Acoustic scattering by one bubble before 1950: Spitzer, Willis, and Division 6

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The solution of the problem of acoustic scattering by a single bubble is an ingredient in many theories used to predict the behavior of bubbly liquids. Significant work on this problem was done in the 1940s, but much of that work was described in unpublished wartime reports. However, it is now possible to access most of those reports, enabling a more coherent summary to be given. That is the purpose of this paper. © *2019 Acoustical Society of America*. https://doi.org/10.1121/1.5120127

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I. INTRODUCTION

Physics of Sound in the Sea,[1] first published in 1946, is a 566-page compilation of results obtained during World War II. The book was edited by Lyman Spitzer and contains 35 chapters. Chapter 28, "Acoustic theory of bubbles,"[2] is of interest here. The author of this chapter is not stated, but it is known now that "chapter 28" was written by Spitzer himself. One reason is that much of chapter 28 is based on a report by Spitzer,[3] written in July 1943. (The differences between the two documents will be examined later.) Another reason is that, much later, Spitzer described his contributions in an interview; see Sec. II, where the genesis of *Physics of Sound in the Sea*[1] is outlined. Some authors misattribute chapter 28 to Wildt; Rupert Wildt (1905–1976), an astronomer, edited Part IV of Ref. 1, comprising Chaps. 26–35.

The first two sections of chapter 28 (Ref. 2) are concerned with the acoustic properties of a single bubble. The results obtained "are only the first step toward the solution of the general problem, the propagation of sound through a medium containing many bubbles" (p. 467 of Ref. 2). The focus here will be on the single-bubble analysis.

Nowadays, the first approach that one would think of for scattering by a sphere would be separation of variables in spherical polar coordinates. In the context of the Helmholtz equation, this approach goes back to Rayleigh (see Sec. III). The results obtained by this approach are reviewed in Sec. IV, paying attention to long-wave approximations. Two reports from the 1940s, by Epstein[4] and by Duvall,[5] are discussed, together with the first mention of the "British Willis Report."

Spitzer preferred to make approximations from the outset, choosing to represent the scattered waves using a monopole source (Sec. V). He did this first for an "ideal bubble" and then for an "actual bubble," where various lossy mechanisms are introduced (Sec. VII). In particular, in his 1943 report,[3] he considers thermal dissipation, and he quotes some results from the "British Willis Report"; the citation is incomplete, with the author given as "Willis" and without a date. Spitzer did state that the results of Willis are similar to those of Pfriem[6] and Saneyosi,[7] and he noted that "[c]ertain assumptions made by Willis and others have been examined critically by C. Herring, who finds that they are valid in the cases of practical importance" (see p. 16 of Ref. 3).

In Sec. VII B, "Willis" is identified as Hector F. Willis, and the "British Willis Report" is identified with a document found in the UK National Archives. The result is a more complete picture of the work done in the 1940s on acoustic scattering by one bubble. Concluding remarks are given in Sec. VIII.

II. LYMAN SPITZER AND DIVISION 6

The National Defense Research Committee (NDRC) was formed in June 1940. In October 1940, a subcommittee was formed "to study the scientific aspects of protection against submarine warfare" (p. 10 of Ref. 8). Its members were E. H. Colpitts (chairman), W. D. Coolidge, H. G. Knox, V. O. Knudsen and L. B. Slichter. In January 1941, the subcommittee made recommendations, urging "that greater attention be paid to the fundamentals of undersea warfare, including …the phenomena of sound propagation in the ocean" (p. 11 of Ref. 8). In April 1941, J. T. Tate was appointed Chairman of the NDRC's Section C-4, "which was the designation of the group that was to be concerned with anti-submarine warfare" (p. 14 of Ref. 8). Section C-4 had research laboratories in San Diego, CA, and New London, CT, and it entered into numerous contracts with companies and universities. Office space was rented at 172 Fulton Street, New York City.

"Originally designated Section C-4 of NDRC, the Underwater Sound Group at the time of NDRC's reorganization on 9 December 1942, became officially Division 6 with Dr. Tate as Division Chief. All research and development work of Division 6 was concentrated in one section, Section 6.1, of which Dr. Colpitts was Chief" (p. 20 of Ref. 8).

"An important adjunct to the central administrative office was a group of scientists employed under a contract with Columbia University. This staff, known as the Program Analysis Group, carried out a continuous analysis of the work in progress" (p. 18 of Ref. 8). Two of its members were L. B. Slichter and W. V. Houston. By the end of 1942,

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its name had been changed to the Special Studies Group and Houston had become its Director. "Particular mention should be made of a section of Special Studies designated as the Sonar Analysis Group, which was set up to analyze and correlate the results of the oceanographic studies conducted at Woods Hole and by the San Diego Laboratory" (p. 18 of Ref. 8).

The Director of the Sonar Analysis Group was Lyman Spitzer (1914–1997). He was interviewed in 1978 for the American Institute of Physics.[9] His recollections are as follows. "When I was at Yale, I got a phone call from [L. B. Slichter] who said he was with Columbia University, Division of War Research, on Fulton St., and wouldn't I come for an interview. So I went to New York…[They] had some staff groups, one under Slichter, one under Bill Houston. Houston I think was in charge of an effort on homing torpedoes, and Slichter had charge of a number of things, but one of them was coordinating underwater sound research. …I assisted in the coordination of the undersea warfare research. …In the final two years of the war, we had a small group set up, of which I was head, called the Sonar Analysis Group, which continued the same thing. …In the closing year of the war, the headquarters of the group, that was me, was moved to Washington, where I had a desk in the Bureau of Ships. But the rest of the group continued in the Empire State Building. …[M]y work involved talking with people who were doing research and telling them what they were doing wrong and what they ought to be doing. I mean, it was scientific administration. And the closing years were devoted to writing one major book, called *The Physics of Sound in the Sea*.[1] …I'm not listed as general editor in the book, but that was essentially my function. I would go over what everybody was writing, and object to it, and I helped write some of it. I became rather intimately involved with theories of underwater sound transmission, reverberation, things of that sort. Pekeris was a member of our group, [also] Leslie Foldy, Henry Primakoff [and] Bob Shankland." In the same interview, Spitzer was asked if he had any contact with British sonar work. He recalled that "some of the British sonar people actually visited us."

III. BEFORE WORLD WAR II

Chapter 28 (Ref. 2) cites three pre-war sources: Minnaert's paper[10] on the natural frequency of bubbles, a translation of the German paper by Meyer and Tamm[11] on the damping of bubble vibrations, and the second edition of Wood's *A Textbook of Sound*.[12] In fact, although the literature on bubbles from this period is sparse, it is well known. Some of it is described briefly now so as to set the scene for later work.

As with most topics within acoustics, the relevant literature starts with Lord Rayleigh. He showed how the method of separation of variables can be used to solve the problem of scattering of a plane wave by a fluid sphere,[13] and he gave a low-frequency approximation. The full solution would not be worked out until 1950 when Anderson[14] used "the full time service of two computers [i.e., people] for a period of about two months," although his choices of parameter values do not cover the case of air bubbles in water.

In more detail, Rayleigh's analysis (Sec. 335 of Ref. 13) is as follows: "Having considered at some length [in Sec. 334] the case of [scattering of a plane wave by] a rigid spherical obstacle, we will now sketch briefly the course of the investigation when the obstacle is gaseous. Although in all natural gases the compressibility is nearly the same, we will suppose for the sake of generality that the matter occupying the sphere differs in compressibility, as well as in density, from the medium in which the plane waves advance." After writing down separated solutions of the Helmholtz equation in spherical polar coordinates and the continuity conditions across the obstacle's surface, Rayleigh goes on: "From these equations the complete solution may be worked out; but we will here confine ourselves to finding the value of the leading terms, when ka, $k'a$ are very small" where the sphere has radius a, k is the exterior wavenumber and k' is the interior wavenumber. … "At a distance from the sphere the disturbance due to it is expressed by"

$$\psi = -\frac{k^2a^3}{3r}\,\mathrm{e}^{\mathrm{i}(\omega t-kr)}\left\{\frac{m'-m}{m'}+3\,\frac{\rho'-\rho}{\rho+2\rho'}\cos\theta\right\}, \qquad (1)$$

where ρ, ρ' "be the natural densities, m, m' the compressibilities [and] $k'^2/k^2 = (\rho'/\rho)(m/m')$." (Minor changes to some of Rayleigh's notation have been made.)

Arnulph Mallock (1851–1933) was an assistant to Rayleigh and a nephew of William Froude (the naval architect).[15] Mallock's 1910 paper[16] gives a formula for the speed of sound, c, in a bubbly liquid. To state it, suppose that the gas (air) in the bubbles has constant density ρ_a and sound speed c_a. In the liquid (water), the sound speed is c_w and the density is ρ_w. Let β be the volume occupied by the gas in a unit volume of the mixture. The density of the mixture, ρ, is given by

$$\rho = \rho_w(1-\beta) + \rho_a\beta. \qquad (2)$$

The bulk modulus of the mixture, K, is given by

$$\frac{1}{K} = \frac{1-\beta}{\rho_w c_w^2} + \frac{\beta}{\rho_a c_a^2}. \qquad (3)$$

Then, Mallock's estimate for c is

$$c^2 = K/\rho \text{ with } \rho \text{ and } K \text{ given by Eqs. (2) and (3)}, \qquad (4)$$

respectively. This equation is called the *Mallock–Wood equation*, *Wood's equation* or *Urick's equation*. Albert Wood (1890–1964) and Robert Urick (1915–1996) were both well-known acousticians. Wood's *A Textbook of Sound* was first published in 1930, with later editions in 1941 (Ref. 12) and 1955. Each edition contains a derivation of Eq. (4); Mallock[16] is cited in the first and second editions but not in the third. Urick was familiar with Wood's book[12] and he[17] made extensive use of Eq. (4).

Equation (4) is only expected to be useful for very low frequency waves (quantified below as $\omega \ll \omega_0$) because it does not take account of resonance effects: such effects are characteristic of air bubbles in water. The natural frequency ω_0 of a spherical gas bubble in a liquid was first calculated by Marcel Minnaert (1893–1970), a Belgian astronomer.[10] For adiabatic conditions, it is given by

$$\omega_0 = \sqrt{\frac{3\gamma p_0}{\rho_{\mathrm{w}} R_0^2}}, \tag{5}$$

where R_0 is the equilibrium radius of the bubble, γ is the ratio of specific heats of the gas, and p_0 is the equilibrium pressure in the bubble. In more detail, if the gas in the bubble has pressure p' and density ρ', we have $p'\rho'^{-\gamma} = p_0 \rho_{\mathrm{a}}^{-\gamma}$. Then, by definition,

$$c_{\mathrm{a}}^2 = \left.\frac{\partial p'}{\partial \rho'}\right|_{\rho'=\rho_{\mathrm{a}}} = \frac{\gamma p_0}{\rho_{\mathrm{a}}}.$$

Thus, from Eq. (5),

$$\omega_0^2 R_0^2 = 3(\rho_{\mathrm{a}}/\rho_{\mathrm{w}}) c_{\mathrm{a}}^2. \tag{6}$$

Some authors[18] refer to ω_0 as the *Minnaert frequency*.

IV. SEPARATION OF VARIABLES

The standard exact approach for the problem of scattering of a plane wave by a bubble of radius R_0 is reviewed here. Use spherical polar coordinates r, θ and φ with the bubble's center at $r=0$, and with $z = r\cos\theta$. The suppressed time dependence is $\mathrm{e}^{-\mathrm{i}\omega t}$. The incident pressure wave is

$$p_{\mathrm{inc}} = \mathcal{P}\,\mathrm{e}^{\mathrm{i}k_{\mathrm{w}} z} = \mathcal{P}\sum_{n=0}^{\infty}(2n+1)\,\mathrm{i}^n j_n(k_{\mathrm{w}} r) P_n(\cos\theta), \tag{7}$$

where $\mathcal{P}$ is a constant, $k_{\mathrm{w}} = \omega/c_{\mathrm{w}}$, j_n is a spherical Bessel function and P_n is a Legendre polynomial. There will be a scattered field, p_{sc}, outside the bubble and a field, p_{int}, inside the bubble. These fields have expansions

$$p_{\mathrm{sc}} = \mathcal{P}\sum_{n=0}^{\infty}(2n+1)\,\mathrm{i}^n A_n h_n(k_{\mathrm{w}} r) P_n(\cos\theta), \quad r \ge R_0, \tag{8}$$

$$p_{\mathrm{int}} = \mathcal{P}\sum_{n=0}^{\infty}(2n+1)\,\mathrm{i}^n B_n j_n(k_{\mathrm{a}} r) P_n(\cos\theta),\ 0 \le r \le R_0, \tag{9}$$

where $k_{\mathrm{a}} = \omega/c_{\mathrm{a}}$ and $h_n \equiv h_n^{(1)}$ is a spherical Hankel function. The (dimensionless) coefficients A_n and B_n are determined using the continuity conditions (also known as transmission conditions) across the bubble surface at $r = R_0$. These are continuity of pressure and continuity of normal (radial) velocity, v.

Given a wave function $\Phi = \mathrm{Re}\{\phi(r,\theta,\varphi)\mathrm{e}^{-\mathrm{i}\omega t}\}$, pressure $= -\rho\,\partial\Phi/\partial t$ and radial velocity $= \partial\Phi/\partial r$. Removing the time dependence, these give $p = \mathrm{i}\omega\rho\phi$ and $v = \partial\phi/\partial r = (\mathrm{i}\omega\rho)^{-1}\partial p/\partial r$. Then the continuity conditions at $r = R_0$ give

$$j_n(k_{\mathrm{w}}R_0) + A_n h_n(k_{\mathrm{w}}R_0) = B_n j_n(k_{\mathrm{a}}R_0),$$
$$\rho_{\mathrm{w}}^{-1} k_{\mathrm{w}}\{j_n'(k_{\mathrm{w}}R_0) + A_n h_n'(k_{\mathrm{w}}R_0)\} = \rho_{\mathrm{a}}^{-1} k_{\mathrm{a}} B_n j_n'(k_{\mathrm{a}}R_0).$$

Solving these gives

$$A_n = \frac{j_n(\eta) j_n'(\nu) - q j_n'(\eta) j_n(\nu)}{\Delta_n}, \quad B_n = \frac{\mathrm{i}\,q}{\eta^2 \Delta_n}, \tag{10}$$

where $\Delta_n = q h_n'(\eta) j_n(\nu) - h_n(\eta) j_n'(\nu)$, $q = \rho_{\mathrm{a}} k_{\mathrm{w}}/(\rho_{\mathrm{w}} k_{\mathrm{a}})$,

$$\nu = k_{\mathrm{a}} R_0 \quad \text{and} \quad \eta = k_{\mathrm{w}} R_0.$$

Anderson[14] writes $A_n = -(1 + \mathrm{i}C_n)^{-1}$ with $C_n = \mathcal{Y}_n/\mathcal{J}_n$, where

$$\mathcal{J}_n = g h j_n'(\eta) j_n(\nu) - j_n(\eta) j_n'(\nu), \tag{11}$$

$$\mathcal{Y}_n = g h y_n'(\eta) j_n(\nu) - y_n(\eta) j_n'(\nu), \tag{12}$$

y_n is another spherical Bessel function ($h_n = j_n + \mathrm{i}y_n$),

$$g = \rho_{\mathrm{a}}/\rho_{\mathrm{w}} \quad \text{and} \quad h = c_{\mathrm{a}}/c_{\mathrm{w}} \tag{13}$$

(so that $\eta = \nu h$ and $q = gh$). However, it is more convenient to write

$$A_n = \frac{\mathrm{i}\mathcal{J}_n}{\mathcal{Y}_n - \mathrm{i}\mathcal{J}_n} \quad \text{and} \quad B_n = \frac{gh/\eta^2}{\mathcal{Y}_n - \mathrm{i}\mathcal{J}_n}. \tag{14}$$

When developing approximations for A_n, it is important to keep energy conservation in mind. (At this stage, all dissipative mechanisms have been excluded, but see Sec. VII.) This implies that A_n must satisfy

$$|A_n|^2 + \mathrm{Re}\,(A_n) = 0; \tag{15}$$

it is easily seen that this constraint is satisfied identically (without knowing anything about the forms of C_n, $\mathcal{J}_n$, or $\mathcal{Y}_n$ except that they are all real).

Approximations will be made assuming that the waves are long compared to the bubble radius, so that (as is reasonable) $\eta < \nu \ll 1$. Before proceeding, note that for air bubbles in water, $g \simeq 0.00126$ and $h \simeq 0.226$.[19] Thus g is small, and this fact will also be used.

A. Monopole component: $n = 0$

Start with $n = 0$, and then expand the spherical functions for small arguments. Using $j_0(z) \sim 1 - \frac{1}{6}z^2$, $j_0'(z) \sim -\frac{1}{3}z$, $y_0(z) \sim -z^{-1} + \frac{1}{2}z$ and $y_0'(z) \sim z^{-2} + \frac{1}{2}$ as $z \to 0$, the following estimates are obtained:

$$\mathcal{J}_0 = \frac{1}{3}\nu(1 - gh^2) + O(\nu^3),$$
$$\mathcal{Y}_0 = \frac{1}{3h}\left\{\frac{3gh^2}{\eta^2} - 1 + \frac{g}{2}(3h^2 - 1)\right\} + O(\nu^2),$$

as $\nu \to 0$. Discarding the error terms, Eq. (14) gives

$$A_0 = \frac{\mathrm{i}\eta^3(1 - gh^2)}{3gh^2 - \eta^2\left[1 - \frac{1}{2}g(3h^2 - 1)\right] - \mathrm{i}\eta^3(1 - gh^2)}. \tag{16}$$

Formally, this shows that $A_0 = O(\nu^3)$ as $\nu \to 0$:

$$A_0 \sim \mathrm{i}\eta^3 \frac{1 - gh^2}{3gh^2}. \tag{17}$$

However, this estimate does not satisfy Eq. (15) and it ignores the fact that the leading term in the denominator of Eq. (16), $3gh^2$, can be comparable to the next term. Thus, noting that $g \ll 1$ for air bubbles in water, discard all the terms in Eq. (16) containing g except the term $3gh^2$. This gives the approximation

$$A_0(\omega) = \frac{\mathrm{i}\eta}{W - 1 - \mathrm{i}\eta}, \tag{18}$$

where, using Eqs. (6) and (13),

$$W(\omega) = \frac{3gh^2}{\eta^2} = \frac{3\rho_a c_a^2}{\rho_w c_w^2 k_w^2 R_0^2} = \frac{\omega_0^2}{\omega^2} \tag{19}$$

and ω_0 is the Minnaert frequency, Eq. (5). Note that $A_0(\omega_0) = -1$ but $A_0(\omega) = O(\omega^3)$ as $\omega \to 0$.

B. Dipole component: $n = 1$

A similar analysis can be given when $n = 1$. For example, it can be shown that $A_1 = O(\nu^3)$ as $\nu \to 0$:

$$A_1 \sim \frac{\mathrm{i}\eta^3(g-1)}{3(2g+1)}. \tag{20}$$

Approximations similar to Eq. (18) can also be developed but details are omitted here.

C. Discussion

According to the strict definition of Rayleigh scattering (calculate the leading behavior as $\omega \to 0$), A_0 and A_1 are comparable, in agreement with Rayleigh's result, Eq. (1), which shows both a monopole and a dipole. However, the dominant contribution comes from the monopole when the frequency ω is close to the natural frequency of the bubble, ω_0. Retaining A_0 with the approximation Eq. (18), substitution in Eq. (8) gives

$$p_{sc} \simeq \mathcal{P} A_0 h_0(k_w r) = \mathbb{B} \frac{R_0}{r} \mathrm{e}^{\mathrm{i}k_w r}, \tag{21}$$

$$\mathbb{B} = \frac{\mathcal{P}}{(\omega_0/\omega)^2 - 1 - \mathrm{i}k_w R_0}, \tag{22}$$

after using $h_0(w) = \mathrm{e}^{\mathrm{i}w}/(\mathrm{i}w)$. For more on the point-source approximation seen here, see Sec. VI.

Higher-order multipoles (with $n \geq 2$) can be investigated but it turns out that they are asymptotically negligible; see Ref. 19 for numerical studies.

The pressure inside the bubble can be calculated using Eqs. (9) and (10). For low frequencies, B_0 is dominant so that, to leading order, $p_{int} = \mathcal{P} B_0$, a constant, inside the bubble.

D. Scattering cross-section

The monopole component of the scattering cross-section is proportional to $|A_0|^2$. From Eq. (18),

$$|A_0|^2 = \eta^2 / \{(W-1)^2 + \eta^2\}, \tag{23}$$

which is Eq. (22) in Ref. 2. However, as discussed in detail by Ainslie and Leighton,[20] this calculation is incomplete because a real quadratic term in η has been omitted from the denominator in Eq. (18):

$$|W - 1 + \alpha\eta^2 - \mathrm{i}\eta|^2 \simeq (W-1)^2 + \eta^2\{1 + 2\alpha(W-1)\}$$

as $\eta \to 0$. The real quantity α can be determined but doing so is beyond the scope of this paper, except for a simplified calculation in Sec. V; see Eq. (28).

E. Connection to Spitzer's articles

Spitzer, in both his 1943 report[3] and in chapter 28 (Ref. 2), notes that the method of separation of variables could be used, and he cites two 1941 papers by the physicist Paul Epstein.[4,21] In the earlier paper, Epstein[21] uses separation of variables for one sphere, with a viscous fluid in the exterior and another viscous fluid or an elastic solid inside. There are two Helmholtz equations to be solved in each region with continuity conditions on the spherical interface. In the second paper[4] (which cites the first), Epstein applies his theory to a bubble (retaining viscosity effects). Low-frequency approximations (waves much longer than the bubble diameter) are developed but resonance effects are not noticed. Some effects of heat conduction are examined. A more complete theory, incorporating viscosity and thermal conduction, was given later in the well-known paper by Epstein and Carhart;[22] that theory uses three Helmholtz equations in each region.

Epstein's second paper[4] bears further examination. It has the following sections:

(1) *On the stability of air bubbles in the sea.*
(2) *The extinction, due to viscosity, of sound in water containing air bubbles.* Separation of variables for one bubble, citing Ref. 21.
(3) *The extinction, due to heat conduction, of sound in water containing air bubbles.* "(t)he investigation [in Sec. 2] was incomplete inasmuch as only the effects of viscosity were taken into consideration and the effects of heat conduction neglected. A complementary investigation, in which only heat conduction is taken into account and viscosity neglected, is contained in the British Willis Report. However, I am not quite satisfied with that work as its methods appear to be somewhat crude …" The "British Willis Report" is not identified.
(4) *Sound absorption in water including the influence of air bubbles.*

The last page of Epstein's paper[4] contains an addendum:

Comment by Dr. William V. Houston

The work of Professor Epstein represents a more refined analysis of the absorption and scattering of sound by air bubbles than that made by Willis. The results are, however, in substantial agreement with those of Willis both in magnitude and in dependence on bubble size and frequency. This suggests that Willis treated the dominant features of the problem. 10/3/41

Spitzer[3] also cites a report by George Duvall.[5] [Duvall (1920–2003) was a young physicist at the time who went on to become well known for his work on shock waves.] Duvall gives results for a bubble "plotted from the infinite series which represents the rigorous solution of the problem";[5] see also Fig. 1 in Ref. 3. He cites Epstein's first paper[21] but he is well aware of the "sharp resonance peak" at the Minnaert frequency ω_0.

V. AN APPROXIMATE METHOD

In principle, the method of separation of variables, described in Sec. IV, is exact, but it is quite complicated. Spitzer preferred to give an approximate method; see Sec. I in Ref. 3 and Sec. 28.1 in Ref. 2. It proceeds as follows.

Consider a pulsating spherical bubble of radius $R(t)$ containing a fixed mass of gas. Under adiabatic conditions, pV^{γ} is constant, where $p(t)$ is the pressure in the bubble and $V = \frac{4}{3}\pi R^3$ is its volume. Thus $pR^{3\gamma} = p_0 R_0^{3\gamma}$, where the right-hand side is a constant. Differentiating and rearranging gives

$$\frac{dp}{dt} = -\frac{3\gamma p}{R}\frac{dR}{dt} \simeq -\frac{3\gamma p_0}{R_0}\frac{dR}{dt},$$

for small disturbances. This relates the pressure to the radial velocity of the bubble surface. The time-harmonic form of this equation is

$$-\mathrm{i}\omega p_{\mathrm{int}} = -(3\gamma p_0/R_0)v \quad \text{on the sphere, } r = R_0.$$

Suppose that the bubble is centered at the origin. Assuming that the bubble is small ($k_{\mathrm{w}}R_0 \ll 1$), approximate the incident wave in its vicinity, giving $p_{\mathrm{inc}} = \mathcal{P}\,\mathrm{e}^{\mathrm{i}k_{\mathrm{w}}z} \simeq \mathcal{P}$. Assume further that the bubble behaves as a monopole, so that the scattered field can be approximated by Eq. (21). (This approximation is discussed further in Sec. VI.) Thus, outside but near the bubble,

$$p_{\mathrm{inc}} + p_{\mathrm{sc}} \simeq \mathcal{P} + \mathcal{P}A_0 h_0(k_{\mathrm{w}}r), \tag{24}$$

$$v = (\mathrm{i}\omega\rho_{\mathrm{w}})^{-1}\mathcal{P}A_0 k_{\mathrm{w}} h_0'(k_{\mathrm{w}}r); \tag{25}$$

as $h_0(w) = \mathrm{e}^{\mathrm{i}w}/(\mathrm{i}w)$, $h_0'(w) = \mathrm{e}^{\mathrm{i}w}(\mathrm{i}w-1)/(\mathrm{i}w^2)$.

Matching the pressures across $r = R_0$, $p_{\mathrm{inc}} + p_{\mathrm{sc}} = p_{\mathrm{int}}$, gives

$$\mathcal{P} + \mathcal{P}A_0 h_0(k_{\mathrm{w}}R_0) = \frac{3\gamma p_0}{\mathrm{i}\omega R_0}v = \frac{3\gamma p_0}{\mathrm{i}\omega R_0}\frac{\mathcal{P}k_{\mathrm{w}}}{\mathrm{i}\omega\rho_{\mathrm{w}}}A_0 h_0'(k_{\mathrm{w}}R_0),$$

an equation for A_0. Thus

$$1 + A_0 h_0(\eta) = -W\eta A_0 h_0'(\eta),$$

where $W = (\omega_0/\omega)^2$. Hence

$$\mathrm{i}\eta/A_0 = \mathrm{e}^{\mathrm{i}\eta}\{(1-\mathrm{i}\eta)W - 1\}. \tag{26}$$

Expanding $\mathrm{e}^{\mathrm{i}\eta}$ for small η gives

$$\mathrm{i}\eta/A_0 = W - 1 - \mathrm{i}\eta + \alpha\eta^2 + O(\eta^3) \tag{27}$$

as $\eta = k_{\mathrm{w}}R_0 \to 0$, where $\alpha = \frac{1}{2}(W+1)$. Discarding the η^2 term (put $\alpha = 0$) shows that A_0 is given by Eq. (18), as obtained previously by a more systematic method. Retaining the η^2 term gives

$$|A_0|^2 = \eta^2/\{(W-1)^2 + \eta^2 W^2\}. \tag{28}$$

Spitzer computed $|A_0|^2$ with $\alpha = 0$ and so obtained an incorrect estimate, Eq. (23); see Eq. (22) in Ref. 2. For a detailed discussion on estimates of $|A_0|^2$, see Ref. 20.

VI. THE POINT-SOURCE APPROXIMATION

The simple idea of representing the scattered field using a simple monopole source can be found in the "British Willis Report." Presumably, it was this approximation that Epstein[4] considered to be "somewhat crude." Nevertheless, it has been used to good effect subsequently. For example, it was the starting point for Foldy's theory of multiple scattering by random collections of small bubbles[23] and the companion experimental paper by Carstensen and Foldy;[24] both papers (or earlier report versions) were cited by Spitzer.[2,3]

VII. DAMPING

The scattering analysis given in Secs. IV and V takes no account of energy absorption, such as conversion into heat. Other effects were ignored, such as those due to viscosity or to surface tension. One quick way to include these effects is to replace the denominator in the formula for $\mathbb{B}$ in Eq. (22) by

$$(\omega_{\mathrm{res}}/\omega)^2 - 1 - \mathrm{i}\delta,$$

where ω_{res} is the resonance frequency of the bubble and δ is a dimensionless damping constant.

Surface tension changes the resonance frequency. Spitzer[3] noted that Smith[25] had calculated this change erroneously and then gave a correction; see also Eq. (26) in Ref. 2 (where Smith is not cited) and Eq. (66) in Ref. 26.

Spitzer[2,3] cites Meyer and Tamm[11] for their experimental results on damping. Another early paper[27] is cited by Epstein.[4] Erwin Meyer (1899–1972) also compiled the book containing his chapter with Skudrzyk[28] on "Sound absorption by gas bubbles." The book itself is a translation of a document that "was written in the British zone of Germany in 1946 by the group of German scientists mentioned on the title page. A copy of the original became available in the United States in the spring of 1947."

A. Thermal dissipation

Spitzer's 1943 report[3] cites three papers on theoretical models of thermal dissipation, by Pfriem,[6] Saneyosi,[7] and Willis: this is the "British Willis Report" mentioned above. Spitzer's citation of this report as follows:

> Willis, British Report, reprinted as Confidential Report Section C4-BrTs-503, *Dissipation of Energy Due to Presence of Air Bubbles in the Sea*.

In chapter 28 of *Physics of Sound in the Sea*,[1] Spitzer refers back to his 1943 report[3] but he does not mention Willis. However, in the companion book, edited by Eckart,[29] the same citation is made (see Reference 5.4 on p. 282 of Ref. 29) except "Willis" is replaced by "F. H. Willis." This is a typographical error: it should be H. F. Willis, as clarified below in Sec. VII B.

Spitzer's 1943 report[3] summarises the results of Willis, and states that similar results were obtained by Pfriem[6] and Saneyosi.[7] (Saneyosi's short paper contains few details.) Devin [Sec. II(b) of Ref. 26] gives a full derivation, following "the derivation as outlined by Pfriem," eventually finding that his result "agrees exactly with Willis' curve as given in the report by Spitzer" (see p. 1662 of Ref. 26). For another detailed exposition, see pp. 175–188 of Leighton's book.[18] One result is that the gas in a bubble is found to behave in a complicated polytropic manner.

In fact, Spitzer does not cite Pfriem,[6] Saneyosi,[7] or Willis in chapter 28 (Ref. 2), claiming that "predicted values of δ are much smaller than the observed ones" and so he omits discussion of theoretical methods (see p. 466 of Ref. 2). For a more recent assessment, see Ref. 30.

B. The "British Willis Report": Who was "Willis"?

A typographical error in Ref. 29 has already been noted: F. H. Willis should be H. F. Willis. This is worth correcting because there was a man called F. H. Willis who worked in acoustics at about the same time!

French Hoke Willis (1904–1981) was American. He obtained a Ph.D. from New York University in 1943; his thesis has the same title as a subsequent journal paper.[31] This paper, and an earlier one,[32] are mainly concerned with experimental investigations. F. H. Willis gave his affiliation on both papers as Bell Telephone Laboratories, New York.

Hector Ford Willis (1909–1989) was British and worked for the Admiralty. His wartime reports came out of the Anti-Submarine Experimental Establishment (HMA/SEE) at Fairlie, Scotland. Subsequently, he was at the Admiralty Research Laboratory in Teddington. In 1947, the U.S, Government awarded H. F. Willis its Medal of Freedom with Silver Palm "for scientific research and development in theoretical physics and particularly the transmission of sound."[33]

Some of the wartime reports of H. F. Willis, such as Ref. 34, are concerned with underwater explosions; these contributions are described in Cole's book.[35] Other contributions are concerned with sound transmission in the sea. Thus Eckart writes (p. 57 of Ref. 29): "A possible explanation for the large [observed] attenuation was proposed by H. F. Willis, who suggested that it could be ascribed to the absorption caused by the suspension of very small air bubbles in the water. Laboratory experiments with ordinary and air-free water do show a difference, but it is not great enough to explain the facts." Later, in a section on scattering by single bubbles (p. 85 of Ref. 29): "There are thus occasions when air or vapor bubbles might be expected to exert an appreciable influence on the transmission of sound. This was first investigated by H. F. Willis, of H. M. A/SEE."

In his 1989 paper,[36] David Weston (1929–2001), who also worked at the Admiralty Research Laboratory, cited a report by Willis, as "Acoustic properties of bubbles in the sea, Unpublished MOD Rept., 1939." This was a decisive clue for searching in the UK National Archives, because such wartime reports are filed by title, not by author's name. A visit to the Archives confirmed that H. F. Willis did write a report[37] with the title quoted by Weston.

The first page of that report is dated 23 November 1944, and contains the following note, confirming that Ref. 37 is the "British Willis Report."

> This report was prepared and given limited circulation in manuscript form in the summer of 1939. In 1941 it was reproduced in the United States from a micro-film copy as "Confidential Report. Section C4-BrTs-503" under the title "Dissipation of Energy due to presence of air bubbles in the sea." It is now being published as an Internal Report by H.M.A/S.E.E., Fairlie, in order that a permanent record may be available in this country.

The report's contents are summarised by Willis as follows

> "This report deals with the behaviour of a bubble of air or other gas in the sea under the influence of an incident sound wave train. In the first part of the paper [Section A (pp. 1–13), entitled "Dissipation of energy due to the presence of air bubbles in the sea"], expressions are obtained for the energy dissipated as a result of heat interchanges, and the energy scattered as sound. In the second part [Section B (pp. 14–20), entitled "Application to attenuation in the sea"], the formula for the energy dissipation enables a theory to be advanced for the anomalously high attenuation of supersonic sound in the sea. The application of the results to artificial Bubble screens, both for damping unwanted signals, or as targets, is indicated."

Inspection of Section A shows that the analysis is indeed similar to that in Pfriem's paper.[6] One difference is that Willis writes the pressure outside the bubble using a monopole source, as in Eq. (24), and then obtains an expression for A_0 giving the amplitude of the scattered waves, Eq. (26); see also Sec. VI.

After the war, Willis co-authored a chapter[38] for a report commissioned by the US National Research Council Committee on Undersea Warfare. For background to the report and its influence, see Ref. 39.

VIII. CONCLUSIONS

The scientific study of bubbles and their acoustic effects advanced significantly during World War II, by necessity. However, some of these studies were not in the open literature, also by necessity. Today, many of these studies are readily discoverable on the internet, and so it is possible to give a coherent account based on original sources.

Two documents cited by Spitzer in his 1943 report were not so easy to find. A copy of the report by Duvall[5] was obtained from the Special Collections at the Library of the University of California, San Diego. The "British Willis Report"[37] was found in the UK National Archives (as noted in Sec. VII B).

In conclusion, much has happened in the world of bubble dynamics since 1950, but this is not the place for a thorough review; such reviews are available.[18,30,40]

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[1]L. Spitzer, Jr. (ed.), *Physics of Sound in the Sea*. Issued in 1946 as Summary Technical Report of Division 6, NDRC, vol. 8. Reprinted in 1949, National Research Council. Reprinted in 1969, Department of the Navy, Washington, DC.

[2]L. Spitzer, Jr., "Acoustic theory of bubbles," Chap. 28 in Ref. 1, pp. 460–477, with separate bibliography on p. 555. Referred to as "Wildt 1946" in Ref. 30.

[3]L. Spitzer, Jr., "Acoustic properties of gas bubbles in a liquid," OSRD 1705, NDRC 6.1-sr20-918, July 1943, 80 pp. The report was declassified in July 1946.

[4]P. S. Epstein, "The stability of air bubbles in the sea and the effect of bubbles and particles on the extinction of sound and light in sea water," Report No. C4-sr30-027, University of California Division of National Defense Research, San Diego, September 1941, 32 pp. The report was declassified in May 1959.

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