{J}}{d t}=-\frac{1}{\tau} \mathcal{J}-\frac{\eta \mathcal{D}}{2}\left(\mathcal{J}+k_{1} * \mathcal{J}\right)+\left(\mathcal{D}(t)-\mathcal{D}_{0}\right) E_{x 0}(0, t) \tag{4.13}
\end{equation*}
$$

Equation (4.12) can be converted into a Volterra integral equation of the second kind. To do this, regard (4.12) as a linear first-order equation for $\mathcal{J}$; solving it gives

$$
\begin{align*}
\mathcal{J}(t) & =\int_{0}^{t} e^{-\left(t-t^{\prime}\right) / \tau}\left\{\mathcal{D}\left(t^{\prime}\right) E_{x}\left(0, t^{\prime}\right)-\mathcal{D}_{0} E_{x 0}\left(0, t^{\prime}\right)\right\} d t^{\prime}  \tag{4.14}\\
& =k_{2} *\left(\mathcal{D} E_{x}-\mathcal{D}_{0} E_{x 0}\right)
\end{align*}
$$

where we set $k_{2}(t)=e^{-t / \tau}$. Now substitute for $E_{x}\left(0, t^{\prime}\right)$ from (4.10) to obtain

$$
\begin{equation*}
\mathcal{J}+(\eta / 2)\left[k_{2} *(\mathcal{D} \mathcal{J})+k_{2} *\left\{\mathcal{D}\left(k_{1} * \mathcal{J}\right)\right\}\right]=k_{2} *\left[\left(\mathcal{D}-\mathcal{D}_{0}\right) E_{x 0}\right] \tag{4.15}
\end{equation*}
$$

Defining a new kernel $k$ by

$$
\begin{equation*}
k(t, s)=k_{2}(t-s) \mathcal{D}(s)+\int_{s}^{t} k_{2}(t-r) \mathcal{D}(r) k_{1}(r-s) d r \tag{4.16}
\end{equation*}
$$

and an operator $K$ by

$$
\begin{equation*}
(K u)(t)=\frac{\eta}{2} \int_{0}^{t} k(t, s) u(s) d s \tag{4.17}
\end{equation*}
$$

we can rewrite (4.15) concisely as

$$
\begin{equation*}
\mathcal{J}+K \mathcal{J}=k_{2} *\left[\left(\mathcal{D}-\mathcal{D}_{0}\right) E_{x 0}\right] \tag{4.18}
\end{equation*}
$$

Finally, for a later numerical method it will be convenient to work directly with (4.10) and (4.14), which give us the system of integral equations

$$
\begin{align*}
& E_{x}(0, t)=-(\eta / 2)\left(\mathcal{J}+k_{1} * \mathcal{J}\right)+E_{x 0}(0, t)  \tag{4.19}\\
& \mathcal{J}(t)=k_{2} *\left(\mathcal{D} E_{x}-\mathcal{D}_{0} E_{x 0}\right)=k_{2} *\left(\mathcal{D} E_{x}\right)+2 e^{-t / \tau}-j_{0}(t) \tag{4.20}
\end{align*}
$$

5. Analysis of the current density equation. In this section we investigate some properties of (4.13) and (4.18). In particular, we will consider existence and uniqueness of solutions as well as derive a priori estimates for $\mathcal{J}$ when $\mathcal{D}$ is of a certain form. Throughout we will assume that $\mathcal{D}>0$. Also, when investigating the properties of the solution of (4.18) we will consider the equation

$$
\begin{equation*}
u+K u=f \tag{5.1}
\end{equation*}
$$

where we take $f$ to be real. Note that while our particular $f$ is complex, we can simply take real and imaginary parts to reduce to the real case because $k$ is real.
5.1. Well-posedness. When $\mathcal{D}$ is continuous, (5.1) is guaranteed to have a unique continuous solution on $[0, \infty)$ [6, Thm. 2.1.1], [5, Thm. 1.2.3]. When $\mathcal{D}$ is not as smooth (we are especially interested in discontinuous $\mathcal{D}$ ), we need to understand some properties of the kernels $k_{1}, k_{2}$, and $k$.

LEmmA 5.1. We have $k_{1}(t)>0$ and $k_{2}(t)>0$ for $t>0$. Furthermore, $k(t, s)>0$ for $t, s>0$.

Proof. The claim is clear for $k_{2}(t)=e^{-t / \tau}$. For $k_{1}$, we use the Poisson integral for $J_{1}(x) / x$ [31, eq. 10.9.4] to get

$$
k_{1}(t)=\int_{0}^{a t} \frac{a}{x} J_{1}(x) d x=\frac{2 a}{\pi} \int_{0}^{1} \sqrt{1-s^{2}} \frac{\sin (a t s)}{s} d s=\frac{4 a}{\pi} \int_{0}^{1} \frac{s}{\sqrt{1-s^{2}}} \operatorname{Si}(a t s) d s
$$

where Si is the sine integral [31, eq. 6.2.9]. The result follows because $\operatorname{Si}(x)>0$ when $x>0$ [31, eq. 6.10.4]. The claim for $k$ follows from (4.16) and the positivity of $k_{1}, k_{2}$, and $\mathcal{D}$.

Now we investigate properties of the operator $K$ from (4.17).
Lemma 5.2. Fix $T>0$ and let $u \in L^{2}(0, T)$. The operator $K: L^{2}(0, T) \rightarrow$ $L^{2}(0, T)$ is compact. Moreover, $K u$ is continuous on $[0, T]$.

Proof. For compactness, note that $K$ is a Hilbert-Schmidt operator [5, Thm. 8.1.3 (b)]. For continuity, note that the kernel $k$ is the sum of $k_{2}(t-s) \mathcal{D}(s)$, which may be discontinuous, and a continuous part, (4.16). For the first part, we consider

$$
\begin{equation*}
e^{-t / \tau} \int_{0}^{t} e^{s / \tau} \mathcal{D}(s) u(s) d s \tag{5.2}
\end{equation*}
$$

this is continuous because the indefinite integral of an $L^{2}$ function is continuous.
THEOREM 5.3. Let $T>0$. Equation (5.1) has a unique solution $u \in L^{2}(0, T)$ for each $f \in L^{2}(0, T)$.

Proof. Suppose there is a $u \in L^{2}(0, T)$ that solves $u+K u=0$. Then

$$
u(t)=-(K u)(t)
$$

and we know from Lemma 5.2 that $u$ is continuous. If we suppose for contradiction that $u \neq 0$, then, replacing $u$ with $-u$ if necessary, there is an interval $\left[0, t_{0}\right]$ on which $u \geq 0$ and $u\left(t_{0}\right)>0$. Since

$$
(K u)\left(t_{0}\right)=\frac{\eta}{2} \int_{0}^{t_{0}} k\left(t_{0}, s\right) u(s)>0
$$

we must have that $u\left(t_{0}\right)<0$, a contradiction. Therefore, the kernel of $I+K$ is trivial. Since $K$ is compact, by the Fredholm alternative [12, Appendix D, Thm. 5] we know that the range of $I+K$ is all of $L^{2}(0, T)$, which completes the proof.
5.2. Regularity. From (4.16) we can see that the smoothness of the kernel $k$ is limited by the smoothness of the Drude weight $\mathcal{D}$. As a result, we expect the regularity of a solution of (5.1) to be limited by the smoothness of $\mathcal{D}$ and $f$. The next theorem shows that we still have some degree of regularity when $\mathcal{D} \in L^{2}$, for example.

Theorem 5.4. Fix $T>0$. If $f$ is continuous on $[0, T]$, then the solution of (5.1) is continuous.

Proof. This follows from $u=-K u+f$ and the continuity of $K u$ in Lemma 5.2. $\square$
When $\mathcal{D}$ and $f$ have more regularity, we may expect that $u$ has more regularity. Standard theory [5, Thm. 2.1.1] gives that $u \in C^{m}[0, T]$ when both $f$ and $\mathcal{D}$ are in the same space. Then, for our particular $k(t, s)$, it is easy to see that $K u$ can be differentiated once more; the relevant piece is (5.2). Hence $u \in C^{m+1}$ if $\mathcal{D} \in C^{m}$ and $f \in C^{m+1}$.

Of more interest to us is the situation where $\mathcal{D}$ is piecewise $C^{m}$ on $[0, T]$.
THEOREM 5.5. Fix $T>0$ and suppose that $\mathcal{D}$ is piecewise $C^{n}$ on $[0, T]$. Then $K u$ is continuous and piecewise $C^{n+1}$ on $[0, T]$. Moreover, if $f$ is also continuous and piecewise $C^{n+1}$ on $[0, T]$, then so is the solution of (5.1).

Proof. The continuity of $K u$ follows from Lemma 5.2. Then suppose that $\mathcal{D}(t)$ is $C^{n}$ except at points $t=a_{p}, p=0,1,2, \ldots$, with $0=a_{0}<a_{1}<\cdots<a_{M-1}<a_{M}=T$, so that $\mathcal{D} \in C^{n}\left[a_{p}, a_{p+1}\right], p=0,1,2, \ldots, M-1$. Then, given $t \in\left[a_{m}, a_{m+1}\right]$, we have

$$
(K u)(t)=\frac{\eta}{2} \sum_{p=1}^{m} \int_{a_{p-1}}^{a_{p}} k(t, s) u(s) d s+\frac{\eta}{2} \int_{a_{m}}^{t} k(t, s) u(s) d s
$$

The first $m$ integrals have constant integration limits; the $p$ th integral defines a function in $C^{n+1}\left[a_{p-1}, a_{p}\right]$ because we can differentiate under the integral sign. The remaining integral is in $C^{n+1}\left[a_{m}, a_{m+1}\right]$ by an argument similar to that used in the proof of Lemma 5.2: use standard theory plus one more derivative due to the special structure of $k(t, s)$.

For the claim on $u$, use $u=-K u+f$ together with properties of $K u$ and $f$.
5.3. A priori estimates. We start by multiplying (4.13) by $\overline{\mathcal{J}}$, the complex conjugate of $\mathcal{J}=j-j_{0}$. Take the complex conjugate of (4.13) and multiply it by $\mathcal{J}$. Adding the resulting expressions gives

$$
\frac{d}{d t}|\mathcal{J}|^{2}=-\frac{2}{\tau}|\mathcal{J}|^{2}-\eta \mathcal{D}\left\{|\mathcal{J}|^{2}+\operatorname{Re}\left(\overline{\mathcal{J}}\left[k_{1} * \mathcal{J}\right]\right)\right\}+2 \operatorname{Re}(\overline{\mathcal{J}} f),
$$

where $f(t)=\left(\mathcal{D}(t)-\mathcal{D}_{0}\right) E_{x 0}(0, t)$. Integrate this expression from 0 to $t$, making use of the notation

$$
\langle u, v\rangle_{L^{2}(0, t)}=\int_{0}^{t} u(s) \overline{v(s)} d s, \quad B_{t}[u, v]=\left\langle u+k_{1} * u, v\right\rangle_{L^{2}(0, t)},
$$

and $\|u\|^{2}=\langle u, u\rangle$. We obtain

$$
\begin{equation*}
\frac{1}{2}|\mathcal{J}|^{2}+\frac{\eta}{2} \operatorname{Re}\left\{B_{t}[\mathcal{J}, \mathcal{D} \mathcal{J}]\right\}=\operatorname{Re}\langle f, \mathcal{J}\rangle_{L^{2}(0, t)}-\frac{1}{\tau}\|\mathcal{J}\|_{L^{2}(0, t)}^{2} \tag{5.3}
\end{equation*}
$$

Assuming that $\mathcal{D}$ is differentiable, we see that an integration by parts gives

$$
B_{t}[\mathcal{J}, \mathcal{D} \mathcal{J}]=\mathcal{D}(t) B_{t}[\mathcal{J}, \mathcal{J}]-\int_{0}^{t} \mathcal{D}^{\prime}(s) B_{s}[\mathcal{J}, \mathcal{J}] d s
$$

Combining this with (5.3) gives the energy identity

$$
\begin{equation*}
\mathcal{E}(t)=\frac{\eta}{2} \int_{0}^{t} \mathcal{D}^{\prime}(s) \operatorname{Re}\left\{B_{s}[\mathcal{J}, \mathcal{J}]\right\} d s+\operatorname{Re}\langle f, \mathcal{J}\rangle_{L^{2}(0, t)}-\frac{1}{\tau}\|\mathcal{J}\|_{L^{2}(0, t)}^{2}, \tag{5.4}
\end{equation*}
$$

where the energy $\mathcal{E}$ is defined as

$$
\mathcal{E}(t)=\frac{1}{2}|\mathcal{J}(t)|^{2}+\frac{1}{2} \eta \mathcal{D}(t) \operatorname{Re}\left\{B_{t}[\mathcal{J}, \mathcal{J}]\right\}
$$

At this point it is not clear that $\mathcal{E}$ is truly an energy, since we do not know if $\operatorname{Re}\left\{B_{t}[\mathcal{J}, \mathcal{J}]\right\} \geq 0$. To prove that this is indeed the case, we begin with some necessary results on the kernel $k_{1}$. First, we separate $k_{1}$ into

$$
\begin{equation*}
k_{1}(t)=a+k_{1}^{\circ}(t) \quad \text { where } \quad k_{1}^{\circ}(t)=-\int_{t}^{\infty} \frac{a}{t^{\prime}} J_{1}\left(a t^{\prime}\right) d t^{\prime}, \tag{5.5}
\end{equation*}
$$

using $\int_{0}^{\infty} x^{-1} J_{1}(a x) d x=1$ [17, eq. 6.561.17].
The functions $k_{1}(t)$ and $k_{1}^{\circ}(t)$ have been defined for $t \geq 0$, but we can extend them to all $t$ by defining them as 0 for $t<0$. Having done this, we can define and compute the Fourier transform of $k_{1}^{\circ}$ as follows:

$$
\begin{align*}
K_{1}^{\circ}(\omega) & =\int_{-\infty}^{\infty} k_{1}^{\circ}(t) e^{-i \omega t} d t=\int_{0}^{\infty} \frac{a}{i \omega x} J_{1}(a x)\left(e^{-i \omega x}-1\right) d x  \tag{5.6}\\
& =\frac{i a}{\omega}-\frac{i a}{\omega} \int_{0}^{\infty} J_{1}(a x) \cos (\omega x) \frac{d x}{x}-\frac{a}{\omega} \int_{0}^{\infty} J_{1}(a x) \sin (\omega x) \frac{d x}{x} .
\end{align*}
$$

The remaining integrals are standard. In particular, using [17, eq. 6.693.1],

$$
\operatorname{Re}\left\{K_{1}^{\circ}(\omega)\right\}= \begin{cases}-1, & 0 \leq \omega \leq a  \tag{5.7}\\ -\left(a^{2} / \omega\right)\left(\omega+\sqrt{\omega^{2}-a^{2}}\right)^{-1}, & \omega \geq a\end{cases}
$$

for $\omega<0$, note that $\operatorname{Re}\left\{K_{1}^{\circ}(\omega)\right\}$ is an even function of $\omega$. The result (5.7) will be used later (Lemma 5.7).

In analogy to the relation between $k_{1}$ and $k_{1}^{\circ}$, we can also define the bilinear form

$$
\begin{equation*}
B_{t}^{\circ}[u, v]=\left\langle u+k_{1}^{\circ} * u, v\right\rangle_{L^{2}(0, t)} . \tag{5.8}
\end{equation*}
$$

Note that $B_{t}$ and $B_{t}^{\circ}$ are related by

$$
\begin{equation*}
B_{t}[u, v]=a\left\langle\int_{0}^{t} u\left(t^{\prime}\right) d t^{\prime}, v\right\rangle_{L^{2}(0, t)}+B_{t}^{\circ}[u, v] . \tag{5.9}
\end{equation*}
$$

We are now ready to investigate some properties of $B_{t}$ and $B_{t}^{\circ}$.
Lemma 5.6. Let $u, v \in L^{2}(0, t)$ and extend them to $\mathbb{R}$ by 0 . Then

$$
\begin{equation*}
B_{t}^{\circ}[u, v]=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left\{1+K_{1}^{\circ}(\omega)\right\} U(\omega) \overline{V(\omega)} d \omega \tag{5.10}
\end{equation*}
$$

where $U$ and $V$ are the Fourier transforms of $u$ and $v$, respectively, and $K_{1}^{\circ}$ is defined by (5.6).

Proof. We have

$$
\begin{aligned}
B_{t}^{\circ}[u, v] & =\langle u, v\rangle_{L^{2}(0, t)}+\int_{0}^{t} \int_{0}^{s} k_{1}^{\circ}(s-r) u(r) d r \overline{v(s)} d s \\
& =\langle u, v\rangle_{L^{2}(\mathbb{R})}+\int_{0}^{t} \int_{-\infty}^{\infty} k_{1}^{\circ}(s-r) u(r) d r \overline{v(s)} d s \\
& =\langle u, v\rangle_{L^{2}(\mathbb{R})}+\int_{-\infty}^{\infty}\left(k_{1}^{\circ} \tilde{*} u\right)(s) \overline{v(s)} d s
\end{aligned}
$$

using $k_{1}^{\circ}(s-r)=0$ for $r>s, u(r)=0$ for $r<0, v(s)=0$ for $s>t$ and for $s<0$, and $\tilde{*}$ denotes standard Fourier convolution,

$$
(f \tilde{\star} g)(x)=\int_{-\infty}^{\infty} f(x-t) g(t) d t=\frac{1}{2 \pi} \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{i \omega x} d \omega
$$

where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f$ and $g$, respectively. Using the Fourier convolution theorem and the Plancherel formula,

$$
\langle F, G\rangle_{L^{2}(\mathbb{R})}=2 \pi\langle f, g\rangle_{L^{2}(\mathbb{R})}
$$

we obtain the result (5.10).
Lemma 5.7. Let $u \in L^{2}(0, t)$. Then $\operatorname{Re}\left\{B_{t}[u, u]\right\} \geq 0$ and $\operatorname{Re}\left\{B_{t}^{\circ}[u, u]\right\} \geq 0$.
Proof. We can reduce to the real case by noting that

$$
\operatorname{Re}\left\{B_{t}[u, u]\right\}=B_{t}[\operatorname{Re}(u), \operatorname{Re}(u)]+B_{t}[\operatorname{Im}(u), \operatorname{Im}(u)]
$$

and similarly for $B_{t}^{\circ}$. Then, assuming now that $u$ is real, the first term on the righthand side of (5.9) is

$$
\begin{equation*}
a \int_{0}^{t} u\left(t^{\prime}\right) \int_{0}^{t^{\prime}} u(r) d r d t^{\prime}=\frac{a}{2}\left(\int_{0}^{t} u\left(t^{\prime}\right) d t^{\prime}\right)^{2} \geq 0 \tag{5.11}
\end{equation*}
$$

Therefore, we can focus our attention on $B_{t}^{\circ}[u, u]$.
If we put $u=v$ in (5.10), we obtain

$$
B_{t}^{\circ}[u, u]=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left\{1+K_{1}^{\circ}(\omega)\right\}|U(\omega)|^{2} d \omega=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left\{1+\operatorname{Re}\left[K_{1}^{\circ}(\omega)\right]\right\}|U(\omega)|^{2} d \omega
$$

because the left-hand side is real. Then, making use of (5.7) we obtain

$$
\begin{equation*}
B_{t}^{\circ}[u, u]=\frac{1}{2 \pi} \int_{|\omega|>a} \sqrt{1-(a / \omega)^{2}}|U(\omega)|^{2} d \omega \geq 0 \tag{5.12}
\end{equation*}
$$

Lemma 5.8. Let $u \in L^{2}(0, t)$. Then

$$
\operatorname{Re}\left\{B_{t}^{\circ}[u, u]\right\} \leq\|u\|_{L^{2}(0, t)}^{2} \quad \text { and } \quad \operatorname{Re}\left\{B_{t}[u, u]\right\} \leq(1+a / 2)\|u\|_{L^{2}(0, t)}^{2}
$$

Proof. For complex $u,\|u\|^{2}=\|\operatorname{Re}(u)\|^{2}+\|\operatorname{Im}(u)\|^{2}$, and so we can reduce to the real case. Then, assuming now that $u$ is real, (5.12) gives

$$
B_{t}^{\circ}[u, u] \leq \frac{1}{2 \pi} \int_{|\omega|>a}|U(\omega)|^{2} d \omega \leq \frac{1}{2 \pi} \int_{-\infty}^{\infty}|U(\omega)|^{2} d \omega=\|u\|_{L^{2}(0, t)}
$$

Combining (5.9) and (5.11) gives

$$
B_{t}[u, u]=\frac{a}{2}\left(\int_{0}^{t} u\left(t^{\prime}\right) d t^{\prime}\right)^{2}+B_{t}^{\circ}[u, u]
$$

Applying Jensen's inequality gives us

$$
\frac{a}{2}\left(\int_{0}^{t} u\left(t^{\prime}\right) d t^{\prime}\right)^{2} \leq \frac{a}{2} \int_{0}^{t}\left[u\left(t^{\prime}\right)\right]^{2} d t^{\prime}=\frac{a}{2}\|u\|_{L^{2}(0, t)}^{2}
$$

and the result follows.
We are now prepared to return to the question of estimating the energy $\mathcal{E}$.
ThEOREM 5.9. Suppose that $\mathcal{D}$ is continuously differentiable and nonincreasing for $t \geq 0$. Then we have the estimates

$$
\begin{align*}
& \mathcal{E}(t) \leq \tau\|f\|_{L^{2}(0, t)}^{2}  \tag{5.13}\\
& \|\mathcal{J}\|_{L^{2}(0, t)} \leq \tau\|f\|_{L^{2}(0, t)} \tag{5.14}
\end{align*}
$$

where $f(t)=\left(\mathcal{D}(t)-\mathcal{D}_{0}\right) E_{x 0}(0, t)$.
Proof. Recall the energy identity (5.4). As $\mathcal{D}^{\prime} \leq 0$ and $\operatorname{Re}\left\{B_{s}[\mathcal{J}, \mathcal{J}]\right\} \geq 0$,

$$
\begin{aligned}
\mathcal{E}(t) & \leq \operatorname{Re}\langle f, \mathcal{J}\rangle_{L^{2}(0, t)}-\tau^{-1}\|\mathcal{J}\|_{L^{2}(0, t)}^{2} \\
& \leq\|f\|_{L^{2}(0, t)}\|\mathcal{J}\|_{L^{2}(0, t)}-\tau^{-1}\|\mathcal{J}\|_{L^{2}(0, t)}^{2} \\
& =\left(\|f\|_{L^{2}(0, t)}-\tau^{-1}\|\mathcal{J}\|_{L^{2}(0, t)}^{2}\right)\|\mathcal{J}\|_{L^{2}(0, t)}
\end{aligned}
$$

We know the left-hand side is nonnegative, so we obtain the estimate (5.14). Substituting this back into the right-hand side then gives the estimate (5.13).

Remark 5.10. The energy estimate takes a different form in the case of no damping, i.e., $\tau \rightarrow \infty$. Even when there is no damping, the system is not conservative since varying $\mathcal{D}(t)$ can inject energy into the system. This fact is made evident in section 7 .
6. A numerical method based on convolution quadrature. In this section we discuss a method for discretizing the system (4.19)-(4.20) directly. The key idea [27] is that we can replace the convolutions by discrete operators, which allow us to devise a time-marching scheme. In convolution quadrature the continuous convolution is approximated by

$$
(f * g)(N h) \approx \sum_{n=0}^{N} \omega_{N-n} g(n h)+\sum_{n=0}^{p-2} w_{N, n} g(n h),
$$

where the convolution weights $\omega_{n}$ and correction weights $w_{N, n}$ depend on $f$. When the correction terms are chosen carefully, the error from replacing the continuous convolution with the discrete convolution is of order $O\left(h^{p}\right)$.

The convolution weights $\omega_{n}$ are defined by the expansion

$$
\hat{f}(\delta(\zeta) / h)=\sum_{n=0}^{\infty} \omega_{n}(h) \zeta^{n}
$$

where $\hat{f}(s)=\mathcal{L}\{f\}$ is the Laplace transform of $f$, and $\delta(\zeta)$ is the generating formula for the $p$ th-order backward difference formula ( BDF ),

$$
\delta(\zeta)=\sum_{n=1}^{p} \frac{1}{n}(1-\zeta)^{n}
$$

The correction weights $w_{N, n}$ are defined by the system

$$
\sum_{n=0}^{p-2}(n h)^{q} w_{N, n}(h)=\int_{0}^{t} f\left(t^{\prime}\right)\left(t-t^{\prime}\right)^{q} d t^{\prime}-\sum_{n=0}^{N} \omega_{n}(h)(t-n h)^{q}
$$

where $q=0,1, \ldots, p-2$ and $t=N h$.
In what follows, we apply the above scheme with $f=k_{\ell}(\ell=1,2)$, and we denote the corresponding weights by $\omega_{n}^{(\ell)}$ and $w_{N, n}^{(\ell)}$.
6.1. Discretization of the governing equations. Assume that $\mathcal{D}$ has $p$ continuous derivatives. Writing $E_{x, N} \sim E_{x}(0, N h)$ and $\mathcal{J}_{N} \sim \mathcal{J}(N h)$, we discretize (4.19) and (4.20) and apply the convolution quadrature formula to get

$$
\begin{align*}
E_{x, N}+\frac{\eta}{2}\left(1+\omega_{0}^{(1)}\right) \mathcal{J}_{N}= & -\frac{\eta}{2}\left(\sum_{n=0}^{N-1} \omega_{N-n}^{(1)} \mathcal{J}_{n}+\sum_{n=0}^{p-2} w_{N, n}^{(1)} \mathcal{J}_{n}\right)+E_{x 0}(0, N h),  \tag{6.1}\\
-\omega_{0}^{(2)} \mathcal{D}(N h) E_{x, N}+\mathcal{J}_{N}= & \sum_{n=0}^{N-1} \omega_{N-n}^{(2)} \mathcal{D}(n h) E_{x, n}+\sum_{n=0}^{p-2} w_{N, n}^{(2)} \mathcal{D}(n h) E_{x, n} \\
& +2 e^{-N h / \tau}-j_{0}(N h) . \tag{6.2}
\end{align*}
$$

Note that for $1 \leq N \leq p-2$ we get a dense system for $E_{x, N}$ and $\mathcal{J}_{N}$ which must be solved by inverting a $(p-2) \times(p-2)$ matrix, but for $N>p-2$ we get a fully explicit numerical method.
6.2. The second-order case. In general, computing the convolution quadrature weights $\omega_{n}^{(\ell)}$ and the correction weights $w_{N, n}^{(\ell)}$ is quite involved; see [28] for an idea of how to do this in general. In the case where $p=2$, however, we can derive fairly explicit formulas for the quadrature weights, which allows for a fast and simple scheme. First, note that when $p=2$ there is only a single correction weight, so the scheme is fully explicit for every time step.

The formula for $\omega_{n}^{(2)}$ is explicit and is given by

$$
\begin{equation*}
\omega_{n}^{(2)}=\frac{h}{\sqrt{1-2 h / \tau}}\left(\frac{1}{r_{-}^{n+1}}-\frac{1}{r_{+}^{n+1}}\right), \quad n=0,1,2, \ldots \tag{6.3}
\end{equation*}
$$

where $r_{ \pm}=2 \pm \sqrt{1-2 h / \tau}$. We derive this and subsequent formulas in Appendix A, where it is also shown that $\omega_{n}^{(2)}$ is proportional to a Chebyshev polynomial $U_{n}$, (A.4).

For $\omega_{n}^{(1)}$, we have the following recursion. Define

$$
\begin{aligned}
& b_{0}=1+\frac{4 a^{2} h^{2}}{9}, \quad b_{n}=a^{2} h^{2}\left(n+\frac{n+4}{3^{n+2}}\right) \\
& a_{0}=\sqrt{b_{0}}, \quad a_{n}=\frac{1}{2 a_{0}}\left(b_{n}-\sum_{k=1}^{n-1} a_{n-k} a_{k}\right)
\end{aligned}
$$

Then $\omega_{0}^{(2)}=a_{0}-1$ and $\omega_{n}^{(2)}=a_{n}$ for $n>0$.
For $p=2$ the correction weights $w_{N, n}^{(\ell)}$ reduce to a single equation for $w_{N, 0}^{(\ell)}$,

$$
\begin{equation*}
w_{N, 0}^{(\ell)}(h)=\int_{0}^{N h} k_{\ell}\left(t^{\prime}\right) d t^{\prime}-\sum_{n=0}^{N} \omega_{n}^{(\ell)}(h) \tag{6.4}
\end{equation*}
$$

which simplifies further, as shown in Appendix A.
We note that for large $t$ there is a simpler method to compute the weights. For $\omega_{n}^{(\ell)}$ we can use [27, eq. (4.2)], which tells us that

$$
\omega_{n}^{(\ell)}(h) \rightarrow h k_{\ell}(n h) \quad \text { as } n h \rightarrow \infty
$$

For the $w_{N, n}^{(\ell)}$ we use [27, Cor. 4.2] to replace $w_{N, n}^{(\ell)}$ with $\omega_{N-n}^{(\ell)} c_{n}$, where the $c_{n}$ depend only on $p$. For the particular case where $p=2$, there is only $c_{0}=-1 / 2$.

Remark 6.1. When computing $k_{1}$, we have found it convenient to use the identity $x^{-1} J_{1}(x)=J_{0}(x)-J_{1}^{\prime}(x)$, which gives

$$
k_{1}(t)=a \int_{0}^{a t} J_{0}\left(t^{\prime}\right) d t^{\prime}-a J_{1}(a t)
$$

In Figure 2 we show a convergence study for convolutional quadrature with $p=2$ in the case where $\mathcal{D}_{0}$ is constant. The dots show the computed data points, while the lines show the line of best fit. From the line of best fit we see that the order of convergence for both $E_{x}$ and $\mathcal{J}$ is 2, as expected.

Further verification of the convolution quadrature scheme for nonconstant $\mathcal{D}(t)$ was done by comparing the computed solution with a simple collocation method based on discretizing (4.18) using the trapezoid rule, which converges according to the general theory in [4]. The two methods yielded results which agree to three digits in several examples.


FIg. 2. Convergence study for convolution quadrature.
7. Temporal control of graphene plasmons. In this section, we study how graphene plasmons can be controlled by varying the Drude weight in time. We begin by deriving a simpler model, an approximation based on a large wave number. The same approximation can be derived in the quasi-static regime using physical arguments based on electrostatics. The simpler approximate equation is easier to analyze and remains accurate in the regime of interest.
7.1. Approximate equation. Using $k_{1}=a+k_{1}^{\circ}$, (5.5), we rewrite (4.11) as

$$
\begin{equation*}
\frac{d j}{d t}+\frac{1}{\tau} j+\frac{\mathcal{D} \xi}{2 \epsilon} \int_{0}^{t} j\left(t^{\prime}\right) d t^{\prime}+\frac{\eta \mathcal{D}}{2}\left(\mathcal{J}+k_{1}^{\circ} * \mathcal{J}\right)=\frac{\mathcal{D} \xi}{2 \epsilon} \int_{0}^{t} j_{0}\left(t^{\prime}\right) d t^{\prime}+\mathcal{D} E_{x 0} \tag{7.1}
\end{equation*}
$$

using $a \eta=\xi / \epsilon$. We will use the Laplace transform representation of the convolution to approximate the fourth term in (7.1). Using $k_{1}=a+k_{1}^{\circ}$ and (4.8), we get

$$
\mathcal{L}\left\{\mathcal{J}+k_{1}^{\circ} * \mathcal{J}\right\}=s^{-1}\left(\sqrt{s^{2}+a^{2}}-a\right) \widehat{\mathcal{J}}
$$

where $\widehat{\mathcal{J}}=\mathcal{L}\{\mathcal{J}\}$. Since we are interested in the regime $a=c \xi \gg 1$, we approximate the square root and obtain

$$
\mathcal{L}\left\{\mathcal{J}+k_{1}^{\circ} * \mathcal{J}\right\}=\frac{s}{2 a} \widehat{\mathcal{J}}+O\left(a^{-3}\right)
$$

Inverting the Laplace transforms, using $\mathcal{J}(0)=0$ gives

$$
\mathcal{J}(t)+k_{1}^{\circ} * \mathcal{J}=\frac{1}{2 a} \frac{d \mathcal{J}}{d t}+O\left(a^{-3}\right)
$$

Since we expect $\mathcal{J}$ to be oscillating at a frequency $\omega \approx \xi^{1 / 2}$ (see (3.9)), we have

$$
\mathcal{J}(t)+k_{1}^{\circ} * \mathcal{J}=O\left(\xi^{-1 / 2}\right)
$$

As $E_{x 0}(0, t)=\gamma_{0}\left(\epsilon s_{0}\right)^{-1} e^{-s_{0} t}$ and $j_{0}(t)=2 e^{-s_{0} t}$, the right-hand side of (7.1) evaluates to

$$
\mathcal{D}(t) E_{x 0}(0,0)\left\{\frac{\xi}{\gamma_{0}}+\left(1-\frac{\xi}{\gamma_{0}}\right) e^{-s_{0} t}\right\}
$$

From (3.10), $\gamma_{0}=\xi+O(1)$ as $\xi \rightarrow \infty$, so we can approximate the quantity inside $\}$ by 1. Hence, discarding the $O\left(\xi^{-1 / 2}\right)$ terms gives us the approximate equation

$$
\begin{equation*}
\frac{d j_{a}}{d t}+\frac{1}{\tau} j_{a}(t)+\frac{\mathcal{D} \xi}{2 \epsilon} \int_{0}^{t} j_{a}(s) d s=\mathcal{D}(t) E_{x 0}(0,0), \quad \text { with } \quad j_{a}(0)=2 \tag{7.2}
\end{equation*}
$$

where we have replaced $j$ with $j_{a}$ to distinguish from the solution of the exact equation.
To conclude this section, note that if we let

$$
u(t)=\int_{0}^{t} j_{a}(s) d s-\frac{2 \epsilon}{\xi} E_{x 0}(0,0)
$$

then we can rewrite (7.2) as the second-order ordinary differential equation

$$
\begin{equation*}
\frac{d^{2} u}{d t^{2}}+\frac{1}{\tau} \frac{d u}{d t}+\frac{\mathcal{D} \xi}{2 \epsilon} u=0 \tag{7.3}
\end{equation*}
$$

with $u(0)=-(2 \epsilon / \xi) E_{x 0}(0,0)$ and $u^{\prime}(0)=2$.
Figure 3 shows plots of the solution determined by the approximate equation (7.2) versus the exact solution obtained from the convolution quadrature methods of section 6. The upper plot shows a small $\xi$, and the lower plot shows a large $\xi$ (see figure caption for exact parameter values). The order of convergence in the $L^{2}$ and $L^{\infty}$ norms as $\xi \rightarrow \infty$ is shown in Figure 4. In both cases the order of convergence is $1 / 2$ as expected since the derivation neglected terms of order $\xi^{-1 / 2}$.

Remark 7.1. In an earlier work [37] we posited that when $\mathcal{D}(t)$ jumps from $\mathcal{D}_{0}$ to $\mathcal{D}_{1}$ at $t=t_{0}, \xi$ is conserved and only $\omega(\xi)$ varies across time $t_{0}$. Using this hypothesis, we devised a propagator matrix framework for modeling plasmon evolution in graphene when the Drude weight $\mathcal{D}(t)$ is time dependent. The approximation above gives formal justification for the theory. In Appendix B, we provide further justification by considering the approximation in the specific case of piecewise constant $\mathcal{D}(t)$.
7.2. Wave reversal. Motivated by the work of Bacot et al. [2], in which wave reversal was demonstrated for shallow water waves, we ask if it is possible to reverse a right-going plasmon by instantaneously changing the Drude weight of the graphene. That is, we consider

$$
\mathcal{D}(t)= \begin{cases}\mathcal{D}_{0}, & t<0 \\ \mathcal{D}_{1}, & t \geq 0\end{cases}
$$

In analogy to Maxwell's equations with time-dependent material parameters [30], we expect that the sudden jump in the Drude weight will cause part of the plasmon to be reflected.

Since $\mathcal{D}$ is constant for $t \geq 0$, we can differentiate (7.2) to get

$$
\begin{equation*}
\frac{d^{2} j_{a}}{d t^{2}}+\frac{1}{\tau} \frac{d j_{a}}{d t}+\frac{\mathcal{D}_{1} \xi}{2 \epsilon} j_{a}=0 \tag{7.4}
\end{equation*}
$$

We retain the initial condition $j_{a}(0)=2$, and we also have

$$
\frac{d j_{a}}{d t}(0)=-\frac{1}{\tau} j_{a}(0)+\mathcal{D}_{1} E_{x 0}(0,0)=-\frac{2}{\tau}+\frac{\mathcal{D}_{1} \gamma_{0}}{\epsilon s_{0}}
$$




Fig. 3. Solution determined by the approximate equation (7.2) versus the exact solution as determined by convolution quadrature. Here $\epsilon=\mu=1$ and $\tau=2$. We set $\mathcal{D}_{0}=2$ and $\mathcal{D}(t)=1$ for $t>0$. The top plot shows $\xi=8$, and the bottom plot shows $\xi=256$.


Fig. 4. Convergence of the approximate solution to the exact solution as $\xi$ is increased.

The general solution of (7.4) is $j_{a}=j_{a}^{+}+j_{a}^{-}$, where $j_{a}^{+}=T e^{-s_{a} t}$ and $j_{a}^{-}=R e^{-\overline{s_{a}} t}$. Here $s_{a}$ is given by the dispersion relation (cf. (3.9))

$$
\begin{equation*}
s_{a}=\frac{1}{2 \tau}+i \sqrt{\frac{\mathcal{D}_{1} \xi}{2 \epsilon}-\frac{1}{4 \tau^{2}}}=i \sqrt{\frac{\mathcal{D}_{1} \xi}{2 \epsilon}}+\frac{1}{2 \tau}+O\left(\xi^{-1 / 2}\right) \quad \text { as } \xi \rightarrow \infty \tag{7.5}
\end{equation*}
$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

To find the coefficients $T$ and $R$ we use the initial conditions and obtain

$$
\begin{equation*}
T=\frac{\mathcal{C}-2 / \tau+2 \overline{s_{a}}}{\overline{s_{a}}-s_{a}}, \quad R=\frac{\mathcal{C}-2 / \tau+2 s_{a}}{s_{a}-\overline{s_{a}}} \quad \text { with } \quad \mathcal{C}=\frac{\mathcal{D}_{1} \gamma_{0}}{\epsilon s_{0}} . \tag{7.6}
\end{equation*}
$$

Expanding these expressions to order $O\left(\xi^{-1 / 2}\right)$ gives the following estimates.
Theorem 7.2. We have

$$
\begin{aligned}
& \frac{j_{a}^{+}(0)}{j_{a}(0)}=\frac{1}{2}\left(1+\sqrt{\frac{\mathcal{D}_{1}}{\mathcal{D}_{0}}}\right)+O\left(\xi^{-1 / 2}\right) \\
& \frac{j_{a}^{-}(0)}{j_{a}(0)}=\frac{1}{2}\left(1-\sqrt{\frac{\mathcal{D}_{1}}{\mathcal{D}_{0}}}\right)+O\left(\xi^{-1 / 2}\right)
\end{aligned}
$$

Proof. We use (3.6) to get

$$
\frac{\mathcal{D}_{1} \gamma_{0}}{\epsilon s_{0}}=\frac{2 \mathcal{D}_{1}}{\mathcal{D}_{0}}\left(\frac{1}{\tau}-s_{0}\right)=\frac{2 \mathcal{D}_{1}}{\mathcal{D}_{0}}\left(\frac{1}{2 \tau}-i \sqrt{\frac{\mathcal{D}_{0} \xi}{2 \epsilon}}\right)+O\left(\xi^{-1 / 2}\right)
$$

Next, note from (7.5) that

$$
s_{a}-\bar{s}_{a}=-2 i \sqrt{\frac{\mathcal{D}_{1} \xi}{2 \epsilon}}+O\left(\xi^{-1 / 2}\right)
$$

Hence, the leading-order terms give

$$
T=1+\sqrt{\mathcal{D}_{1} / \mathcal{D}_{0}}+O\left(\xi^{-1 / 2}\right)
$$

Recalling that $j_{a}(0)=2$ gives us the result. The derivation for $R$ is similar.
The result of Theorem 7.2 tells us that a plasmon propagating to the right when $t<0$ and $\mathcal{D}(t)=\mathcal{D}_{0}$ splits into two components when $t>0$ and $\mathcal{D}(t)=\mathcal{D}_{1}$. The first component is rightward, and the amplitude has been scaled to $j_{a}^{+}(0) / j_{a}(0)$. The second component is leftward, and the amplitude is $j_{a}^{-}(0) / j_{a}(0)$. We can interpret the first component as the transmitted wave and the second as the reflected wave. The reflection is large if $\mathcal{D}_{1} / \mathcal{D}_{0}$ is small.

Finally, we note that an alternative derivation of the results of Theorem 7.2 without using (7.2) is possible, as shown in Appendix B.
7.3. Parametric amplification. In [37] we briefly studied how plasmons can be amplified by periodic excitation. We provide a more detailed treatment of this phenomenon next. Our treatment is inspired by [23, sect. 27]. Assume that $\mathcal{D}$ is of the form

$$
\mathcal{D}(t)=\mathcal{D}_{0}+\Delta \mathcal{D}_{0} \sin \left(\omega_{p} t\right),
$$

where we assume $\Delta \mathcal{D}_{0} / \mathcal{D}_{0} \ll 1$. In this case it is most convenient to work with (7.3). Introducing a change of variables $u=U e^{-t /(2 \tau)}$ in (7.3) gives an equation for $U(t)$,

$$
\begin{equation*}
U^{\prime \prime}+\left(\frac{\mathcal{D} \xi}{2 \epsilon}-\frac{1}{4 \tau^{2}}\right) U=0 \tag{7.7}
\end{equation*}
$$

Substituting our expression for $\mathcal{D}$ allows us to write (7.7) as

$$
U^{\prime \prime}+\omega_{0}^{2}\left[1+h \sin \left(\omega_{p} t\right)\right] U=0
$$

where we have set

$$
\begin{equation*}
\omega_{0}^{2}=\frac{\xi \mathcal{D}_{0}}{2 \epsilon}-\frac{1}{4 \tau^{2}}=\frac{2 \xi \mathcal{D}_{0} \tau^{2}-\epsilon}{4 \epsilon \tau^{2}}, \quad h=\frac{\xi \Delta \mathcal{D}_{0}}{2 \epsilon \omega_{0}^{2}}=\frac{\Delta \mathcal{D}_{0}}{\mathcal{D}_{0}-\epsilon /\left(2 \xi \tau^{2}\right)} \tag{7.8}
\end{equation*}
$$

Note that since $\xi \gg 1$ and $\Delta \mathcal{D}_{0} / \mathcal{D}_{0} \ll 1$, we have $h \ll 1$. We also take $\omega_{p}=2 \omega_{0}+\delta$, where we assume $\delta=O(h)$. Since the coefficients of our solution are periodic, Bloch's theorem tells us that the solution will be of the form $e^{r t}$ multiplied by a periodic function. First, we observe that the amplification should grow with $h$ and that there should be no amplification when $h=0$, so we assume that $r=O(h)$. We expect that the periodic function will be a perturbation of the solution when $h=0$, in which case a basis for the solution space is $\left\{e^{-i \omega_{p} t / 2}, e^{i \omega_{p} t / 2}\right\}$. Since

$$
2 i \sin \left(\omega_{p} t\right) e^{ \pm i(2 n+1) \omega_{p} t / 2}= \pm e^{ \pm i(2 n+3) \omega_{p} t / 2} \mp e^{ \pm i(2 n-1) \omega_{p} t / 2}
$$

we look for a solution in the form

$$
U(t)=e^{r t} \sum_{n=0}^{\infty} h^{n}\left(T_{n} e^{-i(2 n+1) \omega_{p} t / 2}+R_{n} e^{i(2 n+1) \omega_{p} t / 2}\right)
$$

Substituting this into (7.7) and collecting the $e^{-i \omega_{p} t / 2}$ and $e^{i \omega_{p} t / 2}$ terms, we get

$$
\begin{aligned}
\left(-2 i r \omega_{0}+\omega_{0} \delta\right) T_{0} & =-\frac{h}{2 i} R_{0}+O\left(h^{2}\right), \\
\left(2 i r \omega_{0}+\omega_{0} \delta\right) R_{0} & =\frac{h}{2 i} T_{0}+O\left(h^{2}\right) .
\end{aligned}
$$

Keeping only $O(h)$ terms, we get the compatibility condition

$$
\operatorname{det}\left[\begin{array}{cc}
(-2 i r+\delta) \omega_{0} & -i h / 2 \\
i h / 2 & (2 i r+\delta) \omega_{0}
\end{array}\right]=0
$$

so that

$$
r^{2}=\frac{1}{4}\left(\frac{h^{2} \omega_{0}^{2}}{4}-\delta^{2}\right) .
$$

In order for amplification to occur we require $r>1 /(2 \tau)$, so this happens when

$$
\delta^{2}<\frac{h^{2} \omega_{0}^{2}}{4}-\frac{1}{\tau^{2}}=\frac{1}{4}\left(\frac{\Delta \mathcal{D}_{0}}{\mathcal{D}_{0}}\right)^{2} \frac{\mathcal{D}_{0} \xi}{2 \epsilon} \cdot \frac{1}{1-\epsilon /\left(2 \xi \mathcal{D}_{0} \tau^{2}\right)}-\frac{1}{\tau^{2}}
$$

after use of (7.8). Now $\left\{1+\epsilon /\left(2 \xi \mathcal{D}_{0} \tau^{2}\right)\right\}^{-1}=1+O\left(\xi^{-1}\right)$, so neglecting higher-order terms gives

$$
\begin{equation*}
\delta^{2}<\frac{1}{4}\left(\frac{\Delta \mathcal{D}_{0}}{\mathcal{D}_{0}}\right)^{2} \frac{\mathcal{D}_{0} \xi}{2 \epsilon}-\frac{1}{\tau^{2}} \tag{7.9}
\end{equation*}
$$

In particular, for amplification to be possible we require that the quantity on the right-hand side be positive, which gives us the condition

$$
\frac{1}{\tau}<\frac{1}{2} \frac{\Delta \mathcal{D}_{0}}{\mathcal{D}_{0}} \sqrt{\frac{\mathcal{D}_{0} \xi}{2 \epsilon}}
$$



Fig. 5. Predicted and computed amplification.

Figure 5 shows the predicted and computed values of $r-1 /(2 \tau)$ in the amplification region. Here $\Delta \mathcal{D}_{0}$ is $20 \%$ of $\mathcal{D}_{0}$. The computed values are obtained by simulating $j$ using the convolution quadrature methods of section 6 and then finding the location of two successive peaks of $\operatorname{Re}(j)$, say $t_{1}$ and $t_{2}$. At a peak, we expect that $\operatorname{Re}(j(t))=C e^{(r-1 /(2 \tau)) t}$, so that

$$
\operatorname{Re}\left(j\left(t_{2}\right)\right) / \operatorname{Re}\left(j\left(t_{1}\right)\right)=e^{(r-1 /(2 \tau))\left(t_{2}-t_{1}\right)}
$$

Solving this equation gives us

$$
r-\frac{1}{2 \tau}=\frac{\log \left\{\operatorname{Re}\left(j\left(t_{2}\right)\right) / \operatorname{Re}\left(j\left(t_{1}\right)\right)\right\}}{t_{2}-t_{1}}
$$

We note that the actual values are shifted slightly to the left of the predicted values, which could perhaps be fixed by computing an $O\left(h^{2}\right)$ correction to the values of $\delta$. As $\Delta \mathcal{D}_{0}$ grows larger the shift gets worse, as seen in Figure 6, where $\Delta \mathcal{D}_{0}$ is instead taken to be $80 \%$ of $\mathcal{D}_{0}$.

Remark 7.3. It should be noted that while the time-reversal and parametric amplification have been studied in the context of the approximate equation, numerical evidence indicates that they are also present in the full model. We performed several numerical experiments to reproduce amplification and reversal phenomena using both convolution quadrature and Euler's method on (4.11).
8. Discussion. We have studied the propagation of plasmons on graphene. Of interest is the evolution of the electromagnetic field associated with the plasmon when the Drude weight is time varying. We provide an analysis of the equations governing the time evolution of the amplitude of the current density on the graphene. A numerical solution, based on convolution quadrature, is proposed. After deriving an approximate equation for the large wave number regime, we investigated the phenomena of wave reversal and parametric amplification. The results indicate that rich and


Fig. 6. Amplification for a large perturbation.
controllable behavior of plasmons may be achieved by modulating the Drude weight of the graphene.

Appendix A. Weights in the convolution quadratures for $\boldsymbol{p}=\mathbf{2}$. To see how we can compute the convolution weights $\omega_{n}^{(\ell)}$, let $K_{j}$ be the Laplace transform of the kernel $k_{j}$. We need to expand $K_{j}(\delta(\zeta) / h)$ in powers of $\zeta$. We have

$$
\begin{align*}
K_{1}(\delta(\zeta) / h) & =\sqrt{1+\left(\frac{a h}{\delta(\zeta)}\right)^{2}}-1  \tag{A.1}\\
K_{2}(\delta(\zeta) / h) & =\frac{1}{\delta(\zeta) / h+1 / \tau} \tag{A.2}
\end{align*}
$$

Furthermore,

$$
\begin{equation*}
\delta(\zeta)=\frac{3}{2}-2 \zeta+\frac{\zeta^{2}}{2} \tag{A.3}
\end{equation*}
$$

We begin by deriving an expression for the simpler $\omega_{n}^{(2)}$ weights. Combining (A.2) and (A.3) gives us

$$
K_{2}(\delta(\zeta) / h)=\frac{2 h}{\zeta^{2}-4 \zeta+3+2 h / \tau}
$$

The roots of the denominator are $r_{ \pm}=2 \pm \sqrt{1-2 h / \tau}$, so partial fractions give

$$
K_{2}(\delta(\zeta) / h)=\frac{h}{\sqrt{1-2 h / \tau}}\left(\frac{1}{\zeta-r_{+}}-\frac{1}{\zeta-r_{-}}\right)
$$

Expanding using the geometric series then gives (6.3).

An alternative formula, obtained by using [31, eq. 18.12.10], is

$$
\begin{equation*}
\omega_{n}^{(2)}=2 h \lambda^{-n-2} U_{n}(2 / \lambda), \quad n=0,1,2, \ldots \tag{A.4}
\end{equation*}
$$

where $\lambda^{2}=3+2 h / \tau$, and $U_{n}$ is a Chebyshev polynomial of the second kind.
Turning to the $\omega_{n}^{(1)}$, we first note that $\delta(\zeta)=\frac{1}{2}(\zeta-3)(\zeta-1)$, which gives us

$$
K_{1}(s)=\sqrt{1+\frac{4 a^{2} h^{2}}{(\zeta-3)^{2}(\zeta-1)^{2}}}-1
$$

We use partial fractions to decompose

$$
\frac{4}{(\zeta-3)^{2}(\zeta-1)^{2}}=\frac{1}{\zeta-1}+\frac{1}{(\zeta-1)^{2}}-\frac{1}{\zeta-3}+\frac{1}{(\zeta-3)^{2}}=\sum_{n=0}^{\infty}\left(n+\frac{n+4}{3^{n+2}}\right) \zeta^{n}
$$

Now computing $\omega_{n}^{(1)}$ reduces to computing the square root of a formal power series with known coefficients. In general, if we have a series

$$
b(\zeta)=\sum_{n=0}^{\infty} b_{n} \zeta^{n}
$$

and wish to find a series

$$
a(\zeta)=\sum_{n=0}^{\infty} a_{n} \zeta^{n}
$$

so that $a^{2}=b$, then this can be done with the recurrence relation

$$
a_{0}=\sqrt{b_{0}}, \quad a_{n}=\frac{1}{2 a_{0}}\left(b_{n}-\sum_{k=1}^{n-1} a_{n-k} a_{k}\right), \quad n=1,2, \ldots
$$

There are also more efficient algorithms which use the fast Fourier transform; see, for example, [18].

For $p=2$, the correction weights $w_{N, 0}^{(\ell)}$ are determined by (6.4) in terms of

$$
W_{\ell}(t)=\int_{0}^{t} k_{\ell}\left(t^{\prime}\right) d t^{\prime} \quad \text { with } t=N h
$$

For $\ell=2, W_{2}(N h)=\tau\left(1-e^{-N h / \tau}\right)$. For $\ell=1$, let $f(t)=W_{1}(t)-t k_{1}(t)$, whence

$$
f^{\prime}(t)=-t k_{1}^{\prime}(t)=-a J_{1}(a t)=(d / d t) J_{0}(a t) .
$$

Integrating, using $f(0)=0$, we obtain

$$
W_{1}(t)=J_{0}(a t)-1+t k_{1}(t)
$$

Appendix B. Approximate solution for piecewise constant $\mathcal{D}$. When $\mathcal{D}(t)=\mathcal{D}_{1}$ for $t \geq 0$, Laplace-transforming the Drude model (2.6) gives

$$
(s+1 / \tau) \mathcal{L}\{j\}=\mathcal{D}_{1} \widehat{E_{x}}(0, s)+j(0)
$$

Now using (4.7) to substitute for $\widehat{E_{x}}(0, s)$ gives us the equation

$$
Q(s) \mathcal{L}\{j\}=\mathcal{D}_{1} E_{x}(0,0)+s j(0)+\frac{\mathcal{D}_{1}}{\epsilon} \frac{\gamma-\gamma_{0}}{s+s_{0}} \quad \text { with } \quad Q(s)=s^{2}+\frac{s}{\tau}+\frac{\mathcal{D}_{1} \gamma}{2 \epsilon}
$$

Inverting the Laplace transform gives

$$
\begin{equation*}
j(t)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty}\left(\mathcal{D}_{1} E_{x}(0,0)+s j(0)+\frac{\mathcal{D}_{1}}{\epsilon} \frac{\gamma-\gamma_{0}}{s+s_{0}}\right) \frac{e^{s t} d s}{Q(s)} . \tag{B.1}
\end{equation*}
$$

Here the constant $c$ is chosen so that the contour is to the right of all the singularities of the integrand. To understand the singularities, note that if $s$ is real, then

$$
Q(s)>s^{2}+\frac{s}{\tau}+\frac{\mathcal{D}_{1} \xi}{2 \epsilon}>0
$$

when $\xi \gg 1$, so the solutions of

$$
\begin{equation*}
Q(s)=s^{2}+\frac{s}{\tau}+\frac{\mathcal{D}_{1} \gamma}{2 \epsilon}=0 \tag{B.2}
\end{equation*}
$$

are complex. Multiplying (B.2) by $s(s+1 / \tau)-\mathcal{D}_{1} \gamma /(2 \epsilon)$ gives us

$$
p(s):=s^{4}+\frac{2}{\tau} s^{3}+\left(\frac{1}{\tau^{2}}-\frac{\mathcal{D}_{1}^{2} \eta^{2}}{4}\right) s^{2}-\frac{\mathcal{D}_{1}^{2} \xi^{2}}{4 \epsilon^{2}}=0
$$

Note that after the change of variables $s \rightarrow-s$ this is the same as the dispersion relation (3.8), so the two roots of (B.2) are $s=-s_{1}$ and $s=-\overline{s_{1}}$, where

$$
s_{1}=i \sqrt{\frac{\xi \mathcal{D}_{1}}{2 \epsilon}}+\frac{1}{2 \tau}+O\left(\xi^{-1 / 2}\right)
$$

Now we can evaluate (B.1) using the residue theorem. Rearranging gives us

$$
j(t)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \frac{s(s+1 / \tau)-\mathcal{D}_{1} \xi /(2 \epsilon)}{p(s)}\left(\mathcal{D}_{1} E_{x}(0,0)+s j(0)+\frac{\mathcal{D}_{1}}{\epsilon} \frac{\gamma-\gamma_{0}}{s+s_{0}}\right) e^{s t} d s
$$

Observe that the residues of

$$
\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \frac{s(s+1 / \tau)-\mathcal{D}_{1} \xi /(2 \epsilon)}{p(s)} \cdot \frac{\mathcal{D}_{1}}{\epsilon} \frac{\gamma-\gamma_{0}}{s+s_{0}} e^{s t} d s
$$

are $O\left(\xi^{-1}\right)$ at the singularities $-s_{1},-\overline{s_{1}}$, and $-s_{0}$. Therefore, we have that

$$
j(t)=T e^{-s_{1}}+R e^{-\overline{s_{1}}}+O\left(\xi^{-1}\right)
$$

where $T$ and $R$ are given as follows:

$$
\begin{aligned}
& T=\lim _{s \rightarrow-s_{1}}\left(s+s_{1}\right) \frac{s(s+1 / \tau)-\mathcal{D}_{1} \gamma /(2 \epsilon)}{p(s)}\left[\mathcal{D}_{1} E_{x}(0,0)+s j(0)\right] \\
& R=\lim _{s \rightarrow-\overline{s_{1}}}\left(s+\overline{s_{1}}\right) \frac{s(s+1 / \tau)-\mathcal{D}_{1} \gamma /(2 \epsilon)}{p(s)}\left[\mathcal{D}_{1} E_{x}(0,0)+s j(0)\right]
\end{aligned}
$$

Multiplying and dividing by the appropriate root of $p(s)$ gives us

$$
\begin{aligned}
& T=\lim _{s \rightarrow-s_{1}} \frac{D_{1} E_{x}(0,0)+s j(0)}{s+\bar{s}_{1}}=\frac{\mathcal{D}_{1} \gamma_{0} /\left(\epsilon s_{0}\right)-2 s_{1}}{\overline{s_{1}}-s_{1}}, \\
& R=\lim _{s \rightarrow-\overline{s_{1}}} \frac{\mathcal{D}_{1} E_{x}(0,0)+s j(0)}{s+s_{1}}=\frac{\mathcal{D}_{1} \gamma_{0} /\left(\epsilon s_{0}\right)-2 \overline{s_{1}}}{s_{1}-\overline{s_{1}}}
\end{aligned}
$$

Finally, we note that $-s_{1}=\overline{s_{1}}-2 / \tau$ and $-\overline{s_{1}}=s_{1}-2 / \tau$, so we have

$$
T=\frac{\mathcal{D}_{1} \gamma_{0} /\left(\epsilon s_{0}\right)-2 / \tau+2 \overline{s_{1}}}{\overline{s_{1}}-s_{1}}, \quad R=\frac{\mathcal{D}_{1} \gamma_{0} /\left(\epsilon s_{0}\right)-2 / \tau+2 s_{1}}{s_{1}-\overline{s_{1}}}
$$

These are simply (7.6) with $s_{1}$ instead of $s_{a}$, but since by (7.5) we know $s_{1}$ and $s_{a}$ agree up to $O\left(\xi^{-1 / 2}\right)$, we again recover the result

$$
\begin{equation*}
j(t)=\left(1+\sqrt{\mathcal{D}_{1} / \mathcal{D}_{0}}\right) e^{-s_{1} t}+\left(1-\sqrt{\mathcal{D}_{1} / \mathcal{D}_{0}}\right) e^{-\overline{s_{1}} t}+O\left(\xi^{-1 / 2}\right) \tag{B.3}
\end{equation*}
$$

We can view (B.3) as a special case of Theorem 7.2 when $\mathcal{D}(t)$ is piecewise constant. We note that we obtained (B.3) without using the approximate equation (7.2).

Acknowledgments. This research was initiated by discussions with Tony Low, to whom we express our gratitude. We also thank Misun Min for helpful discussions.

## REFERENCES

[1] P. Avouris, T. F. Heinz, and T. Low, 2 D Materials, Cambridge University Press, Cambridge, UK, 2017.
[2] V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort, Time reversal and holography with spacetime transformations, Nature Phys., 12 (2016), pp. 972-977.
[3] D. N. Basov, M. M. Fogler, and F. J. García de Abajo, Polaritons in van der Waals materials, Science, 354 (2016), aag1992.
[4] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, Cambridge, UK, 2004.
[5] H. Brunner, Volterra Integral Equations, Cambridge University Press, Cambridge, UK, 2017.
[6] T. A. Burton, Volterra Integral and Differential Equations, Academic Press, New York, 1983.
[7] E. Carrasco and J. Perruisseau-Carrier, Reflectarray antenna at terahertz using graphene, IEEE Antennas Wireless Propag. Lett., 12 (2013), pp. 253-256.
[8] A. Chabchoub and M. Fink, Time-reversal generation of rogue waves, Phys. Rev. Lett., 112 (2014), 124101.
[9] J. Chen, M. Badioli, P. Alonso-González, S. Thongrattanasiri, F. Huth, J. Osmond, M. Spasenović, A. Centeno, A. Pesquera, P. Godignon, A. Zurutuza Elorza, N. Camara, F. J. García de Abajo, R. Hillenbrand, and F. H. L. Koppens, Optical nano-imaging of gate-tunable graphene plasmons, Nature, 487 (2012), pp. 77-81.
[10] A. V. Chumak, V. S. Tiberkevich, A. D. Karenowska, A. A. Serga, J. F. Gregg, A. N. Slavin, and B. Hillebrands, All-linear time reversal by a dynamic artificial crystal, Nature Comm., 1 (2010), 141.
[11] C. Draeger and M. Fink, One-channel time reversal of elastic waves in a chaotic $2 D$-silicon cavity, Phys. Rev. Lett., 79 (1997), pp. 407-410.
[12] L. C. Evans, Partial Differential Equations, 2nd ed., American Mathematical Society, New York, 2010.
[13] Z. Fei, A. S. Rodin, G. O. Andreev, W. Bao, A. S. McLeod, M. Wagner, L. M. Zhang, Z. Zhao, M. Thiemens, G. Dominguez, M. M. Fogler, A. H. Castro Neto, C. N. Lau, F. Keilmann, and D. N. Basov, Gate-tuning of graphene plasmons revealed by infrared nano-imaging, Nature, 487 (2012), pp. 82-85.
[14] M. Fink, Time reversal of ultrasonic fields. I. Basic principles, IEEE Trans. Ultrasonics Ferroelectrics Frequency Control, 39 (1992), pp. 555-566.
[15] M. Freitag, T. Low, W. Zhu, H. Yan, F. Xia, and P. Avouris, Photocurrent in graphene harnessed by tunable intrinsic plasmons, Nature Comm., 4 (2013), 1951.
[16] F. J. García de Abajo, Graphene plasmonics: Challenges and opportunities, ACS Photonics, 1 (2014), pp. 135-152.
[17] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 2014.
[18] P. Henrici, Fast Fourier methods in computational complex analysis, SIAM Rev., 21 (1979), pp. 481-527, https://doi.org/10.1137/1021093.
[19] H. Hu, X. Yang, F. Zhai, D. Hu, R. Liu, K. Liu, Z. Sun, and Q. Dai, Far-field nanoscale infrared spectroscopy of vibrational fingerprints of molecules with graphene plasmons, Nature Comm., 7 (2016), 12334.
[20] L. Ju, B. Geng, J. Horng, C. Girit, M. Martin, Z. Hao, H. A. Bechtel, X. Liang, A. Zettl, Y. Ron Shen, and F. Wang, Graphene plasmonics for tunable terahertz metamaterials, Nature Nanotech., 6 (2011), pp. 630-634.
[21] F. H. L. Koppens, T. Mueller, P. Avouris, A. C. Ferrari, M. S. Vitiello, and M. Polini, Photodetectors based on graphene, other two-dimensional materials and hybrid systems, Nature Nanotech., 9 (2014), pp. 780-793.
[22] F. H. L. Koppens, D. E. Chang, and F. J. Garcia de Abajo, Graphene plasmonics: A platform for strong light-matter interactions, Nano Lett., 11 (2011), pp. 3370-3377.
[23] L. D. Landau and E. M. Lifshitz, Mechanics, 3rd ed., Butterworth-Heinemann, Oxford, 1976.
[24] G. Lerosey, J. De Rosny, A. Tourin, A. Derode, G. Montaldo, and M. Fink, Time reversal of electromagnetic waves, Phys. Rev. Lett., 92 (2004), 193904.
[25] T. Low and P. Avouris, Graphene plasmonics for terahertz to mid-infrared applications, ACS Nano, 8 (2014), pp. 1086-1101.
[26] T. Low, A. Chaves, J. D. Caldwell, A. Kumar, N. X. Fang, P. Avouris, T. F. Heinz, F. Guinea, L. Martin-Moreno, and F. Koppens, Polaritons in layered two-dimensional materials, Nature Materials, 16 (2017), pp. 182-194.
[27] C. Lubich, Convolution quadrature and discretized operational calculus. I, Numer. Math., 52 (1988), pp. 129-145.
[28] C. Lubich, Convolution quadrature and discretized operational calculus. II, Numer. Math., 52 (1988), pp. 413-425.
[29] A. H. Mohammadian, V. Shankar, and W. F. Hall, Computation of electromagnetic scattering and radiation using a time-domain finite-volume discretization procedure, Comput. Phys. Comm., 68 (1991), pp. 175-196.
[30] F. R. Morgenthaler, Velocity modulation of electromagnetic waves, IRE Trans. Microwave Theory Techn., 6 (1958), pp. 167-172.
[31] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds., NIST Digital Library of Mathematical Functions, https://dlmf.nist.gov, 2019.
[32] J. B. Pendry, Time reversal and negative refraction, Science, 322 (2008), pp. 71-73.
[33] A. Przadka, S. Feat, P. Petitjeans, V. Pagneux, A. Maurel, and M. Fink, Time reversal of water waves, Phys. Rev. Lett., 109 (2012), 064501.
[34] D. Rodrigo, O. Limaj, D. Janner, D. Etezadi, F. J. García de Abajo, V. Pruneri, and H. Altug, Mid-infrared plasmonic biosensing with graphene, Science, 349 (2015), pp. 165168.
[35] Y. Sivan and J. B. Pendry, Time reversal in dynamically tuned zero-gap periodic systems, Phys. Rev. Lett., 106 (2011), 193902.
[36] J. F. M. Werra, C. Wolff, C. Matyssek, and K. Busch, Current sheets in the discontinuous Galerkin time-domain method: An application to graphene, in Metamaterials X, Prague, Proc. SPIE 9502, 2015, 95020E.
[37] J. Wilson, F. Santosa, M. Min, and T. Low, Temporal control of graphene plasmons, Phys. Rev. B, 98 (2018), 081411.
[38] H. Yan, X. Li, B. Chandra, G. Tulevski, Y. Wu, M. Freitag, W. Zhu, P. Avouris, and F. XiA, Tunable infrared plasmonic devices using graphene/insulator stacks, Nature Nanotech., 7 (2012), pp. 330-334.
[39] H. Yan, T. Low, W. Zhu, Y. Wu, M. Freitag, X. Li, F. Guinea, P. Avouris, and F. XIA, Damping pathways of mid-infrared plasmons in graphene nanostructures, Nature Photonics, 7 (2013), pp. 394-399.


[^0]:    *Received by the editors November 14, 2018; accepted for publication (in revised form) March 27, 2019; published electronically June 20, 2019.
    http://www.siam.org/journals/siap/79-3/M122688.html
    Funding: This work was partially supported by the National Science Foundation through awards DMS 1211884 and DMS 1440471.
    ${ }^{\dagger}$ School of Mathematics, University of Minnesota, Minneapolis, MN 55455-0134 (wil00108@umn. edu, santosa@math.umn.edu).
    $\ddagger$ Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401-1887 (pamartin@mines.edu).

