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Antiplane elastic waves in an anisotropic half-space: Fundamental solution, multipoles and scattering problems

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ABSTRACT

The effects of the traction-free boundary of a half-space can often be incorporated using images. In the simplest situations, a point force in the half-space induces an image force at the mirror-image point outside the half-space. However, for antiplane deformations of anisotropic media, the image point has a different location, as shown by T. C. T. Ting for static problems. This observation leads to formulas for a time-harmonic half-space fundamental solution and for time-harmonic half-space multipole solutions.

1. Introduction

Time-harmonic antiplane motions of a homogeneous anisotropic elastic solid are governed by the following partial differential equation,

$$C_{55}\frac{\partial^2 w}{\partial x^2} + 2C_{45}\frac{\partial^2 w}{\partial x\,\partial y} + C_{44}\frac{\partial^2 w}{\partial y^2} + \rho\omega^2 w = 0, \tag{1}$$

where *w(x,y)* is the out-of-plane displacement, C_{55}, C_{45} and C_{44} are stiffnesses, ρ is the density and ω is the frequency. (More details are given in Section 2.) We are interested in solving Eq. (1) in a half-space $y > -h$ $(h > 0)$ with a traction-free boundary condition at $y = -h$ and a scatterer of some kind (such as a circular cavity) within the half-space. For simplicity, assume that the origin is inside the scatterer.

The simplest situation is *isotropy*, for which $C_{44} = C_{55}$ and $C_{45} = 0$. Then Eq. (1) reduces to the Helmholtz equation, $(\nabla^2 + k^2)w = 0$ with $k^2 = \rho\omega^2/C_{55}$. Standard separated solutions include $H_n(kr)\,\mathrm{e}^{\mathrm{i}n\theta}$, where r and θ are plane polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), n is an integer and $H_n \equiv H_n^{(1)}$ is a Hankel function. These solutions are singular at $r = 0$. The boundary condition, $\partial w/\partial y = 0$ at $y = -h$, can be incorporated by adding an image term,

$$w(x, y) = H_n(kr)\,\mathrm{e}^{\mathrm{i}n\theta} + H_n(k\hat{r})\,\mathrm{e}^{-\mathrm{i}n\hat{\theta}}, \tag{2}$$

where $x = \hat{r}\cos\hat{\theta}$ and $y + 2h = \hat{r}\sin\hat{\theta}$. The extra term is singular at the image point $(x, y) = (0, -2h)$, which is the mirror image of the origin in the "mirror" at $y = -h$.

A slightly more complicated situation is *orthotropy*, for which $C_{44} \neq C_{55}$ and $C_{45} = 0$. In this case, we can reduce Eq. (1) to the Helmholtz equation by scaling x, y or both. For example, putting $x' = x/\alpha$ with $\alpha = \sqrt{C_{55}/C_{44}}$ gives $\partial^2 w/\partial x'^2 + \partial^2 w/\partial y^2 + (\alpha k)^2 w = 0$ with k as before. This scaling does not move the flat boundary at $y = -h$ but it does deform the shape of the scatterer. Alternatively, put $y' = \alpha y$ giving $\partial^2 w/\partial x^2 + \partial^2 w/\partial y'^2 + k^2 w = 0$. Stretching y is closer to what is usually done in the context of anisotropic elasticity (see Eq. (6) below) but it moves the flat boundary to $y' = -\alpha h$. Once the stretching has been done, we can reuse known solutions for the Helmholtz equation. In particular, for a solution singular at the origin, we can incorporate the boundary condition at $y = -h$ by adding an appropriate solution that is singular at the mirror-image point. (The relevant solutions can be recovered from formulas given below by putting $C_{45} = 0$ therein.)

For the general *anisotropic* case, governed by Eq. (1) with $C_{45} \neq 0$, we could transform Eq. (1) into the Helmholtz equation using an appropriate scaling and rotation of coordinates, the rotation being needed so as to eliminate the second term in Eq. (1), the one with the mixed derivative. The implication is that solutions involving Hankel (or Bessel) functions of certain arguments will appear. This approach is convenient for full-space problems but less so for half-space problems because the required transformation will also move the boundary of the half-space.

Instead, we first construct the full-space solutions (as has been done by others) and express them in terms of the original independent variables, x and y. We then introduce corresponding solutions singular at an appropriate image point. The location of this point was found by Ting [1, Section 3.5]: *it is not the mirror-image point unless* $C_{45} = 0$. Ting was concerned with static problems, but introducing dynamics does not change the *location* of the image point, just the kind of solutions that are to be singular at that point. Once this observation has been made, the rest is mere calculation. In Section 3.1, we construct a fundamental solution for Eq. (1) that

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is shown to satisfy the traction-free boundary condition at $y = -h$. This solution, given by Eq. (23), could be used to derive boundary integral equations for scattering problems posed in a half-space. Then, in Section 3.2, we construct multipole solutions, analogous to Eq. (2); these are given by Eq. (27). Their use in the context of scattering problems is discussed in Section 4.

2. Governing equations

We start with background material taken mainly from Chapter 3 of Ting's book [1]. The displacement vector is $\boldsymbol{u} = (u_1, u_2, u_3)$ with respect to Cartesian coordinates $Ox_1x_2x_3$. For special anisotropic materials, antiplane deformations are possible. These have the form $u_1 = u_2 = 0$ with $u_3 \equiv u$ independent of x_3. The equation of motion is

$$C_{55}\frac{\partial^2 u}{\partial x^2} + 2C_{45}\frac{\partial^2 u}{\partial x\,\partial y} + C_{44}\frac{\partial^2 u}{\partial y^2} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{3}$$

where C_{55}, C_{45} and C_{44} are the relevant stiffnesses, ρ is the mass density, $x \equiv x_1$ and $y \equiv x_2$. Eq. (3) is [2, Eq. (2.1)], [3, Eq. (2)] and [4, Eq. (6)]; the static version is [1, Eq. (3.3–5)]. The non-trivial stresses are (see [1, Eq. (3.3-3)])

$$\sigma_{31} = C_{55}\frac{\partial u}{\partial x} + C_{45}\frac{\partial u}{\partial y}, \quad \sigma_{32} = C_{45}\frac{\partial u}{\partial x} + C_{44}\frac{\partial u}{\partial y}. \tag{4}$$

The stiffnesses satisfy

$$C_{44} > 0, \quad C_{55} > 0 \quad \text{and} \quad C_{44}C_{55} - C_{45}^2 > 0. \tag{5}$$

When $C_{45} = 0$, the solid is said to be *orthotropic*. This case is simpler because Eq. (3) can be converted into the wave equation by rescaling x and y. In the more general case, with $C_{45} \neq 0$, such a conversion can be achieved, but a rotation is also required; see Eq. (13) below.

2.1. Complex variables

Change the (real) independent variables from x and y to

$$z = x + py \quad \text{and} \quad \bar{z} = x + \bar{p}y, \tag{6}$$

where p is a complex constant and the overbar denotes complex conjugation; p will be chosen later. (Our z and p are denoted by χ and γ, respectively, in [3,4].) Using the chain rule,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial z\,\partial \bar{z}} + \frac{\partial^2 u}{\partial \bar{z}^2},$$
$$\frac{\partial^2 u}{\partial y^2} = p^2\frac{\partial^2 u}{\partial z^2} + 2|p|^2\frac{\partial^2 u}{\partial z\,\partial \bar{z}} + \bar{p}^2\frac{\partial^2 u}{\partial \bar{z}^2},$$
$$\frac{\partial^2 u}{\partial x\,\partial y} = p\frac{\partial^2 u}{\partial z^2} + (p+\bar{p})\frac{\partial^2 u}{\partial z\,\partial \bar{z}} + \bar{p}\frac{\partial^2 u}{\partial \bar{z}^2}.$$

The equation of motion, Eq. (3), becomes

$$A\frac{\partial^2 u}{\partial z^2} + 2B\frac{\partial^2 u}{\partial z\,\partial \bar{z}} + \overline{A}\frac{\partial^2 u}{\partial \bar{z}^2} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{7}$$

where

$$A = C_{55} + 2pC_{45} + p^2C_{44}, \quad B = C_{55} + (p+\bar{p})C_{45} + |p|^2C_{44}.$$

Choose p so that $A = 0$ (and $\overline{A} = 0$),

$$C_{55} + 2pC_{45} + p^2C_{44} = 0. \tag{8}$$

(This is [1, Eq. (3.3–9)].) Then, using $A + \overline{A} = 0$, we find that $B = 2C_{44}(\operatorname{Im} p)^2$. But solving Eq. (8) gives

$$p = (\mathrm{i}\mu - C_{45})/C_{44} \text{ with } \mu = \sqrt{C_{44}C_{55} - C_{45}^2} > 0 \tag{9}$$

(see Eq. (5) and [1, Eq. (3.3–10)]), whence $B = 2\mu^2/C_{44}$ and Eq. (7) becomes

$$\frac{4\mu^2}{C_{44}}\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \rho\frac{\partial^2 u}{\partial t^2}. \tag{10}$$

This equation can be found in [5, Eq. (2–8)] and [6, Eq. (12)]. The time-harmonic version (with a time-dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$) is [2, Eq. (3.2)], [3, Eq. (6)], [4, Eq. (9)], [5, Eq. (2–11)], [6, Eq. (15)] and [7, Eq. (3)].

2.2. Polar coordinates

Define real quantities R and Θ by

$$z = x + py = R\,\mathrm{e}^{\mathrm{i}\Theta} \tag{11}$$

whence $R^2 = z\bar{z}$ and $\mathrm{e}^{2\mathrm{i}\Theta} = z/\bar{z}$. Using Eq. (9),

$$\{R(x,y)\}^2 = \left(C_{44}x^2 - 2C_{45}xy + C_{55}y^2\right)/C_{44}, \tag{12}$$

so that $R(x, y) =$ constant defines an ellipse in the xy-plane. Thus R and Θ are polar coordinates in the z-plane but not in the xy-plane!

It is worth noting that the origin can be moved: we can replace z by $z_0 = (x - x_0) + p(y - y_0)$, so that $R = 0$ would then correspond to $(x, y) = (x_0, y_0)$.

The chain rule gives

$$4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{\partial^2 u}{\partial R^2} + \frac{1}{R}\frac{\partial u}{\partial R} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \Theta^2}.$$

Using this formula in Eq. (10) gives

$$\frac{\partial^2 u}{\partial R^2} + \frac{1}{R}\frac{\partial u}{\partial R} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \Theta^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{13}$$

where $c = \mu/\sqrt{\rho C_{44}}$. We recognise Eq. (13) as the two-dimensional wave equation for $u(R, \Theta, t)$.

A simple time-harmonic solution of Eq. (13) is

$$u(x, y, t) = \operatorname{Re}\left\{H_0(kR)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{14}$$

where $H_n \equiv H_n^{(1)}$ is a Hankel function and $k = \omega/c$; there is a logarithmic singularity at $R = 0$. The solution Eq. (14) and others with H_0 replaced by $H_n\,\mathrm{e}^{\mathrm{i}n\Theta}$ have been used in several papers by Liu Diankui and co-authors, including [2,5,6], and they are used in [3,4,7]. We can also replace H_n by the Bessel function J_n.

Evidently, Eq. (14) could be used as a fundamental solution for problems posed in a full-space, leading to boundary integral equations for scattering problems.

However, we are mainly interested here in half-space problems, with a traction-free boundary at $y = -h$, where $\sigma_{32} = 0$. That is the focus of the next section.

3. Half-space problems and images

3.1. A fundamental solution (Green's function)

Let $u_0 = H_0(kR)$. The corresponding traction σ_{32} is given by Eq. (4) as

$$C_{45}\frac{\partial u_0}{\partial x} + C_{44}\frac{\partial u_0}{\partial y} = kH_0'(kR)\left(C_{45}\frac{\partial R}{\partial x} + C_{44}\frac{\partial R}{\partial y}\right);$$

denote this quantity by $\sigma_{32}^0(x, y)$. From Eq. (12),

$$\frac{\partial R}{\partial x} = \frac{C_{44}x - C_{45}y}{RC_{44}} \quad \text{and} \quad \frac{\partial R}{\partial y} = \frac{C_{55}y - C_{45}x}{RC_{44}}, \tag{15}$$

whence (as $H_0' = -H_1$)

$$\sigma_{32}^0(x, y) = -kH_1(kR)\,\frac{\mu^2 y}{RC_{44}}. \tag{16}$$

In particular,

$$\sigma_{32}^0(x, -h) = H_1(kR_h)\,\frac{\mu^2 kh}{R_hC_{44}} \text{ with } R_h = R(x, -h). \tag{17}$$

We want to cancel the traction Eq. (17) by introducing an image singularity. In [3,4,7], it is claimed that this can be done by placing a singularity at the mirror-image point, which is at $(x, y) =$

$(0, -2h)$: this is incorrect, in general, as can be checked easily. (The authors of [3,4,7] do not justify their claim; this is another example of "misuse of the method of images" [8].)

The correct location for the image singularity is given by Ting [1, Section 3.5, especially Fig. 3.3]: it is at

$$(x, y) = (\hat{x}, \hat{y}) = (2p'h, -2h) \tag{18}$$

with $p' = \operatorname{Re} p = (p + \bar{p})/2 = -C_{45}/C_{44}$. Quoting Ting [1, p. 78], "the image singularity is located outside the half-space $[y > -h]$. Its location is not the mirror image of [the origin] where the real singularity is located unless $p' = 0$. Notice that the imaginary part of p plays no role in Eq. (18)." We say that Eq. (18) defines the *Ting point*. It becomes the mirror-image point when the solid is orthotropic ($C_{45} = 0$).

Define $\hat{z}$, $\widehat{R}$ and $\widehat{\Theta}$ by

$$\hat{z} = (x - \hat{x}) + p(y - \hat{y}) = \widehat{R}\, e^{i\widehat{\Theta}} \tag{19}$$

so that

$$\begin{aligned}\left\{\widehat{R}(x, y)\right\}^2 &= \hat{z}\bar{\hat{z}} = x^2 + 2p'xy + |p|^2(y + 2h)^2 - 4hp'^2(y + h)\\ &= \left\{C_{44}x^2 - 2C_{45}xy + C_{55}(y + 2h)^2\right.\\ &\quad \left. - 4h(y + h)C_{45}^2/C_{44}\right\}/C_{44}.\end{aligned} \tag{20}$$

Comparing with Eq. (12), we observe that $\widehat{R}(x, -h) = R(x, -h)$: it is this property that enables the image calculation to succeed. Thus let $\hat{u}_0 = H_0(k\widehat{R})$ and

$$\hat{\sigma}_{32}^{0}(x, y) = C_{45}\frac{\partial \hat{u}_0}{\partial x} + C_{44}\frac{\partial \hat{u}_0}{\partial y} = kH_0'(k\widehat{R})\left(C_{45}\frac{\partial \widehat{R}}{\partial x} + C_{44}\frac{\partial \widehat{R}}{\partial y}\right).$$

From Eq. (20),

$$\frac{\partial \widehat{R}}{\partial x} = \frac{C_{44}x - C_{45}y}{\widehat{R}C_{44}} \quad \text{and} \tag{21}$$

$$\frac{\partial \widehat{R}}{\partial y} = \frac{C_{55}(y + 2h) - C_{45}x - 2hC_{45}^2/C_{44}}{\widehat{R}C_{44}}, \tag{22}$$

whence

$$\hat{\sigma}_{32}^{0}(x, y) = -kH_1(k\widehat{R})\,\frac{\mu^2(y + 2h)}{\widehat{R}C_{44}}.$$

In particular,

$$\hat{\sigma}_{32}^{0}(x, -h) = -H_1(kR_h)\,\frac{\mu^2 kh}{R_h C_{44}},$$

using $\widehat{R}(x, -h) = R(x, -h) = R_h$. This cancels with Eq. (17) when the two are added.

We conclude that the function

$$G_0(x, y) = H_0(kR) + H_0(k\widehat{R}) \tag{23}$$

is a fundamental solution: it satisfies the time-harmonic form of Eq. (3) (which is Eq. (1)) everywhere in the anisotropic half-space $y > -h$ except for a logarithmic singularity at the origin, and it satisfies the traction-free boundary condition at $y = -h$. An erroneous formula for G_0 is given in [4, Section 5]. In the special case of orthotropy, Eq. (23) reduces to a formula in Kausel's book [9, Eq. (5.10)].

3.2. Multipoles

Let $u_n(x, y) = H_n(kR)\, e^{in\Theta}$ and

$$\begin{aligned}\sigma_{32}^{n}(x, y) &= C_{45}\frac{\partial u_n}{\partial x} + C_{44}\frac{\partial u_n}{\partial y} = kH_n'(kR)\left(C_{45}\frac{\partial R}{\partial x} + C_{44}\frac{\partial R}{\partial y}\right)e^{in\Theta}\\ &\quad + inH_n(kR)\left(C_{45}\frac{\partial \Theta}{\partial x} + C_{44}\frac{\partial \Theta}{\partial y}\right)e^{in\Theta}.\end{aligned}$$

From $e^{2i\Theta} = z/\bar{z}$, we obtain

$$\frac{\partial \Theta}{\partial x} = -\frac{\mu y}{R^2 C_{44}} \quad \text{and} \quad \frac{\partial \Theta}{\partial y} = \frac{\mu x}{R^2 C_{44}}.$$

Hence, using Eq. (15), we obtain

$$\sigma_{32}^{n}(x, y) = kH_n'(kR)\,\frac{\mu^2 y}{RC_{44}}\, e^{in\Theta} + inH_n(kR)\,\frac{\mu}{R^2 C_{44}}(C_{44}x - C_{45}y)e^{in\Theta}.$$

But, from Eqs. (9) and (11),

$$R\cos\Theta = x - yC_{45}/C_{44} \quad \text{and} \quad R\sin\Theta = \mu y/C_{44},$$

whence (using [10, Eq. 9.1.27])

$$\begin{aligned}\sigma_{32}^{n}(x, y) &= \frac{\mu}{R}\left\{kRH_n'(kR)\sin\Theta + inH_n(kR)\cos\Theta\right\}e^{in\Theta}\\ &= \frac{\mu}{2iR}\left\{kRH_n'(kR) - nH_n(kR)\right\}e^{i(n+1)\Theta}\\ &\quad - \frac{\mu}{2iR}\left\{kRH_n'(kR) + nH_n(kR)\right\}e^{i(n-1)\Theta}\\ &= \frac{i\mu k}{2}\left\{H_{n+1}(kR)\, e^{i(n+1)\Theta} + H_{n-1}(kR)\, e^{i(n-1)\Theta}\right\}.\end{aligned} \tag{24}$$

(When $n = 0$, this formula reduces to Eq. (16).) On the traction-free boundary, we use $R(x, -h)$ and $\Theta(x, -h)$, where the latter is defined by

$$e^{2i\Theta(x,-h)} = \frac{x - ph}{x - \bar{p}h}. \tag{25}$$

Next consider the image. We know that it is singular at the Ting point, so consider $\hat{u}_n = H_n(k\widehat{R})\, e^{-in\widehat{\Theta}}$ with $\widehat{R}$ and $\widehat{\Theta}$ defined by Eq. (19). Proceeding as above, we have

$$\begin{aligned}\hat{\sigma}_{32}^{n}(x, y) &= C_{45}\frac{\partial \hat{u}_n}{\partial x} + C_{44}\frac{\partial \hat{u}_n}{\partial y} = kH_n'(k\widehat{R})\left(C_{45}\frac{\partial \widehat{R}}{\partial x} + C_{44}\frac{\partial \widehat{R}}{\partial y}\right)e^{-in\widehat{\Theta}}\\ &\quad - inH_n(k\widehat{R})\left(C_{45}\frac{\partial \widehat{\Theta}}{\partial x} + C_{44}\frac{\partial \widehat{\Theta}}{\partial y}\right)e^{-in\widehat{\Theta}}.\end{aligned}$$

From $e^{2i\widehat{\Theta}} = \hat{z}/\bar{\hat{z}}$, we obtain

$$\frac{\partial \widehat{\Theta}}{\partial x} = -\frac{\mu(y + 2h)}{\widehat{R}^2 C_{44}} \quad \text{and} \quad \frac{\partial \widehat{\Theta}}{\partial y} = \frac{\mu}{\widehat{R}^2 C_{44}}\left(x + 2h\frac{C_{45}}{C_{44}}\right).$$

Hence, using Eqs. (21) and (22),

$$\begin{aligned}\hat{\sigma}_{32}^{n}(x, y) &= kH_n'(k\widehat{R})\,\frac{\mu^2(y + 2h)}{\widehat{R}C_{44}}\, e^{-in\widehat{\Theta}}\\ &\quad - inH_n(k\widehat{R})\,\frac{\mu}{\widehat{R}^2 C_{44}}(xC_{44} - yC_{45})e^{-in\widehat{\Theta}}\\ &= \frac{\mu}{\widehat{R}}\left\{k\widehat{R}H_n'(k\widehat{R})\sin\widehat{\Theta} - inH_n(k\widehat{R})\cos\widehat{\Theta}\right\}e^{-in\widehat{\Theta}}\\ &= \frac{\mu}{2i\widehat{R}}\left\{k\widehat{R}H_n'(k\widehat{R}) + nH_n(k\widehat{R})\right\}e^{-i(n-1)\widehat{\Theta}}\\ &\quad - \frac{\mu}{2i\widehat{R}}\left\{k\widehat{R}H_n'(k\widehat{R}) - nH_n(k\widehat{R})\right\}e^{-i(n+1)\widehat{\Theta}}\\ &= -\frac{i\mu k}{2}\left\{H_{n+1}(k\widehat{R})\, e^{-i(n+1)\widehat{\Theta}} + H_{n-1}(k\widehat{R})\, e^{-i(n-1)\widehat{\Theta}}\right\}.\end{aligned} \tag{26}$$

From Eq. (19)

$$e^{2i\widehat{\Theta}(x,y)} = \frac{x - (p + \bar{p})h + p(y + 2h)}{x - (p + \bar{p})h + \bar{p}(y + 2h)}$$

so that

$$e^{2i\widehat{\Theta}(x,-h)} = \frac{x - \bar{p}h}{x - ph}.$$

Comparison with Eq. (25) then gives $\widehat{\Theta}(x, -h) = -\Theta(x, -h)$. As $\widehat{R}(x, -h) = R(x, -h)$, we see from Eqs. (24) and (26) that $\sigma_{32}^n(x, -h) + \hat{\sigma}_{32}^n(x, -h) = 0$: we have constructed multipole solutions

$$G_n(x, y) = H_n(kR)\,\mathrm{e}^{\mathrm{i}n\Theta} + H_n(k\widehat{R})\,\mathrm{e}^{-\mathrm{i}n\widehat{\Theta}}. \tag{27}$$

These functions satisfy the time-harmonic form of Eq. (3) everywhere in the anisotropic half-space $y > -h$ apart from a singularity at the origin, and they satisfy the zero-traction condition on the flat boundary at $y = -h$. Erroneous formulas for G_n are given in [3, Eq. (12)], [4, Eq. (13)] and [7, Eq. (4)].

4. Discussion

We have already mentioned that G_0 could be used as an ingredient in a boundary integral equation method for scattering problems. Such a problem arises when an incident plane wave is reflected by the flat boundary and scattered by the defect (which could be a cavity or an inclusion).

Another option [2,3,5] is to use the multipole solutions Eq. (27), writing the scattered field, w_{sc}, as

$$w_{\mathrm{sc}}(x, y) = \sum_{n=-\infty}^{\infty} c_n\, G_n(x, y), \tag{28}$$

with coefficients to be determined using the boundary condition on the scatterer. We recall that one piece of G_n contains $H_n(kR)$ with $R(x,\, y)$ defined by Eq. (12). Recall further that $R =$ constant represents an ellipse; the size and orientation of the ellipse depend on the elastic stiffnesses. The infinite series Eq. (28) is expected to converge for $R > R_0$ for some R_0; whether the ellipse $R = R_0$ encloses the scatterer or not is related to what is known as the *Rayleigh hypothesis*; see [11] for further discussion.

As the coefficients c_n must be found numerically, we could truncate the series Eq. (28) and then seek an approximation, writing

$$w_{\mathrm{sc}}(x, y) \simeq \sum_{n=-N}^{N} c_n(N)\, G_n(x, y). \tag{29}$$

The $2N + 1$ coefficients $c_n(N)$ (which could depend on N) may then be computed perhaps by applying the boundary condition at $2M + 1$ points (with $M \geq N$) or by using a Fourier-type method [3,5]; as noted above, the numerical results in [3,4,7] were obtained with an incorrect choice of image point (except when $C_{45} = 0$). We are not aware of any numerical analysis of these approximation schemes.

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