

## Two-dimensional Brinkman flows and their relation to analogous Stokes flows

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2D Stokes flows often exhibit the Stokes paradox: logarithmic growth of the fluid velocity in the far field. Analogous Brinkman flows are governed by the same equations apart from an additional term involving a parameter  $\alpha$ . Although these equations reduce to those for Stokes flow when  $\alpha = 0$ , we show that the Brinkman solutions do not approach the corresponding Stokes solutions as  $\alpha \rightarrow 0$ ; instead, logarithmic divergence with  $\alpha$  is found. We also show that Brinkman flows do not exhibit a Stokes-like paradox. These results are given in detail for two specific problems, namely flow past a rigid circular cylinder and flow past a thin rigid strip.

**Keywords:** Stokes flow; Brinkman flow.

### 1. Introduction

Creeping flow past objects has been investigated extensively. The governing equations of creeping flow are obtained when the Navier–Stokes equations are non-dimensionalized and the inertial terms are discarded so that only the linear-steady Stokes equations remain. These equations are an appropriate model for flow of an incompressible viscous fluid when the Reynolds number is small; i.e. viscous effects dominate inertial ones and there are no oscillatory mechanisms within the flow. In particular, they have been used widely to model the swimming of small creatures; see, e.g. [Lighthill \(1975, Chapter 3\)](#) and [Lauga & Powers \(2009\)](#).

If oscillatory mechanisms exist within the flow, then the convective terms within the Navier–Stokes equations may be discarded but the time derivative of the velocity remains; this results in the unsteady Stokes equations. Oscillatory flows past objects have been explored previously; see, e.g., [Tuck \(1969\)](#), [Pozrikidis \(1989\)](#) and [Avudainayagam & Geetha \(1998\)](#). One important application is atomic force microscopy, a method whereby a tiny oscillating cantilever detects surface characteristics at very small scales ([Green & Sader, 2002](#); [Clarke \*et al.\*, 2005](#); [Tung \*et al.\*, 2008](#)).

The unsteady Stokes equations can be reduced exactly to the Brinkman equations, which model steady flow through porous media with high porosity ([Brinkman, 1947](#); [Childress, 1972](#); [Durlafsky & Brady, 1987](#); [Auriault, 2009](#); [Cortez \*et al.\*, 2010](#)). In non-dimensional forms, the difference between the unsteady Stokes and Brinkman equations is one parameter; the unsteady Stokes parameter is imaginary and characterizes frequency of oscillations whereas the real-valued Brinkman parameter inversely relates to the permeability of the material through which fluid is flowing. In this study, we denote the Brinkman parameter by  $\alpha$  and define it in (4.2) below. The Brinkman equation has been used to model various biological flows, e.g. flows through the endothelial surface layer ([Damiano \*et al.\*, 1996](#); [Leiderman \*et al.\*, 2008](#)), biofilms ([Kapellos \*et al.\*, 2007](#)), blood clots ([Leiderman & Fogelson, 2011, 2013](#)) and flagellar motion in gels ([Leiderman & Olson, 2016](#)).

The steady Stokes equations can be recovered exactly from the Brinkman equations by setting  $\alpha = 0$ . In addition, for 3D problems, solutions can be expanded in powers of  $\alpha$  so that Brinkman solutions approach the corresponding Stokes solutions as  $\alpha \rightarrow 0$ . Interestingly, this is not the case for 2D problems. In this study, we are interested in what happens in 2D when  $\alpha$  is small. For 2D problems, the situation is more complicated because of the presence of logarithms. Logarithms are expected because of the famous Stokes paradox: uniform Stokes flow past a 2D object induces a ‘perturbation’ to the flow that grows as  $\log r$  with  $r$ , where  $r$  is distance from the object. Logarithms arise in Brinkman flows too, but there are additional terms involving  $\log \alpha$ .

Similar situations arise in 2D acoustic scattering, where the governing equation is the Helmholtz equation,  $(\nabla^2 + k^2)u = 0$ , and the parameter  $k$  is small; for an overview, see [Martin \(2006\)](#), Section 8.2. We exploit this insight so as to investigate the small- $\alpha$  behaviour of Brinkman flows. The results are also applicable to low-frequency, oscillatory Stokes flows.

An outline of the paper is as follows: Sections 2 and 3 are concerned with 2D Stokes flows. Much of this material is known, but we need to collect these results so that we can compare with analogous 2D Brinkman flows. In particular, we recall the solution for 2D Stokes flow past a circle (Section 2.1) and past a thin flat strip (Section 3.2). We also give a method for reducing the problem of Stokes flow past a thin curved strip to a system of integral equations (Section 3.1), and we suggest a numerical method for solving these integral equations (Appendix A); as far as we know, this approach has not been used for Stokes flow, and it will be the subject of future investigation. The study of Stokes flows in the presence of thin objects has a long history going back to a paper by [Taylor \(1951\)](#); see also [Lauga & Powers \(2009\)](#) and [Montenegro-Johnson & Lauga \(2014\)](#).

Sections 4 and 5 are concerned with 2D Brinkman flows. Section 4 parallels Section 2: the governing equations for Brinkman flows are stated, together with their connection to oscillatory Stokes flows, and associated fundamental solutions (Brinkmanlets) are introduced. Then we recall the exact solution for 2D Brinkman flow past a circle (Section 4.4). This solution is compared with the exact solution for 2D Stokes flow past a circle (Section 2.1). We show that if the parameter  $\alpha$  is fixed and non-zero, then there is no Stokes-like paradox; i.e. there is no logarithmic growth with  $r$ . However, the limit  $\alpha \rightarrow 0$  for fixed  $r$  reveals complications; some of these had been noted previously by [Smith \(1997\)](#) in the context of oscillatory Stokes flows. In particular, we show that the Stokes solution is not obtained as  $\alpha \rightarrow 0$ .

We then consider the problem of 2D Brinkman flow past a thin flat strip (Section 5). This problem is more difficult. Although it cannot be solved exactly, an asymptotic approximation for small  $\alpha$  is obtained. Again, it is shown that there is no large- $r$  Stokes-like paradox but the Brinkman flow does not approach the corresponding Stokes flow (Section 3.2) as  $\alpha \rightarrow 0$ .

To put these results for small  $\alpha$  into perspective, let us recall some known facts about the Helmholtz equation,  $(\nabla^2 + k^2)u = 0$ , which becomes Laplace’s equation,  $\nabla^2 u_0 = 0$ , when  $k = 0$ . In 3D, the simplest singular solutions of these equations are  $u = r^{-1} \cos kr$  and  $u_0 = r^{-1}$ . Evidently,  $u \rightarrow u_0$  as  $k \rightarrow 0$  for fixed  $r > 0$ , and  $u = O(r^{-1})$  as  $r \rightarrow \infty$  for fixed  $k > 0$ , in agreement with  $u_0$ . In 2D, the corresponding solutions are  $u = Y_0(kr)$  and  $u_0 = \log r$ , where  $Y_0$  is a Bessel function. As  $Y_0(x) \sim (2/\pi) \log x$  as  $x \rightarrow 0$  and  $Y_0(x) = O(x^{-1/2})$  as  $x \rightarrow \infty$ , we see that  $u$  diverges logarithmically as  $k \rightarrow 0$  for fixed  $r$  whereas  $u$  decays as  $r \rightarrow \infty$  for fixed  $k$ . This peculiar behaviour with the 2D Helmholtz and Laplace equations is exactly what we see when we compare 2D Stokes flows with 2D Brinkman flows. Thus 2D Brinkman flows are regularizing in the sense that they do not exhibit logarithmic (Stokes-like) growth as  $r \rightarrow \infty$ . However, such flows do not reduce to the corresponding Stokes flows as  $\alpha \rightarrow 0$ . Additional concluding remarks are in Section 6.

## 2. Stokes flows

We consider slow flow of an incompressible viscous fluid in two dimensions (2D). In the absence of body forces, the governing equations are (Pozrikidis, 1992, Section 1.1)

$$\rho \frac{\partial u_i}{\partial t} = \mu \nabla^2 u_i - \frac{\partial p}{\partial x_i}, \quad i = 1, 2, \quad \text{with} \quad \frac{\partial u_j}{\partial x_j} = 0, \quad (2.1)$$

where  $\mathbf{u} = (u_1, u_2)$  is the velocity,  $\rho$  is the density,  $p$  is the pressure,  $\mu$  is the viscosity coefficient and (as in the last equation) repeated subscripts are summed over 1 and 2. For steady flows, there is no dependence on time and (2.1) reduces to

$$\nabla^2 u_i - \frac{1}{\mu} \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2, \quad \text{with} \quad \frac{\partial u_j}{\partial x_j} = 0. \quad (2.2)$$

Taking the divergence of (2.2)<sub>1</sub>, making use of (2.2)<sub>2</sub>, shows that  $\nabla^2 p = 0$ . The corresponding stresses are given by

$$\sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.3)$$

The net force acting on an object with boundary curve  $\mathcal{C}$  is  $\mathbf{F}$  with components

$$F_i = \int_{\mathcal{C}} \sigma_{ij} n_j \, ds, \quad (2.4)$$

where  $\mathbf{n}$  is the unit normal vector on  $\mathcal{C}$  pointing into the fluid.

A fundamental solution of (2.2) is (see, e.g. Pozrikidis, 1992, Section 2.6, or Hsiao & Wendland, 2008, Section 2.3)

$$G_{ij}^0(\mathbf{x}; \mathbf{x}') = \delta_{ij} \log \frac{R}{L} - \frac{R_i R_j}{R^2}, \quad P_j(\mathbf{x}; \mathbf{x}') = -\frac{2R_j}{R^2}, \quad T_{ikj}^0(\mathbf{x}; \mathbf{x}') = \frac{4}{R^4} R_i R_j R_k, \quad (2.5)$$

where  $R_i = x_i - x'_i$ ,  $R^2 = R_i R_i$  and  $L$  is a length scale. These formulas mean that, for  $j = 1$  and  $j = 2$ ,  $u_i = U G_{ij}^0$  and  $p = \mu U P_j$  solve (2.2) for  $R \neq 0$ , where  $U$  is a velocity scale. The corresponding stresses at  $\mathbf{x}$  are  $\sigma_{ik} = \mu U T_{ikj}^0$ . The solution  $G_{ij}^0(\mathbf{x}; \mathbf{x}')$  is known as a (2D) ‘Stokeslet’.

For another singular solution, define

$$D_{ij}^0(\mathbf{x}; \mathbf{x}') = \nabla^2 G_{ij}^0 = -2 \frac{\delta_{ij}}{R^2} + 4 \frac{R_i R_j}{R^4}. \quad (2.6)$$

This is similar to  $G_{ij}^0$ ; both have the form  $f(R)\delta_{ij} + g(R)R_i R_j$ . The fields  $u_i = U D_{ij}^0$  and  $p = 0$  solve the Stokes equations (2.2). They define a so-called ‘potential dipole’ (Pozrikidis, 1992, p. 196). The similarity between (2.5) and (2.6) means that the problem of uniform Stokes flow past a circular cylinder can be solved readily.

### 2.1 Uniform Stokes flow past a circular cylinder and the Stokes paradox

Consider a circular cylinder; its cross-section has radius  $a$  and is centred at the origin. There is a uniform ambient flow with constant velocity  $\mathbf{U}$  and constant pressure  $p^\infty$ , and a no-slip boundary condition,  $\mathbf{u} + \mathbf{U} = \mathbf{0}$  on the cylinder, where  $\mathbf{u}$  is the perturbation caused by the presence of the cylinder.

To find  $\mathbf{u}$ , we place a Stokeslet and a potential dipole at the origin, giving

$$u_i(\mathbf{x}) = U G_{ij}^0(\mathbf{x}; \mathbf{0}) f_j^0 + U L^2 D_{ij}^0(\mathbf{x}; \mathbf{0}) g_j^0, \quad p(\mathbf{x}) = \mu U P_j(\mathbf{x}; \mathbf{0}) f_j^0, \quad (2.7)$$

where  $\mathbf{f}_0 = (f_1^0, f_2^0)$  and  $\mathbf{g}_0 = (g_1^0, g_2^0)$  are constant vectors to be found. Take  $L = a$  in (2.5). On the cylinder,  $R = a$ ,  $\mathbf{x} = a\hat{\mathbf{x}}$  (where  $\hat{\mathbf{x}}$  is a unit vector) and  $\mathbf{u} + \mathbf{U} = \mathbf{0}$ . These give

$$\mathbf{U}/U = 2\mathbf{g}_0 + \{(\mathbf{f}_0 - 4\mathbf{g}_0) \cdot \hat{\mathbf{x}}\} \hat{\mathbf{x}}. \quad (2.8)$$

We can satisfy (2.8) by taking  $\mathbf{f}_0 = 4\mathbf{g}_0$  and  $2U\mathbf{g}_0 = \mathbf{U}$ . Then  $\mathbf{u}$  and  $p$  are given by (2.7) as

$$\mathbf{u}(\mathbf{x}) = 2\mathbf{U} \log(r/a) - \mathbf{U}(a/r)^2 - 2\hat{\mathbf{x}}(\mathbf{U} \cdot \hat{\mathbf{x}})(1 - [a/r]^2), \quad p(\mathbf{x}) = -4(\mu/r)(\mathbf{U} \cdot \hat{\mathbf{x}}). \quad (2.9)$$

Computing the stresses from (2.3), we find that  $\sigma_{ij} = 4(\mu/a)\{U_i\hat{x}_j + U_j\hat{x}_i + (\delta_{ij} - 2\hat{x}_i\hat{x}_j)(\mathbf{U} \cdot \hat{\mathbf{x}})\}$  on  $r = a$ , so that the net force (see (2.4)) is  $\mathbf{F} = 8\pi\mu\mathbf{U}$ .

We conclude that (because of the presence of  $G_{ij}^0$  in (2.7)) the ‘perturbation’  $\mathbf{u}$  grows logarithmically with distance from the cylinder: this exemplifies the ‘Stokes paradox’. For a good discussion, see Childress (2009, Section 7.4). In detail, from (2.9), we obtain  $p(\mathbf{x}) = O(r^{-1})$  and

$$\mathbf{u}(\mathbf{x}) = 2\mathbf{U} \log(r/a) - 2\hat{\mathbf{x}}(\mathbf{U} \cdot \hat{\mathbf{x}}) + O(r^{-2}) \quad \text{as } r = |\mathbf{x}| \rightarrow \infty.$$

We shall return to the problem of uniform flow past a circular cylinder at the end of Section 2.2.

### 2.2 Summary of theoretical results

In this section, we recall some known results for Stokes flow past an object with cross-sectional boundary curve  $\mathcal{C}$ . In particular, how should a well-posed boundary-value problem be formulated? We shall clarify the far-field behaviour and recall that the magnitude of the logarithmic term is proportional to the net force on  $\mathcal{C}$ .

Let  $\mathcal{C}_\infty$  be a large circle that encloses  $\mathcal{C}$  and let  $\mathbf{u}$  be the velocity of a Stokes flow. An application of the Lorentz reciprocal theorem (from 1896, translated as Lorentz, 1996) in the region between  $\mathcal{C}$  and  $\mathcal{C}_\infty$  yields (Pozrikidis, 1992, Equation (2.6.24))

$$u_i(\mathbf{x}') = \frac{1}{4\pi} \int_{\mathcal{C}} \Omega_i^0(\mathbf{x}; \mathbf{x}') \, ds(\mathbf{x}) + \frac{1}{4\pi} \int_{\mathcal{C}_\infty} \Omega_i^0(\mathbf{x}; \mathbf{x}') \, ds(\mathbf{x}), \quad (2.10)$$

where  $\mathbf{x}'$  locates a point between  $\mathcal{C}$  and  $\mathcal{C}_\infty$ ,  $\Omega_i^0(\mathbf{x}; \mathbf{x}') = \mu^{-1} t_j(\mathbf{x}) G_{ji}^0(\mathbf{x}; \mathbf{x}') - u_j(\mathbf{x}) T_{ji}^0(\mathbf{x}; \mathbf{x}') n_k(\mathbf{x})$  and  $t_i = \sigma_{ij} n_j$  is the traction with  $\mathbf{n}$  pointing into the fluid on  $\mathcal{C}$  and inwards (towards  $\mathcal{C}$ ) on  $\mathcal{C}_\infty$ .

The term on the left-hand side of (2.10) comes from excising a small disc centred at  $\mathbf{x}'$  followed by letting this disc shrink to  $\mathbf{x}'$ . The second term on the right-hand side of (2.10) does not depend on the

radius of  $\mathcal{C}_\infty$ , and it defines a valid velocity field everywhere inside  $\mathcal{C}_\infty$  (including inside  $\mathcal{C}$ ); denote it by  $u_i^\infty(\mathbf{x}')$ .

Let  $\mathbf{u}^l(\mathbf{x})$  be the (total) velocity of a Stokes flow and let  $p^l(\mathbf{x})$  be the corresponding pressure. Using (2.10) for  $\mathbf{u}^l$  gives

$$u_i^l(\mathbf{x}') = u_i^\infty(\mathbf{x}') + \frac{1}{4\pi\mu} \int_{\mathcal{C}} t_j^l(\mathbf{x}) G_{ji}^0(\mathbf{x}; \mathbf{x}') \, ds(\mathbf{x}) - \frac{1}{4\pi} \int_{\mathcal{C}} u_j^l(\mathbf{x}) T_{jki}^0(\mathbf{x}; \mathbf{x}') n_k(\mathbf{x}) \, ds(\mathbf{x}); \quad (2.11)$$

we can regard  $\mathbf{u}^\infty(\mathbf{x})$  as being the ambient (or incident) flow in the absence of the object.

In the far field, (2.5) gives

$$\mathbf{u}^l(\mathbf{x}) - \mathbf{u}^\infty(\mathbf{x}) = (4\pi\mu)^{-1} \{ \mathbf{F} \log(r/L) - \hat{\mathbf{x}}(\hat{\mathbf{x}} \cdot \mathbf{F}) \} + O(r^{-1}) \quad \text{as } r = |\mathbf{x}| \rightarrow \infty, \quad (2.12)$$

where  $\mathbf{F} = \int_{\mathcal{C}} \mathbf{t} \, ds = \int_{\mathcal{C}} \mathbf{t} \, ds$  is the net force on  $\mathcal{C}$  and  $\hat{\mathbf{x}} = \mathbf{x}/r$ . (The ambient flow does not contribute to  $\mathbf{F}$  because it solves the Stokes equations inside  $\mathcal{C}$ .) For the pressure, (2.2) gives

$$p^l(\mathbf{x}) - p^\infty(\mathbf{x}) = -(2\pi r)^{-1} (\hat{\mathbf{x}} \cdot \mathbf{F}) + O(r^{-2}) \quad \text{as } r \rightarrow \infty, \quad (2.13)$$

where  $p^\infty$  is the pressure corresponding to  $\mathbf{u}^\infty$ . We see that the logarithmic term in (2.12) vanishes when the net force on  $\mathcal{C}$  is zero.

Chang & Finn (1961, Theorem 1) have proved that if  $\mathbf{u}^l$  is required to grow sublinearly,  $|\mathbf{u}^l(\mathbf{x})| = o(r)$  as  $r \rightarrow \infty$ , then  $\mathbf{u}^\infty$  is a constant vector, whence  $p^\infty$  is a constant.

Suppose that

$$\mathbf{u}^l(\mathbf{x}) = \mathbf{u}^\infty(\mathbf{x}) + \mathbf{u}(\mathbf{x}), \quad p^l(\mathbf{x}) = p^\infty(\mathbf{x}) + p(\mathbf{x}), \quad (2.14)$$

where  $(\mathbf{u}^\infty, p^\infty)$  is a valid Stokes flow. Suppose that  $\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$  on  $\mathcal{C}$ , with  $\mathbf{u}(\mathbf{x}) = \mathbf{A} \log r + O(1)$  and  $p(\mathbf{x}) = o(1)$  as  $r \rightarrow \infty$ , where  $\mathbf{A}$  is a 'given' constant vector. (This is the Dirichlet or resistance problem.) Then the Stokes problem has exactly one solution  $(\mathbf{u}, p)$  (Hsiao & Kress, 1985, Corollary 1); from (2.12),  $\mathbf{A} = (4\pi\mu)^{-1} \mathbf{F}$ . For the boundary condition of prescribed traction on  $\mathcal{C}$  (Neumann or mobility problem), see Hsiao & Wendland (2008, Section 2.3.2). For information on the use of boundary integral equations for various 2D Stokes-flow problems, see Pozrikidis (1992), Capuani *et al.* (2005), Rachh & Greengard (2016) and references therein.

Let us return to the problem of uniform flow past a circular cylinder (Section 2.1). According to the result of Hsiao & Kress (1985), we can specify the constant vector  $\mathbf{A}$  in the far-field behaviour ( $\mathbf{u} \sim \mathbf{A} \log r$  as  $r \rightarrow \infty$ ) whereas we did not do that when we arrived at the Solution (2.9) (in which  $\mathbf{A} = 2\mathbf{U}$ ). Indeed, if we add  $C\mathbf{U}_i$  to the formula for  $u_i$  in (2.7) (where  $C$  is an arbitrary constant), the no-slip boundary conditions gives  $\mathbf{f}_0 = 4\mathbf{g}_0$  (as before),  $2\mathbf{U}\mathbf{g}_0 = (1 + C)\mathbf{U}$  and  $\mathbf{u} \sim 2(1 + C)\mathbf{U} \log r$  as  $r \rightarrow \infty$ .

### 3. Stokes flow past a thin object

In this section, we consider 2D Stokes flow past thin plates on which there is a no-slip boundary condition. This gives another example of the Stokes paradox, one that we shall compare with a corresponding 2D Brinkman flow in Section 5.

### 3.1 Stokes flow past a curved plate

Let  $\Gamma$  be an open smooth curve, parametrized as follows:

$$\Gamma = \{\mathbf{x} = (x_1, x_2) : x_1 = x_1(\xi), x_2 = x_2(\xi), -1 \leq \xi \leq 1\}. \quad (3.1)$$

There is an ambient flow past  $\Gamma$ , with velocity  $\mathbf{u}^\infty(\mathbf{x}) = (U_1, U_2)$  and pressure  $p^\infty(\mathbf{x})$ . There is a no-slip boundary condition,  $\mathbf{u}^\dagger = \mathbf{u}^\infty + \mathbf{u} = \mathbf{0}$  on  $\Gamma$ , where  $\mathbf{u}$  is caused by the presence of  $\Gamma$ .

We can represent  $\mathbf{u}$  using  $G_{ij}^0$ ,

$$u_i(\mathbf{x}) = \frac{U}{L} \int_{\Gamma} G_{ij}^0(\mathbf{x}; \mathbf{x}') h_j(\mathbf{x}') \, ds(\mathbf{x}'), \quad i = 1, 2, \quad (3.2)$$

where  $h_1$  and  $h_2$  are (dimensionless) functions to be found. This is a single-layer representation for  $\mathbf{u}$ . Such a representation is continuous as  $\mathbf{x} \rightarrow \Gamma$  whence the boundary condition gives

$$\frac{U}{L} \int_{\Gamma} G_{ij}^0(\mathbf{x}; \mathbf{x}') h_j(\mathbf{x}') \, ds(\mathbf{x}') = -U_i(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad i = 1, 2, \quad (3.3)$$

which is a pair of integral equations for  $h_1$  and  $h_2$ . Inserting the parametrization (3.1) will lead to integral equations over the interval  $[-1, 1]$ ; see Appendix A for details.

Far from  $\Gamma$ , (3.2) gives

$$\mathbf{u}(\mathbf{x}) \sim U\mathbf{H} \log r \quad \text{as } r \rightarrow \infty, \quad \text{where } \mathbf{H} = \frac{1}{L} \int_{\Gamma} \mathbf{h}(\mathbf{x}') \, ds(\mathbf{x}')$$

and  $\mathbf{h} = (h_1, h_2)$ . This is another example of the Stokes paradox.

### 3.2 Uniform Stokes flow past a flat strip

Let us specialize the approach in Section 3.1 to a thin flat plate. As we shall see, the resulting integral equations can be solved exactly.

Thus, we consider a flat strip  $\Gamma$  along the  $x_1$ -axis. For a strip of length  $2a$ , we can take  $x_1(\xi) = a\xi$  and  $x_2(\xi) = 0$ . Locate  $\mathbf{x}'$  on  $\Gamma$  using the parameter  $\xi'$ . For simplicity, we take a uniform ambient flow,  $\mathbf{u}^\infty = \mathbf{U} = U(\cos \psi, \sin \psi)$ , which makes an angle  $\psi$  with the positive  $x_1$ -axis. Take  $L = a$  and write  $h_j(\mathbf{x}') = h_j(\xi')$ . Using  $ds = a \, d\xi'$ ,  $R_i = a(\xi - \xi')\delta_{i1}$  and  $R = a|\xi - \xi'|$ , the system (3.3) decouples into

$$\int_{-1}^1 h_1(\xi') \{ \log |\xi - \xi'| - 1 \} \, d\xi' = -\cos \psi, \quad -1 < \xi < 1, \quad (3.4)$$

$$\int_{-1}^1 h_2(\xi') \log |\xi - \xi'| \, d\xi' = -\sin \psi, \quad -1 < \xi < 1. \quad (3.5)$$

A closed form solution of these integral equations will be given in Section 3.2.2, but first we will give the solution of a more general problem.

3.2.1 *A classical integral equation.* Let us solve a slightly more general integral equation,

$$\int_{-1}^1 f(\xi') \log |\xi - \xi'| d\xi' = q(\xi), \quad -1 < \xi < 1, \quad (3.6)$$

where  $q$  is given and  $f$  is to be found. To solve (3.6), we expand both  $f$  and  $q$  using Chebyshev polynomials,  $T_n(x)$ ,

$$f(\xi) = \frac{1}{\sqrt{1-\xi^2}} \sum_{n=0}^{\infty} c_n T_n(\xi), \quad q(\xi) = \sum_{m=0}^{\infty} q_m T_m(\xi). \quad (3.7)$$

By definition,  $T_n(\cos \vartheta) = \cos n\vartheta$ , and we have orthogonality,

$$\int_{-1}^1 T_n(\xi') T_m(\xi') \frac{d\xi'}{\sqrt{1-\xi'^2}} = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0. \end{cases} \quad (3.8)$$

However, the main reason for using  $T_n$  is the expansion

$$\log |\xi - \xi'| = -\log 2 - \sum_{n=1}^{\infty} \frac{2}{n} T_n(\xi) T_n(\xi'). \quad (3.9)$$

Substituting these expansions in (3.6), and making use of orthogonality gives

$$-\pi c_0 \log 2 = q_0, \quad -(\pi/n) c_n = q_n, \quad n = 1, 2, \dots \quad (3.10)$$

These formulas give the exact solution of (3.6). In most of our applications, the function on the right-hand side,  $q(\xi)$ , will be constant, and so we shall only be interested then in the coefficient  $c_0$ .

3.2.2 *Application to integral equations (3.4) and (3.5).* Equation (3.5) is solved immediately: as  $T_0(\xi) = 1$ , we obtain

$$h_2(\xi) = c_0^{(2)} (1 - \xi^2)^{-1/2} \quad \text{with} \quad \pi c_0^{(2)} \log 2 = \sin \psi. \quad (3.11)$$

For (3.4), define  $H_1 = \int_{-1}^1 h_1(\xi) d\xi$  and write (3.4) as

$$\int_{-1}^1 h_1(\xi') \log |\xi - \xi'| d\xi' = H_1 - \cos \psi, \quad -1 < \xi < 1.$$

Solving this equation gives

$$h_1(\xi) = c_0^{(1)} (1 - \xi^2)^{-1/2} \quad \text{with} \quad -\pi c_0^{(1)} \log 2 = H_1 - \cos \psi. \quad (3.12)$$

Integrating the first of these gives

$$H_1 = c_0^{(1)} \int_{-1}^1 \frac{d\xi}{\sqrt{1-\xi^2}} = \pi c_0^{(1)}.$$

Substituting this in the second of (3.12) then yields

$$\pi c_0^{(1)} (1 + \log 2) = \cos \psi, \quad (3.13)$$

and this gives  $h_1(\xi)$ . Now that  $h_1$  and  $h_2$  have been found,  $\mathbf{u}$  is given by (3.2).

#### 4. Brinkman flows

The Brinkman equations were originally intended to model slow flow of an incompressible viscous fluid through a porous medium (Brinkman, 1947) but they have been used in other contexts (see Leiderman & Olson, 2016, for references). The relevant equations are as for Stokes flow but with an additional frictional resistance term due to the porous medium. For steady 2D flow in the absence of body forces, the governing equations are

$$\nabla^2 u_i - k^2 u_i - \frac{1}{\mu} \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2, \quad \text{with} \quad \frac{\partial u_j}{\partial x_j} = 0, \quad (4.1)$$

where  $k^2 = K_D^{-1}$  and the constant  $K_D$  is the Darcy permeability of the porous medium. Evidently, (4.1) reduces to the Stokes equations (2.2) when  $k = 0$ . We also have  $\nabla^2 p = 0$ . Using a length scale  $L$ , we can define a (real and positive) dimensionless permeability factor

$$\alpha = kL = L/\sqrt{K_D}. \quad (4.2)$$

Formally, the Brinkman equations (4.1) are also obtained if solutions of the ‘unsteady’ Stokes equations (2.1) are sought that are proportional to  $e^{st}$  (Pozrikidis, 1992, p. 17). In this context,  $s$  could be a Laplace-transform variable and  $k = (\rho s/\mu)^{1/2}$ . For oscillatory Stokes flows, we can take  $s = -i\omega$ , where  $\omega$  is the frequency. Then  $k = (-i\rho\omega/\mu)^{1/2}$  so that the corresponding value of  $\alpha$  is complex for such flows.

In this section, we set up the basic equations for 2D Brinkman flows, culminating in an analysis of flow past a circular cylinder. Flow past a thin strip is discussed in Section 5.

##### 4.1 Ambient Brinkman flows

For Stokes flow, we considered examples with a uniform ambient flow, meaning that the velocity is a constant vector  $\mathbf{U}$  and the pressure is a constant  $p^\infty$ . For comparable Brinkman flows, we can take a constant velocity  $\mathbf{U}$  but, from (4.1), the pressure is  $p^\infty(\mathbf{x}) = -\mu k^2 \mathbf{U} \cdot \mathbf{x}$  plus an arbitrary constant. Thus, this ambient flow has a large pressure at infinity. As an alternative, we could suppose that  $p^\infty$  is a constant but then (4.1) gives  $\mathbf{u}^\infty(\mathbf{x}) = \mathbf{U} \exp(\mathbf{k} \cdot \mathbf{x})$  where  $\mathbf{U}$  and  $\mathbf{k}$  are constant vectors satisfying  $|\mathbf{k}| = k$  and  $\mathbf{U} \cdot \mathbf{k} = 0$ . This solution is not attractive because of the exponential growth in certain directions, and so we choose to compare with the first ambient flow: constant velocity and pressure growing linearly with  $r = |\mathbf{x}|$ .



#### 4.2 Fundamental solution

A fundamental solution of the Brinkman equations (4.1) is

$$G_{ij}(\mathbf{x}; \mathbf{x}') = \delta_{ij} A(kR) + \frac{R_i R_j}{R^2} B(kR), \quad P_j(\mathbf{x}; \mathbf{x}') = -\frac{2R_j}{R^2} \quad (4.3)$$

where

$$A(x) = -K_0(x) - K_2(x) + 2x^{-2}, \quad B(x) = 2K_2(x) - 4x^{-2} \quad (4.4)$$

and  $K_n(x)$  is a modified Bessel function. These formulas are well known; see Yano *et al.* (1991), Avudainayagam & Geetha (1993), Kohr *et al.* (2008) and references therein. For convenience, we have provided a derivation in Appendix B. Note that  $P_j$  is the same as with Stokes flow; see (2.5).

The corresponding stresses  $\sigma_{ik}$  at  $\mathbf{x}$  are  $\mu UT_{ikj}$  where

$$\begin{aligned} T_{i\ell j} = & 2\delta_{i\ell} \frac{R_j}{R^2} \{1 + B(kR)\} + \frac{2}{R^2} (\delta_{ij} R_\ell + \delta_{j\ell} R_i) \{kRK_1(kR) + B(kR)\} \\ & - \frac{4}{R^4} R_i R_j R_\ell \{kRK_1(kR) + 2B(kR)\}. \end{aligned} \quad (4.5)$$

For small  $x$ , we have

$$K_0(x) = -\log(x/2) - \gamma + o(1) \quad \text{and} \quad K_2(x) = 2x^{-2} - \frac{1}{2} + o(1) \quad \text{as } x \rightarrow 0, \quad (4.6)$$

where  $\gamma \simeq 0.5772$  is Euler's constant. Hence,

$$A(x) = \log x - \log 2 + \gamma + \frac{1}{2} + o(1) \quad \text{and} \quad B(x) = -1 + o(1) \quad \text{as } x \rightarrow 0. \quad (4.7)$$

As we require  $A(kR)$  in (4.3), we see that  $G_{ij} \not\rightarrow G_{ij}^0$  as  $kL \rightarrow 0$ ; in detail,

$$G_{ij} - G_{ij}^0 \sim E(kL) \delta_{ij} \quad \text{as } kL \rightarrow 0, \text{ with } E(kL) = \log(kL/2) + \gamma + \frac{1}{2}. \quad (4.8)$$

A similar difficulty occurs when solving 2D acoustic scattering problems, governed by the Helmholtz equation. On the other hand, comparing (2.5) and (4.5), and using  $xK_1(x) \sim 1$  as  $x \rightarrow 0$ , we find that  $T_{i\ell j} \rightarrow T_{i\ell j}^0$  as  $kL \rightarrow 0$ .

#### 4.3 A potential dipole

For uniform Brinkman flow past a circular cylinder (Section 4.4), we shall need a potential dipole, defined by  $D_{ij} = \nabla^2 G_{ij}$ . From (B.3) and (B.4),

$$\begin{aligned} D_{ij}(\mathbf{x}; \mathbf{x}') &= \left(k^2 A + \frac{P}{R^2}\right) \delta_{ij} + \frac{R_i R_j}{R^2} \left(k^2 B + \frac{kP'}{R} - \frac{2P}{R^2}\right) \\ &= -k^2 (K_0(kR) + K_2(kR)) \delta_{ij} + 2k^2 R_i R_j R^{-2} K_2(kR). \end{aligned} \quad (4.9)$$

We note that  $D_{ij} \rightarrow D_{ij}^0$  (defined by (2.6)) as  $kL \rightarrow 0$ , even though  $G_{ij} \not\rightarrow G_{ij}^0$  as  $kL \rightarrow 0$ .

#### 4.4 Uniform Brinkman flow past a circular cylinder

For uniform Brinkman flow past a circular cylinder, we can proceed as in Section 2.1 and write (see also Leiderman & Olson, 2016, Section III.A.1)

$$u_i(\mathbf{x}) = UG_{ij}(\mathbf{x}; \mathbf{0})f_j + Ua^2D_{ij}(\mathbf{x}; \mathbf{0})g_j, \quad p(\mathbf{x}) = \mu UP_j(\mathbf{x}; \mathbf{0})f_j, \quad (4.10)$$

where  $\mathbf{f} = (f_1, f_2)$  and  $\mathbf{g} = (g_1, g_2)$  are constant vectors to be found so that  $\mathbf{u} + \mathbf{U} = \mathbf{0}$  on the circle. Applying this boundary condition gives

$$\begin{aligned} \mathbf{U}/U = & \{K_0(\alpha) + K_2(\alpha) - 2\alpha^{-2}\}\mathbf{f} + \alpha^2\{K_0(\alpha) + K_2(\alpha)\}\mathbf{g} \\ & - 2\{K_2(\alpha) - 2\alpha^{-2}\}(\mathbf{f} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}} - 2\alpha^2K_2(\alpha)(\mathbf{g} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}, \end{aligned}$$

where  $\alpha = ka$ . We can remove the last two terms by choosing

$$\alpha^2K_2(\alpha)\mathbf{g} = \{2\alpha^{-2} - K_2(\alpha)\}\mathbf{f}.$$

Then, the remaining terms give

$$U\mathbf{f} = \frac{\alpha^2K_2(\alpha)}{2K_0(\alpha)}\mathbf{U} \quad \text{and} \quad U\mathbf{g} = \frac{2\alpha^{-2} - K_2(\alpha)}{2K_0(\alpha)}\mathbf{U}. \quad (4.11)$$

Substitution in (4.10) gives the velocity everywhere outside the cylinder:

$$\mathbf{u}(\mathbf{x}) = \frac{\mathbf{U}}{K_0(\alpha)} \left\{ \frac{a^2}{r^2}K_2(\alpha) - K_2(kr) - K_0(kr) \right\} + \frac{2\hat{\mathbf{x}}}{K_0(\alpha)}(\mathbf{U} \cdot \hat{\mathbf{x}}) \left\{ K_2(kr) - \frac{a^2}{r^2}K_2(\alpha) \right\}. \quad (4.12)$$

The pressure perturbation is found to be

$$p = -2\mu U \frac{\hat{\mathbf{x}} \cdot \mathbf{f}}{r} = -\mu \frac{\alpha^2K_2(\alpha)}{K_0(\alpha)} \frac{\mathbf{U} \cdot \hat{\mathbf{x}}}{r}. \quad (4.13)$$

The problem can also be solved by introducing a stream function; see Pop & Cheng (1992) and Smith (1997). We also mention H. A. Stone's appendix to Smith's paper.

If we fix  $\alpha$  and let  $r = |\mathbf{x}| \rightarrow \infty$ , noting that  $K_n(x)$  decays exponentially as  $x \rightarrow \infty$ , we find that  $\mathbf{u}$  is (algebraically) small at infinity and thus there is no analogue of the Stokes paradox. In detail,

$$\mathbf{u}(\mathbf{x}) \sim \frac{a^2K_2(\alpha)}{r^2K_0(\alpha)} \{ \mathbf{U} - 2(\mathbf{U} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}} \} \quad \text{as } r \rightarrow \infty. \quad (4.14)$$

Instead of fixing  $\alpha$  and letting  $r \rightarrow \infty$ , suppose we fix  $r/a$  and let  $\alpha \rightarrow 0$ . We have

$$U\mathbf{f} \sim -\frac{1}{\log \alpha}\mathbf{U} \quad \text{and} \quad U\mathbf{g} \sim -\frac{1}{4\log \alpha}\mathbf{U}.$$

(The relation  $\mathbf{f} \sim 4\mathbf{g}$  accords with  $\mathbf{f}_0 = 4\mathbf{g}_0$  as found for uniform Stokes flow past a circle; see below (2.8).) As  $kr = \alpha(r/a)$  is also small, we obtain  $G_{ij}(\mathbf{x}; \mathbf{0}) \sim \delta_{ij} \log \alpha$  and  $a^2 D_{ij}(\mathbf{x}; \mathbf{0}) = O(1)$ . It follows that  $\mathbf{u}(\mathbf{x}) \sim -\mathbf{U}$  as  $\alpha \rightarrow 0$ , for all finite values of  $r = |\mathbf{x}|$ . In detail, using (4.6) in (4.12),

$$\mathbf{u}(\mathbf{x}) \sim -\mathbf{U} \left\{ 1 + \frac{\log(r/a) + \frac{1}{2}(1 - [a/r]^2)}{\log \alpha} \right\} + \frac{\hat{\mathbf{x}}(\mathbf{U} \cdot \hat{\mathbf{x}})}{\log \alpha} \left( 1 - \frac{a^2}{r^2} \right) \quad (4.15)$$

as  $\alpha \rightarrow 0$  for fixed  $r/a$ . Evidently, this limiting flow is not the same as the Stokes flow found in Section 2.1; see (2.9). Indeed, this explicit calculation provides a counterexample to a result of MacCamy (1966, Theorem 5).

Smith (1997) made a detailed study of the small- $\alpha$  solution when  $r/a$  is fixed (in the context of oscillatory Stokes flow) and showed that it should be regarded as an inner solution to be matched to an appropriate outer solution.

The results for flow past a cylinder were obtained by exploiting the availability of an exact solution so that various asymptotic limits could be explored readily. Such is not the case for the problem of Brinkman flow past a strip, which we present in Section 5. First, we briefly review what is known about Brinkman flows past objects of other shapes.

#### 4.5 Summary of theoretical results

For the Brinkman equations, we can proceed exactly as in Section 2.2, arriving at the integral representation (2.11) but with  $G_{ij}^0$  and  $T_{ijk}^0$  replaced by  $G_{ij}$  and  $T_{ijk}$ , respectively. However, the far-field behaviour is very different: we obtain  $\mathbf{u}^l(\mathbf{x}) = \mathbf{u}^\infty(\mathbf{x}) + O(r^{-2})$  as  $r \rightarrow \infty$ , which should be compared with (2.12). Thus the presence of  $\mathcal{C}$  causes a small perturbation to the velocity field far away from  $\mathcal{C}$ . For the pressure, we obtain  $p^l(\mathbf{x}) = p^\infty(\mathbf{x}) + O(r^{-1})$ , which has the same form as for Stokes flow, (2.13). Note that, in these estimates,  $(\mathbf{u}^\infty, p^\infty)$  is assumed to be a valid solution of the Brinkman equations.

If we define  $\mathbf{u}$  and  $p$  by (2.14), and require that  $\mathbf{u}(\mathbf{x}) = O(r^{-2})$  and  $p(\mathbf{x}) = O(r^{-1})$  as  $r \rightarrow \infty$ , then the standard boundary-value problems (velocity or traction prescribed on  $\mathcal{C}$ ) have exactly one solution Varnhorn (2004). Note that the far-field conditions on  $\mathbf{u}$  and  $p$  are satisfied by the representations (4.10) used to solve the Brinkman flow problem in Section 4.4.

### 5. Uniform Brinkman flow past a flat strip

For Brinkman flow past a thin object, we can proceed as in Section 3 but with  $G_{ij}$  instead of  $G_{ij}^0$  in the single-layer representation (3.2) and in the integral equations (3.3). Thus, we write

$$u_i(\mathbf{x}) = \frac{U}{L} \int_{\Gamma} G_{ij}(\mathbf{x}; \mathbf{x}') h_j(\mathbf{x}') ds(\mathbf{x}'), \quad i = 1, 2, \quad (5.1)$$

and then putting  $\mathbf{x}$  on  $\Gamma$  gives an integral equation for  $\mathbf{h} = (h_1, h_2)$ . Far from  $\Gamma$ , (4.3) and (5.1) give  $\mathbf{u}(\mathbf{x}) = O(r^{-2})$  as  $r \rightarrow \infty$ .

As in Section 3.2, we consider a uniform flow past a flat strip  $\Gamma$ , lying along the  $x_1$ -axis. We parametrize as before, with  $x_1(\xi) = a\xi$  and  $x_2(\xi) = 0$  for  $-1 < \xi < 1$ . We take  $L = a$  and write  $h_j(\mathbf{x}') = h_j(\xi')$ . Using  $ds = a d\xi'$ ,  $R_i = a(\xi - \xi')\delta_{i1}$ ,  $R = a|\xi - \xi'|$  and (4.3), the system (3.3) decouples

into the following integral equations:

$$\int_{-1}^1 h_1(\xi') \{A(ka|\xi - \xi'|) + B(ka|\xi - \xi'|)\} d\xi' = -\cos \psi, \quad -1 < \xi < 1, \quad (5.2)$$

$$\int_{-1}^1 h_2(\xi') A(ka|\xi - \xi'|) d\xi' = -\sin \psi, \quad -1 < \xi < 1, \quad (5.3)$$

with  $A$  and  $B$  given by (4.4). These integral equations should be compared with (3.4) and (3.5).

For normal incidence ( $\psi = \frac{1}{2}\pi$ ), only  $h_2$  is of interest. From (4.4), the kernel of (5.3) is given by

$$A(x) = 2K'_1(x) + 2x^{-2} = 2(K_1(x) - x^{-1})' = -2(K_0(x) - \log x)'';$$

this shows that (5.3) is the same as an integral equation derived and solved numerically by Tuck (1969); see his Equations (5.6) and (5.7). For later work along the same lines, see Van Eysden & Sader (2006) and references therein.

Suppose now that  $ka = \alpha$  is small. From (4.7),  $B \sim -1$  and

$$A(ka|\xi - \xi'|) \sim \log |\xi - \xi'| + E(ka) \quad \text{with} \quad E(ka) = \log(ka/2) + \gamma + \frac{1}{2}.$$

Then, as in Section 3.2.2, we are faced with solving

$$\int_{-1}^1 f(\xi') \{\log |\xi - \xi'| + \mathcal{A}\} d\xi' = \mathcal{B}, \quad -1 < \xi < 1,$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are given constants. With  $F = \int_{-1}^1 f(\xi) d\xi$ , we have

$$\int_{-1}^1 f(\xi') \log |\xi - \xi'| d\xi' = \mathcal{B} - F\mathcal{A}, \quad -1 < \xi < 1.$$

Thus,

$$f(\xi) = \mathcal{D} (1 - \xi^2)^{-1/2} \quad \text{with} \quad -\pi \mathcal{D} \log 2 = \mathcal{B} - F\mathcal{A}.$$

But integrating gives  $F = \pi \mathcal{D}$  whence the constant  $\mathcal{D}$  is determined:

$$\pi \mathcal{D} (\log 2 - \mathcal{A}) = -\mathcal{B}. \quad (5.4)$$

This method can be found in a paper by (Noble, 1962, Section 3), where it is used for acoustic scattering by a thin flat strip.

Returning to (5.2) and (5.3), we can write

$$h_j(\xi) \simeq \mathcal{D}_j (1 - \xi^2)^{-1/2}, \quad j = 1, 2, \quad (5.5)$$

to leading order in  $\alpha$ . The method described above gives

$$\pi \mathcal{D}_1(\log 2 - E + 1) = \cos \psi \quad \text{and} \quad \pi \mathcal{D}_2(\log 2 - E) = \sin \psi.$$

Thus  $\mathcal{D}_j \not\rightarrow c_0^{(j)}$  as  $ka \rightarrow 0$ , where  $c_0^{(j)}$  are the corresponding coefficients for Stokes flow past a strip, given (exactly) by (3.11) and (3.13). The far field is

$$u_i(\mathbf{x}) \sim \frac{2\pi U}{(kr)^2} \left( \delta_{ij} - 2\hat{x}_i \hat{x}_j \right) \mathcal{D}_j \quad \text{as } r \rightarrow \infty. \quad (5.6)$$

Alternatively, as in Section 4.4, let us fix  $r/a$  but let  $\alpha \rightarrow 0$ . Using  $E \sim \log \alpha$ , we have

$$\pi \mathcal{D}_1 \sim -(\log \alpha)^{-1} \cos \psi \quad \text{and} \quad \pi \mathcal{D}_2 \sim -(\log \alpha)^{-1} \sin \psi.$$

Then, from (5.1), (5.5) and  $G_{ij}(\mathbf{x}; \mathbf{x}') \sim \delta_{ij} \log \alpha$ ,

$$\mathbf{u}(\mathbf{x}) \sim U \int_{-1}^1 (\mathcal{D}_1, \mathcal{D}_2) \frac{\log \alpha \, d\xi}{\sqrt{1 - \xi^2}} \sim -\mathbf{U} \quad \text{as } \alpha \rightarrow 0.$$

This result means that the leading-order limiting Brinkman flow exactly cancels the ambient flow everywhere in the vicinity of the strip, not just on the strip itself. The same result was also obtained in Section 4.4 for Brinkman flow past a cylinder. Corrections, of order  $(\log \alpha)^{-1}$ , can be derived.

## 6. Discussion and conclusions

As in several 2D problems involving a small parameter (frequency  $\omega$  in oscillatory motions or permeability factor  $\alpha$  in Brinkman flows), the limit as the parameter tends to zero may not give the solution of the problem with the parameter equal to zero: examples are the Helmholtz equation which becomes Laplace's equation at zero frequency, and the Brinkman equations which become the steady Stokes equations when  $\alpha = 0$ . We have shown this to be the case for Brinkman flows (and for oscillatory Stokes flows). We did this by examining two specific problems, flow past a cylinder and flow past a thin straight strip. It is expected that a similar result can be proved for arbitrary geometries, using layer potentials and boundary integral equations (Hsiao & Wendland, 2008); our expectation is based on properties of the associated fundamental solutions, namely  $G_{ij} \not\rightarrow G_{ij}^0$  as  $\alpha \rightarrow 0$ .

In a recent paper, Ahmadi *et al.* (2017, p. 76) replace the Brinkmanlet  $G_{ij}$  by

$$G_{ij} - E(kL) \delta_{ij} = \tilde{G}_{ij},$$

so that, from (4.8) (where  $E$  is defined),  $\tilde{G}_{ij} \rightarrow G_{ij}^0$  as  $\alpha = kL \rightarrow 0$ . Then, from (4.1),  $P_j$  should be replaced by  $P_j + k^2 E(\alpha) R_j = \tilde{P}_j$ . Ahmadi *et al.* (2017, p. 76) state that the effect of using  $\tilde{G}_{ij}$  'is to produce a constant flow that can be subtracted'. The first part of this statement is correct (see Section 4.1, noting the linear growth of  $\tilde{P}_j$ ) but the second part is not: the flow to be subtracted becomes infinitely large as  $\alpha \rightarrow 0$  (for fixed  $r$ ). Indeed, the problems of Stokes flow and Brinkman flow past an object are well posed (with appropriate far-field conditions, see Sections 2.2 and 4.5), and one does not approach

the other as  $\alpha \rightarrow 0$ ; this phenomenon cannot be eliminated by changing the fundamental solution. See also the discussion in the last paragraph of Section 1.

In conclusion, it is notable that the Brinkman model gives a regularization of the flow problem in the sense that it does not exhibit a Stokes-like paradox. It is also used within the ‘Brinkman penalization’ framework, an established technique to regularize certain numerical methods for investigating fluid–solid interactions (Verma *et al.*, 2017).

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## Appendix A. Stokes flow past a curved strip

For this problem, the governing system of integral equations is (3.3). Put

$$v(\xi) = L^{-1} \left( [x'_1(\xi)]^2 + [x'_2(\xi)]^2 \right)^{1/2} \quad \text{and} \quad h_j(\xi) = h_j(\mathbf{x}(\xi)) v(\xi)$$

so that  $h_j(\mathbf{x}) ds(\xi) = Lh_j(\xi) d\xi$ . Also, define

$$G_{ij}^0(\mathbf{x}, \mathbf{x}') = \delta_{ij} \log |\xi - \xi'| + K_{ij}^0(\xi, \xi') \quad \text{with} \quad K_{ij}^0(\xi, \xi') = \delta_{ij} \log \frac{R}{L|\xi - \xi'|} - \frac{R_i R_j}{R^2}$$

so that  $K_{ij}^0$  is not singular at  $\xi = \xi'$  ( $\mathbf{x} = \mathbf{x}'$ ). The system (3.3) becomes

$$\int_{-1}^1 h_1(\xi') \log |\xi - \xi'| d\xi' + \int_{-1}^1 \left\{ K_{11}^0(\xi, \xi') h_1(\xi') + K_{12}^0 h_2(\xi') \right\} d\xi' = -\frac{1}{U} U_1(\mathbf{x}(\xi)), \quad (\text{A.1})$$

$$\int_{-1}^1 h_2(\xi') \log |\xi - \xi'| d\xi' + \int_{-1}^1 \left\{ K_{21}^0(\xi, \xi') h_1(\xi') + K_{22}^0 h_2(\xi') \right\} d\xi' = -\frac{1}{U} U_2(\mathbf{x}(\xi)) \quad (\text{A.2})$$

for  $-1 < \xi < 1$ . Proceeding as in Section 3.2.1, we expand  $h_j$  as

$$h_j(\xi) = \frac{1}{\sqrt{1 - \xi^2}} \sum_{n=0}^{\infty} c_n^{(j)} T_n(\xi), \quad (\text{A.3})$$

whence (3.8) and (3.9) give

$$\int_{-1}^1 h_j(\xi') \log |\xi - \xi'| d\xi' = -\pi c_0^{(j)} \log 2 - \sum_{n=1}^{\infty} \frac{\pi}{n} c_n^{(j)} T_n(\xi).$$

This gives an analytical treatment of the logarithmic part of the kernel. Then, one could develop an ‘expansion–collocation method’: truncate the expansions (A.3) at  $n = N$ , say, and then evaluate the integral equations (A.1) and (A.2) at  $N + 1$  points in the interval  $-1 < \xi < 1$ . Such methods have been used and analyzed in other contexts; see (Martin, 2006, Section 6.7.1) for some references.

## Appendix B. Verifying the Brinkmanlet solution (4.3)

Look for solutions in the form

$$G_{ij} = \delta_{ij} A(kR) + \frac{R_i R_j}{R^2} B(kR), \quad P_j = \frac{R_j}{R^2} P(kR). \quad (\text{B.1})$$

Some calculation gives

$$\begin{aligned} \frac{\partial G_{ij}}{\partial x_\ell} &= \delta_{ij} \frac{\partial A}{\partial x_\ell} + \left( \delta_{i\ell} \frac{R_j}{R^2} + \delta_{j\ell} \frac{R_i}{R^2} - 2 \frac{R_i R_j R_\ell}{R^4} \right) B + \frac{R_i R_j}{R^2} \frac{\partial B}{\partial x_\ell}, \\ \nabla^2 G_{ij} &= \delta_{ij} \nabla^2 A + \frac{\partial}{\partial x_i} \left( \frac{R_j B}{R^2} \right) + \frac{\partial}{\partial x_j} \left( \frac{R_i B}{R^2} \right) - 2 \frac{\partial}{\partial x_\ell} \left( \frac{R_i R_j R_\ell B}{R^4} \right) \\ &\quad + \left( \delta_{i\ell} \frac{R_j}{R^2} + \delta_{j\ell} \frac{R_i}{R^2} - 2 \frac{R_i R_j R_\ell}{R^4} \right) \frac{\partial B}{\partial x_\ell} + \frac{R_i R_j}{R^2} \nabla^2 B. \end{aligned}$$

In particular,

$$\frac{\partial G_{ij}}{\partial x_i} = \frac{\partial A}{\partial x_j} + \frac{R_j}{R^2} B + \frac{R_i R_j}{R^2} \frac{\partial B}{\partial x_i} = \frac{R_j}{R^2} \{ kR(A' + B') + B \}. \quad (\text{B.2})$$



We have

$$\begin{aligned}\frac{\partial}{\partial x_i} \left( \frac{R_j B}{R^2} \right) &= \frac{\partial}{\partial x_j} \left( \frac{R_i B}{R^2} \right) = \delta_{ij} \frac{B}{R^2} + \frac{R_i R_j}{R^2} \left( \frac{kB'}{R} - \frac{2B}{R^2} \right), \\ \frac{\partial}{\partial x_\ell} \left( \frac{R_i R_j R_\ell}{R^4} \right) &= \frac{\delta_{i\ell} R_j R_\ell + \delta_{j\ell} R_i R_\ell + 2R_i R_j}{R^4} - 4 \frac{R_i R_j R_\ell R_\ell}{R^6} = 0, \\ \frac{\partial}{\partial x_\ell} \left( \frac{R_i R_j R_\ell B}{R^4} \right) &= \frac{R_i R_j R_\ell}{R^4} \frac{\partial B}{\partial x_\ell} = \frac{R_i R_j}{R^3} kB', \\ \left( \delta_{i\ell} \frac{R_j}{R^2} + \delta_{j\ell} \frac{R_i}{R^2} - 2 \frac{R_i R_j R_\ell}{R^4} \right) \frac{\partial B}{\partial x_\ell} &= \left( 2 \frac{R_i R_j}{R^3} - 2 \frac{R_i R_j R_\ell R_\ell}{R^5} \right) kB' = 0.\end{aligned}$$

Hence,

$$\nabla^2 G_{ij} = \left( \nabla^2 A + \frac{2B}{R^2} \right) \delta_{ij} + \frac{R_i R_j}{R^2} \left( \nabla^2 B - \frac{4B}{R^2} \right). \quad (\text{B.3})$$

Also,

$$\frac{\partial P_j}{\partial x_i} = \delta_{ij} \frac{P}{R^2} + \frac{R_i R_j}{R^2} \left( \frac{kP'}{R} - \frac{2P}{R^2} \right).$$

Insert  $u_i = UG_{ij}$  and  $p = \mu UP_j$  in (4.1)<sub>1</sub>. For these equations to be satisfied, we require

$$\nabla^2 A - k^2 A + \frac{2B}{R^2} - \frac{P}{R^2} = 0 \quad \text{and} \quad \nabla^2 B - k^2 B - \frac{4B}{R^2} - \frac{kP'}{R} + \frac{2P}{R^2} = 0. \quad (\text{B.4})$$

As  $\nabla^2 A(kR) = k^2 A''(kR) + (k/R)A'(kR)$ , multiplying (B.4) by  $R^2$  gives

$$x^2 A''(x) + xA'(x) - x^2 A(x) = P(x) - 2B(x), \quad (\text{B.5})$$

$$x^2 B''(x) + xB'(x) - (x^2 + 4)B(x) = xP'(x) - 2P(x). \quad (\text{B.6})$$

These are to be solved, together with the continuity equation, which (using (B.2)) reduces to

$$x \{A'(x) + B'(x)\} + B(x) = 0. \quad (\text{B.7})$$

Let us start with (B.6). Assume that  $P(x) = 2P_0$ , a constant. Then

$$B(x) = 2\mathcal{B}K_2(x) + 4P_0x^{-2}, \quad (\text{B.8})$$

where  $\mathcal{B}$  is an arbitrary constant. Equation (B.5) becomes

$$x^2 A''(x) + xA'(x) - x^2 A(x) = 2P_0(1 - 4x^{-2}) - 4\mathcal{B}K_2(x),$$

whence

$$A(x) = \mathcal{A}K_0(x) - \mathcal{B}K_2(x) - 2P_0x^{-2}, \quad (\text{B.9})$$

where  $\mathcal{A}$  is another arbitrary constant. Substituting for  $A$  and  $B$  in (B.7), making use of  $K'_0(x) = -K_1(x)$  and  $xK'_2(x) = -xK_1(x) - 2K_2(x)$ , we obtain  $(\mathcal{A} + \mathcal{B})xK_1(x) = 0$ . This gives  $\mathcal{B} = -\mathcal{A}$ .

Next, let us investigate the behaviour of  $A(x)$  and  $B(x)$  as  $x \rightarrow 0$ . Use of (4.6) gives

$$A(x) \sim -\mathcal{A} \log x + 2(\mathcal{A} - P_0)x^{-2} \quad \text{and} \quad B(x) \sim -4(\mathcal{A} - P_0)x^{-2} \quad \text{as } x \rightarrow 0.$$

To eliminate the strong singularities at  $x = 0$ , we take  $\mathcal{A} - P_0 = 0$ . Finally, we take  $\mathcal{A} = -1$  so that  $A(x) \sim \log x$  as  $x \rightarrow 0$ , in agreement with the Stokes solution, (2.5). Thus,  $\mathcal{B} = 1$ ,  $P_0 = -1$ ,

$$A(x) = -K_0(x) - K_2(x) + 2x^{-2}, \quad B(x) = 2K_2(x) - 4x^{-2}, \quad (\text{B.10})$$

together with  $P = -2$  in (B.1). These formulas agree with Kohr *et al.* (2008, Equation (2.13)).