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A Stroh Formalism for Small-on-Large Problems in Spherical Polar Coordinates

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Abstract The governing equations for small-on-large analysis of an incompressible hyperelastic solid are reduced to a coupled system of six first-order ordinary differential equations with respect to the radial coordinate in spherical polar coordinates. This reduction to Stroh form does not assume a particular form for the strain-energy function.

Keywords Nonlinear elasticity · Small-on-large · Incremental equations

Mathematics Subject Classification (2010) 74B15 · 74B20 · 35A09 · 35J57

1 Introduction

In a recent paper, the scattering of sound waves by a rubber spherical shell is studied, using analytical, computational and experimental approaches [9]; for the experiments, the authors used "a commercially manufactured American handball". Another recent paper [13] considers the oscillations of a thin pre-stressed elastic spherical shell, using analytical and experimental approaches; for their experiments, the authors used a "helium-filled novelty balloon". Our study began with the following related (but more difficult) problem: how is a sound wave scattered by an inflated (pre-stressed) rubber balloon?

To formulate the problem, suppose that the balloon is spherical and that it is made from an incompressible hyperelastic isotropic material. Inflate the balloon quasistatically: the resulting large deformation can be calculated. (This calculation is reviewed briefly in Sect. 2.) We then consider small dynamic perturbations about the static configuration, leading to a small-on-large analysis. In more detail, suppose that the regions inside and outside the

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spherical balloon are filled with an inviscid compressible fluid. There is a time-harmonic plane wave incident on the sphere, resulting in a scattered wave, motions of the balloon, and a transmitted wave motion inside the sphere. The small motions of the balloon are governed by the incremental equations; see [8, Chap. 4], [16, Chap. 6] and Sect. 3. These equations are linear but they are complicated, in general. The main focus of this paper is on solving the incremental equations.

There is some previous work on scattering in the presence of pre-stress although we are unaware of previous work on acoustic scattering by an inflated balloon. Problems involving cavities in a pre-stressed solid have been studied [18, 19] but such problems have the technical difficulty of specifying the elastic behaviour far from the cavity. The balloon problem has the advantage of a simpler far-field behaviour (where the acoustic waves are governed by the Helmholtz equation) but at the cost of a more difficult (but tractable) quasistatic problem.

Our purpose is to develop a Stroh formalism in spherical polar coordinates because this provides an efficient way of representing the small-on-large solution, as explained below (1.1). A related development was given by Norris and Shuvalov [15] for elastodynamic problems in radially inhomogeneous materials with spherical anisotropy. For our small-on-large problem, the inhomogeneous anisotropy is generated by the underlying quasistatic deformation. Another difference is we have to ensure that the incompressibility constraint is imposed.

We have no need to specify the underlying strain-energy function, so that our formalism is quite general. A Stroh formalism for a restricted class of strain-energy functions has been given by Ciarletta [1]; see the text below (4.42) for more details.

The derivation starts by following Norris and Shuvalov [15], expanding the incremental displacement using vector spherical harmonics; the unknown coefficients are functions of the radial coordinate r and they are collected in a column vector $\mathbf{U}$. The corresponding coefficients for the radial traction vector are collected in a column vector $\mathbf{T}$. Then the Stroh formalism reduces the governing equations to

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}r} = \frac{\mathrm{i}}{r^2}\mathbf{N}(r)\,\boldsymbol{\eta}(r), \tag{1.1}$$

where $\boldsymbol{\eta} = (\mathbf{U}, \mathrm{i}r^2\mathbf{T})^T$ is a 6-component column vector and the 6×6 matrix $\mathbf{N}$ is given explicitly below. Thus (1.1) consists of a coupled system of 6 first-order ordinary differential equations. Efficient methods for solving such systems are available [1, 3]. Evidently, they will require boundary conditions on spherical surfaces. As these are usually specified in terms of displacements and tractions, it is seen that the Stroh formalism is convenient. We note again that the structure of (1.1) does not depend on the choice of strain-energy function employed in the solution of the finite-deformation static problem.

The entries of the matrix $\mathbf{N}$ are functions of r; they involve the incremental constitutive coefficients (denoted by $\mathcal{A}_{0ijkl}$ in [16]) and the hydrostatic pressure (denoted by p_{hp} below) introduced so as to satisfy the incompressibility constraint in the quasistatic large-deformation problem; all these quantities are assumed to be known.

The first of the 6 equations in (1.1) enforces incompressibility. The remaining equations come from the incremental constitutive equation and the incremental equation of motion. Notably absent is p^*, the incremental form of the hydrostatic pressure p_{hp}; if needed, p^* can be calculated later, after (1.1) has been solved.

Our derivation of (1.1) is given in Sect. 4. It is followed by a discussion of alternative formulations and methods, with most details relegated to an Online Resource (Supplementary Material). The application of the Stroh formalism to the balloon problem will be the subject of future work.

2 The Quasistatic Problem: Inflation of a Thick Spherical Shell

Consider a material body in its reference configuration $\mathcal{B}_{\text{ref}}$. A typical point in $\mathcal{B}_{\text{ref}}$ has position vector $\mathbf{X}$. We use Grad and Div to denote the gradient and divergence operators with respect to $\mathbf{X}$. We deform $\mathcal{B}_{\text{ref}}$ into configuration $\mathcal{B}$ so that $\mathbf{X} \in \mathcal{B}_{\text{ref}}$ is taken to $\mathbf{x} \in \mathcal{B}$ by a deformation $\boldsymbol{\chi}$, with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. The deformation gradient tensor is $\mathbf{F} = \text{Grad}\,\boldsymbol{\chi}$.

For $\mathcal{B}_{\text{ref}}$, we consider a thick spherical shell made from an isotropic, incompressible, hyperelastic material. Locate $\mathbf{X} \in \mathcal{B}_{\text{ref}}$ using spherical polar coordinates R, Θ and Φ, where Θ is the polar coordinate and Φ is the azimuthal coordinate. The undeformed shell occupies the region $A < R < B$, $0 \le \Theta \le \pi$, $-\pi < \Phi \le \pi$. After deformation, a point at (R, Θ, Φ) moves to $(r, \theta, \phi) \in \mathcal{B}$. We consider a spherically-symmetric deformation given by

$$r = Rf(R), \qquad \theta = \Theta, \qquad \phi = \Phi. \tag{2.1}$$

For such a deformation, $\mathbf{F}$ is diagonal with respect to polar coordinates,

$$\mathbf{F} = \text{diag}\big(Rf'(R) + f(R), f(R), f(R)\big) = \text{diag}(\lambda_1, \lambda_2, \lambda_3), \tag{2.2}$$

where λ_i is the principal stretch in the ith direction. We have assumed that $i = 1$, 2 and 3 correspond to r, θ and ϕ, respectively.

If the deformed sphere occupies the region $a < r < b$, incompressibility implies that $f(R) = \{1 + (a^3 - A^3)/R^3\}^{1/3}$. One way to see this is to note that the volume between radii A and R, $\frac{4}{3}\pi(R^3 - A^3)$, is unchanged after deformation, and so equals $\frac{4}{3}\pi(r^3 - a^3)$; for more details, see [16, Problem 2.2.16 & §5.3.2]. Alternatively, for incompressible materials, $\det \mathbf{F} = \lambda_1\lambda_2\lambda_3 = 1$, and then (2.2) gives $Rf^2 f' + f^3 = 1$ with general solution $f^3(R) = 1 + CR^{-3}$; the constant C is determined using $a = Af(A)$.

Two stress tensors are of interest, the Cauchy stress $\boldsymbol{\sigma}$ and the nominal stress $\mathbf{S}$. They are related by $\boldsymbol{\sigma} = \mathbf{FS}$; $\boldsymbol{\sigma}$ is symmetric but $\mathbf{S}$ is not, in general.

The configuration $\mathcal{B}$ is in static equilibrium (no body forces): equivalent statements are

$$\text{div}\,\boldsymbol{\sigma} = \mathbf{0}, \qquad \text{Div}\,\mathbf{S} = \mathbf{0}, \tag{2.3}$$

where div is the divergence operator with respect to $\mathbf{x} \in \mathcal{B}$.

For hyperelastic materials, there is a strain-energy function $W(\mathbf{F})$. As our material is also incompressible, we have

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p_{\text{hp}}\mathbf{I}, \qquad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p_{\text{hp}}\mathbf{F}^{-1},$$

where p_{hp} is an arbitrary hydrostatic pressure. For isotropic materials, we can express W in terms of the stretches only, see (2.2), giving $W(\lambda_1, \lambda_2, \lambda_3)$. Then the principal Cauchy stresses are given by [16, Eq. (4.3.49)]

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p_{\text{hp}}, \quad i = 1, 2, 3 \text{ (no sum on } i) \tag{2.4}$$

where $\sigma_1 \equiv \sigma_{rr}$, $\sigma_2 \equiv \sigma_{\theta\theta}$ and $\sigma_3 \equiv \sigma_{\phi\phi}$.

For an incompressible material subject to the deformation (2.1), we have

$$\lambda_1 = \lambda^{-2} \quad \text{and} \quad \lambda_2 = \lambda_3 = \lambda, \quad \text{where } \lambda = r/R = f(R). \tag{2.5}$$

As the three stretches can all be defined in terms of λ, define $\hat{W}(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$. The chain rule gives

$$\frac{\mathrm{d}\hat{W}}{\mathrm{d}\lambda} = \frac{\partial W}{\partial \lambda_1}\frac{\mathrm{d}\lambda_1}{\mathrm{d}\lambda} + \frac{\partial W}{\partial \lambda_2}\frac{\mathrm{d}\lambda_2}{\mathrm{d}\lambda} + \frac{\partial W}{\partial \lambda_3}\frac{\mathrm{d}\lambda_3}{\mathrm{d}\lambda} = -\frac{2}{\lambda^3}\frac{\partial W}{\partial \lambda_1} + \frac{\partial W}{\partial \lambda_2} + \frac{\partial W}{\partial \lambda_3}$$

whence

$$\lambda\frac{\mathrm{d}\hat{W}}{\mathrm{d}\lambda} = -2\lambda_1\frac{\partial W}{\partial \lambda_1} + \lambda_2\frac{\partial W}{\partial \lambda_2} + \lambda_3\frac{\partial W}{\partial \lambda_3} = -2\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{\phi\phi}. \tag{2.6}$$

Note that p_{hp} does not appear in this equation.

Suppose the thick shell is inflated by an internal pressure p_{int}. There is also an external pressure p_{ext}. In equilibrium, both pressures are constant, with $p_{\mathrm{int}} > p_{\mathrm{ext}}$. To relate p_{int} and p_{ext} to the deformation, we have to solve the equilibrium equations (2.3). This was first done by Green and Shield [7, §7]; see also [21, §57, Family 4], [8, §3.10], [16, §5.3.2] and [20, §7.2]. There are three equilibrium equations [16, Eq. (1.5.54)], but two of them are satisfied identically because there is no dependence on θ and ϕ in our problem. The remaining equation is

$$\sigma_{rr}'(r) + r^{-1}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}) = 0, \quad a < r < b. \tag{2.7}$$

We integrate (2.7) between $r = a$ and $r = b$. The boundary conditions are $\sigma_{rr}(a) = -p_{\mathrm{int}}$ and $\sigma_{rr}(b) = -p_{\mathrm{ext}}$. Also, from (2.6), $2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} = -\lambda\hat{W}'(\lambda)$. Hence

$$p_{\mathrm{int}} - p_{\mathrm{ext}} = \int_a^b \hat{W}'(\lambda)\frac{\lambda\,\mathrm{d}r}{r}.$$

Using $\lambda(r) = r/R(r)$ and $R^3 - A^3 = r^3 - a^3$, we find $(\lambda/r)\,\mathrm{d}r/\mathrm{d}\lambda = (1-\lambda^3)^{-1}$, whence [11, Eq. (72)], [16, Eq. (5.3.21)], [6, Eq. (3.19)], [2, Eq. (47)]

$$p_{\mathrm{int}} - p_{\mathrm{ext}} = \int_{\lambda_b}^{\lambda_a} \frac{\hat{W}'(\lambda)}{\lambda^3 - 1}\,\mathrm{d}\lambda, \tag{2.8}$$

where $\lambda_a = \lambda(a) = a/A$ and $\lambda_b = \lambda(b) = b/B$ are related: $\lambda_a^3 - 1 = (B/A)^3(\lambda_b^3 - 1)$. Equation (2.8) relates the pressure jump across the shell to the inner radius a.

The hydrostatic pressure $p_{\mathrm{hp}}(r)$ can be calculated by integrating (2.7) between $r = a$ and r itself giving

$$\lambda_1\frac{\partial W}{\partial \lambda_1} - p_{\mathrm{hp}}(r) + p_{\mathrm{int}} = \int_{\lambda(r)}^{\lambda_a} \frac{\hat{W}'(\lambda)}{\lambda^3 - 1}\,\mathrm{d}\lambda. \tag{2.9}$$

We remark that the solution given above is for an *incompressible* material. The analogous problem for a *compressible* material is more complicated, but some progress can be made; see [16, §5.2.2], [20, Chap. 8] and the review by Horgan [12].

3 Incremental Equations

The basic quantities of interest in a small-on-large analysis are the incremental displacement $\boldsymbol{u}(\mathbf{x}, t)$ and the incremental nominal stress $\boldsymbol{\Sigma}(\mathbf{x}, t)$ (which is not symmetric, in general).

They are related by the incremental equation of motion [16, Eq. (6.4.1)]

$$\operatorname{div} \boldsymbol{\Sigma} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{3.1}$$

where ρ is the mass density, which is constant due to incompressibility. We also have the incremental constitutive equation [11, Eq. (79)], [16, Eq. (6.3.3)], [2, Eq. (29)]

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_A + p_{\text{hp}}\boldsymbol{\Gamma} - p^*\mathbf{I} \quad \text{with} \quad \boldsymbol{\Sigma}_A = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma}, \tag{3.2}$$

where $\boldsymbol{\mathcal{A}}_0$ is the fourth-order tensor of elastic moduli, $\boldsymbol{\Gamma} = \operatorname{grad} \boldsymbol{u}$ is the displacement gradient, and p^* is the incremental form of p_{hp}. The incremental form of the incompressibility constraint reduces to

$$\operatorname{tr} \boldsymbol{\Gamma} \equiv \operatorname{div} \boldsymbol{u} = 0. \tag{3.3}$$

3.1 Use of Spherical Polar Coordinates

Let $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ be the spherical polar unit vectors. We write $\boldsymbol{u} = u_1\hat{\boldsymbol{r}} + u_2\hat{\boldsymbol{\theta}} + u_3\hat{\boldsymbol{\phi}}$ and then put the components in a column vector $\mathbf{u} = (u_1, u_2, u_3)^T$. We shall introduce additional column vectors later.

Let us write the governing equations in terms of spherical polar coordinates, starting with the equation of motion (3.1), as recorded by Ogden [16, Eq. (1.5.54)] and by Fu [4, Eq. (A10)],

$$\frac{\partial \Sigma_{j1}}{\partial \xi_j} + \frac{1}{r}(2\Sigma_{11} - \Sigma_{22} - \Sigma_{33}) + \frac{\cot\theta}{r}\Sigma_{21} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{3.4}$$

$$\frac{\partial \Sigma_{j2}}{\partial \xi_j} + \frac{1}{r}(2\Sigma_{12} + \Sigma_{21}) + \frac{\cot\theta}{r}(\Sigma_{22} - \Sigma_{33}) = \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{3.5}$$

$$\frac{\partial \Sigma_{j3}}{\partial \xi_j} + \frac{1}{r}(2\Sigma_{13} + \Sigma_{31}) + \frac{\cot\theta}{r}(\Sigma_{23} + \Sigma_{32}) = \rho\frac{\partial^2 u_3}{\partial t^2}, \tag{3.6}$$

where $\partial/\partial\xi_1 = \partial/\partial r$, $\partial/\partial\xi_2 = r^{-1}\partial/\partial\theta$ and $\partial/\partial\xi_3 = (r\sin\theta)^{-1}\partial/\partial\phi$.

The constitutive equation (3.2) becomes [4, Eq. (A6)]

$$\Sigma_{ji} = \Sigma^A_{ji} + p_{\text{hp}}\gamma_{ji} - p^*\delta_{ji} \quad \text{with} \quad \Sigma^A_{ji} = \mathcal{A}_{jilk}\gamma_{kl}, \tag{3.7}$$

where we have denoted the components of $\boldsymbol{\mathcal{A}}_0$ by $\mathcal{A}_{ijkl}$; they are given by [2, §3.1]

$$\mathcal{A}_{ijkl} = F_{ip}F_{kq}\frac{\partial^2 W}{\partial F_{jp}\partial F_{lq}} = \mathcal{A}_{klij}, \tag{3.8}$$

but simplified formulas will be given later (Sect. 3.4). For $p_{\text{hp}}(r)$, see (2.9).

The displacement gradients γ_{ij} are given by [10, Eq. (3.7)], [4, Eq. (A9)]

$$\gamma_{11} = \frac{\partial u_1}{\partial r}, \qquad \gamma_{12} = \frac{1}{r}\frac{\partial u_1}{\partial \theta} - \frac{u_2}{r}, \qquad \gamma_{13} = \frac{1}{r\sin\theta}\frac{\partial u_1}{\partial \phi} - \frac{u_3}{r}, \tag{3.9}$$

$$\gamma_{21} = \frac{\partial u_2}{\partial r}, \qquad \gamma_{22} = \frac{1}{r}\frac{\partial u_2}{\partial \theta} + \frac{u_1}{r}, \qquad \gamma_{23} = \frac{1}{r\sin\theta}\frac{\partial u_2}{\partial \phi} - \frac{\cot\theta}{r}u_3, \tag{3.10}$$

$$\gamma_{31} = \frac{\partial u_3}{\partial r}, \qquad \gamma_{32} = \frac{1}{r}\frac{\partial u_3}{\partial \theta}, \qquad \gamma_{33} = \frac{1}{r\sin\theta}\frac{\partial u_3}{\partial \phi} + \frac{u_1}{r} + \frac{\cot\theta}{r}u_2. \tag{3.11}$$

The incompressibility condition (3.3) becomes [10, Eq. (3.8)]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_1\right) + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(u_2\sin\theta) + \frac{\partial u_3}{\partial \phi}\right\} = 0. \tag{3.12}$$

3.2 Rewriting the Equation of Motion

Inspection of (3.4)–(3.6) reveals the common quantity

$$\frac{\partial \Sigma_{jm}}{\partial \xi_j} + \frac{2}{r}\Sigma_{1m} + \frac{\cot\theta}{r}\Sigma_{2m} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\Sigma_{1m}\right) + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(\Sigma_{2m}\sin\theta) + \frac{\partial}{\partial \phi}\Sigma_{3m}\right\}.$$

This suggests that we can write the equation of motion compactly by introducing three traction vectors $\mathbf{t}_j$ with components Σ_{jm},

$$\mathbf{t}_1 = \begin{pmatrix}\Sigma_{11}\\ \Sigma_{12}\\ \Sigma_{13}\end{pmatrix}, \qquad \mathbf{t}_2 = \begin{pmatrix}\Sigma_{21}\\ \Sigma_{22}\\ \Sigma_{23}\end{pmatrix}, \qquad \mathbf{t}_3 = \begin{pmatrix}\Sigma_{31}\\ \Sigma_{32}\\ \Sigma_{33}\end{pmatrix}. \tag{3.13}$$

The result is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathbf{t}_1\right) + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(\mathbf{t}_2\sin\theta) + \frac{\partial}{\partial \phi}\mathbf{t}_3 + \sin\theta\,\mathbf{K}\mathbf{t}_2 + \mathbf{H}\mathbf{t}_3\right\} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{3.14}$$

where

$$\mathbf{K} = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \mathbf{H} = \begin{pmatrix}0 & 0 & -\sin\theta\\ 0 & 0 & -\cos\theta\\ \sin\theta & \cos\theta & 0\end{pmatrix}.$$

This has the same form as an equation obtained by Norris and Shuvalov [15, Eq. (3.1)] for a different elastodynamic problem (one in which the stress tensor is symmetric). For later use, we note that, for any column vector $\mathbf{v} = (v_1, v_2, v_3)^T$,

$$\mathbf{K}\mathbf{v} = \begin{pmatrix}-v_2\\ v_1\\ 0\end{pmatrix}, \qquad \mathbf{H}\mathbf{v} = \begin{pmatrix}-v_3\sin\theta\\ -v_3\cos\theta\\ v_1\sin\theta + v_2\cos\theta\end{pmatrix}. \tag{3.15}$$

3.3 Rewriting the Constitutive Relation

The first piece of the constitutive relation (3.7) is $\Sigma^A_{ji} = \mathcal{A}_{jilk}\gamma_{kl}$, with γ_{ij} given by (3.9)–(3.11). Define column vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ by

$$\mathbf{e}_1 = \frac{\partial \mathbf{u}}{\partial r}, \qquad r\,\mathbf{e}_2 = \frac{\partial \mathbf{u}}{\partial \theta} + \mathbf{K}\mathbf{u} = \left(\frac{\partial u_1}{\partial \theta} - u_2,\ \frac{\partial u_2}{\partial \theta} + u_1,\ \frac{\partial u_3}{\partial \theta}\right)^T, \tag{3.16}$$

$$\begin{aligned}(r\sin\theta)\,\mathbf{e}_3 &= \frac{\partial \mathbf{u}}{\partial \phi} + \mathbf{H}\mathbf{u}\\ &= \left(\frac{\partial u_1}{\partial \phi} - u_3\sin\theta,\ \frac{\partial u_2}{\partial \phi} - u_3\cos\theta,\ \frac{\partial u_3}{\partial \phi} + u_1\sin\theta + u_2\cos\theta\right)^T\end{aligned} \tag{3.17}$$

whence $(\gamma_{1m}, \gamma_{2m}, \gamma_{3m})^T = \mathbf{e}_m$, $m = 1, 2, 3$. From (3.7),

$$\begin{aligned}\Sigma^A_{ji} &= \mathcal{A}_{ji11}\gamma_{11} + \mathcal{A}_{ji12}\gamma_{21} + \mathcal{A}_{ji13}\gamma_{31} + \mathcal{A}_{ji21}\gamma_{12} + \mathcal{A}_{ji22}\gamma_{22} \\ &\quad + \mathcal{A}_{ji23}\gamma_{32} + \mathcal{A}_{ji31}\gamma_{13} + \mathcal{A}_{ji32}\gamma_{23} + \mathcal{A}_{ji33}\gamma_{33} \\ &\quad = (\mathcal{A}_{ji11}, \mathcal{A}_{ji12}, \mathcal{A}_{ji13})\mathbf{e}_1 + (\mathcal{A}_{ji21}, \mathcal{A}_{ji22}, \mathcal{A}_{ji23})\mathbf{e}_2 + (\mathcal{A}_{ji31}, \mathcal{A}_{ji32}, \mathcal{A}_{ji33})\mathbf{e}_3.\end{aligned} \tag{3.18}$$

Then, defining vectors $\mathbf{t}^A_j$ using (3.13) but with Σ_{ij} replaced by Σ^A_{ij} therein, and using the symmetry relation $\mathcal{A}_{ijkl} = \mathcal{A}_{klij}$ (3.8), we obtain

$$\begin{pmatrix}\mathbf{t}^A_1\\ \mathbf{t}^A_2\\ \mathbf{t}^A_3\end{pmatrix} = \begin{pmatrix}\mathbf{Q}_A & \mathbf{R}_A & \mathbf{L}_A\\ \mathbf{R}^T_A & \mathbf{J}_A & \mathbf{S}_A\\ \mathbf{L}^T_A & \mathbf{S}^T_A & \mathbf{M}_A\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \tag{3.19}$$

where

$$\mathbf{Q}_A = \begin{pmatrix}\mathcal{A}_{1111} & \mathcal{A}_{1112} & \mathcal{A}_{1113}\\ \mathcal{A}_{1112} & \mathcal{A}_{1212} & \mathcal{A}_{1213}\\ \mathcal{A}_{1113} & \mathcal{A}_{1213} & \mathcal{A}_{1313}\end{pmatrix}, \qquad \mathbf{R}_A = \begin{pmatrix}\mathcal{A}_{1121} & \mathcal{A}_{1122} & \mathcal{A}_{1123}\\ \mathcal{A}_{1221} & \mathcal{A}_{1222} & \mathcal{A}_{1223}\\ \mathcal{A}_{1321} & \mathcal{A}_{1322} & \mathcal{A}_{1323}\end{pmatrix}, \tag{3.20}$$

$$\mathbf{L}_A = \begin{pmatrix}\mathcal{A}_{1131} & \mathcal{A}_{1132} & \mathcal{A}_{1133}\\ \mathcal{A}_{1231} & \mathcal{A}_{1232} & \mathcal{A}_{1233}\\ \mathcal{A}_{1331} & \mathcal{A}_{1332} & \mathcal{A}_{1333}\end{pmatrix}, \qquad \mathbf{J}_A = \begin{pmatrix}\mathcal{A}_{2121} & \mathcal{A}_{2122} & \mathcal{A}_{2123}\\ \mathcal{A}_{2122} & \mathcal{A}_{2222} & \mathcal{A}_{2223}\\ \mathcal{A}_{2123} & \mathcal{A}_{2223} & \mathcal{A}_{2323}\end{pmatrix}, \tag{3.21}$$

$$\mathbf{S}_A = \begin{pmatrix}\mathcal{A}_{2131} & \mathcal{A}_{2132} & \mathcal{A}_{2133}\\ \mathcal{A}_{2231} & \mathcal{A}_{2232} & \mathcal{A}_{2233}\\ \mathcal{A}_{2331} & \mathcal{A}_{2332} & \mathcal{A}_{2333}\end{pmatrix}, \qquad \mathbf{M}_A = \begin{pmatrix}\mathcal{A}_{3131} & \mathcal{A}_{3132} & \mathcal{A}_{3133}\\ \mathcal{A}_{3132} & \mathcal{A}_{3232} & \mathcal{A}_{3233}\\ \mathcal{A}_{3133} & \mathcal{A}_{3233} & \mathcal{A}_{3333}\end{pmatrix}. \tag{3.22}$$

Equation (3.19) has the same form as [15, Eq. (3.2)].

We note that $\mathbf{Q}_A$, $\mathbf{J}_A$ and $\mathbf{M}_A$ are symmetric matrices. Thus there are $3 \times 6 + 3 \times 9 = 45$ independent entries in the 6 matrices. We will see later that, for our problem, there are just 9 distinct (but related) entries, in general.

The second piece of the constitutive relation (3.7) is $p_{\rm hp}\gamma_{ji}$. The column vector on the left-hand side of (3.19) is $(\Sigma^A_{11}, \Sigma^A_{12}, \Sigma^A_{13}, \Sigma^A_{21}, \Sigma^A_{22}, \Sigma^A_{23}, \Sigma^A_{31}, \Sigma^A_{32}, \Sigma^A_{33})^T$ to which we must add $p_{\rm hp}(\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{31}, \gamma_{32}, \gamma_{33})^T$. But the column vector on the right-hand side of (3.19) is $(\gamma_{11}, \gamma_{21}, \gamma_{31}, \gamma_{12}, \gamma_{22}, \gamma_{32}, \gamma_{13}, \gamma_{23}, \gamma_{33})^T$, so that some simple reordering is required, leading to one change in each of the matrices (3.20)–(3.22). Finally, taking account of the last term in (3.7), we obtain

$$\begin{pmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \mathbf{t}_3\end{pmatrix} = \begin{pmatrix}\mathbf{Q} & \mathbf{R} & \mathbf{L}\\ \mathbf{R}^T & \mathbf{J} & \mathbf{S}\\ \mathbf{L}^T & \mathbf{S}^T & \mathbf{M}\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} - p^*\begin{pmatrix}\mathbf{i}_1\\ \mathbf{i}_2\\ \mathbf{i}_3\end{pmatrix} \tag{3.23}$$

where $\mathbf{i}_1 = (1, 0, 0)^T$, $\mathbf{i}_2 = (0, 1, 0)^T$, $\mathbf{i}_3 = (0, 0, 1)^T$,

$$\mathbf{Q} = \begin{pmatrix}\mathcal{A}^p_{1111} & \mathcal{A}_{1112} & \mathcal{A}_{1113}\\ \mathcal{A}_{1112} & \mathcal{A}_{1212} & \mathcal{A}_{1213}\\ \mathcal{A}_{1113} & \mathcal{A}_{1213} & \mathcal{A}_{1313}\end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix}\mathcal{A}_{1121} & \mathcal{A}_{1122} & \mathcal{A}_{1123}\\ \mathcal{A}^p_{1221} & \mathcal{A}_{1222} & \mathcal{A}_{1223}\\ \mathcal{A}_{1321} & \mathcal{A}_{1322} & \mathcal{A}_{1323}\end{pmatrix}, \tag{3.24}$$

$$\mathbf{L} = \begin{pmatrix}\mathcal{A}_{1131} & \mathcal{A}_{1132} & \mathcal{A}_{1133}\\ \mathcal{A}_{1231} & \mathcal{A}_{1232} & \mathcal{A}_{1233}\\ \mathcal{A}^p_{1331} & \mathcal{A}_{1332} & \mathcal{A}_{1333}\end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix}\mathcal{A}_{2121} & \mathcal{A}_{2122} & \mathcal{A}_{2123}\\ \mathcal{A}_{2122} & \mathcal{A}^p_{2222} & \mathcal{A}_{2223}\\ \mathcal{A}_{2123} & \mathcal{A}_{2223} & \mathcal{A}_{2323}\end{pmatrix}, \tag{3.25}$$

$$\mathbf{S} = \begin{pmatrix} \mathcal{A}_{2131} & \mathcal{A}_{2132} & \mathcal{A}_{2133} \\ \mathcal{A}_{2231} & \mathcal{A}_{2232} & \mathcal{A}_{2233} \\ \mathcal{A}_{2331} & \mathcal{A}^{p}_{2332} & \mathcal{A}_{2333} \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} \mathcal{A}_{3131} & \mathcal{A}_{3132} & \mathcal{A}_{3133} \\ \mathcal{A}_{3132} & \mathcal{A}_{3232} & \mathcal{A}_{3233} \\ \mathcal{A}_{3133} & \mathcal{A}_{3233} & \mathcal{A}^{p}_{3333} \end{pmatrix} \tag{3.26}$$

and we have defined

$$\mathcal{A}^{p}_{ijkl} = \mathcal{A}_{ijkl} + p_{\text{hp}} \tag{3.27}$$

with $p_{\text{hp}}(r)$ given by (2.9).

Alternatively, we can write

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{t}^A_1 \\ \mathbf{t}^A_2 \\ \mathbf{t}^A_3 \end{pmatrix} + p_{\text{hp}} \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} - p^* \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix} \tag{3.28}$$

with $\mathbf{g}_m = (\gamma_{m1}, \gamma_{m2}, \gamma_{m3})^T$; see (3.9)–(3.11) and (3.19). Compare the definition of $\mathbf{g}_m$ with that of $\mathbf{e}_m = (\gamma_{1m}, \gamma_{2m}, \gamma_{3m})^T$.

3.4 The Components of $\mathcal{A}_0$

The components of $\mathcal{A}_0$ are defined by (3.8) in terms of certain derivatives of the strain-energy function W. For an isotropic material, W may be taken as a function of the three principal invariants I_1, I_2 and I_3, defined in terms of the principal stretches by [16, Eq. (4.3.51)]

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2;$$

for an incompressible material, $I_3 = 1$. The chain rule gives

$$\frac{\partial W}{\partial \lambda_p} = \frac{\partial W}{\partial I_m}\frac{\partial I_m}{\partial \lambda_p} = 2\lambda_p \frac{\partial W}{\partial I_1} + 2\lambda_p\left(I_1 - \lambda_p^2\right)\frac{\partial W}{\partial I_2} + \frac{2I_3}{\lambda_p}\frac{\partial W}{\partial I_3}, \tag{3.29}$$

$$\begin{aligned} \frac{\partial^2 W}{\partial \lambda_p\,\partial \lambda_q} &= 2\delta_{pq}\frac{\partial W}{\partial I_1} + 2\left(\delta_{pq}I_1 + 2\lambda_p\lambda_q - 3\lambda_p^2\delta_{pq}\right)\frac{\partial W}{\partial I_2} \\ &\quad + 2I_3\left(\frac{2}{\lambda_p\lambda_q} - \frac{1}{\lambda_p^2}\delta_{pq}\right)\frac{\partial W}{\partial I_3} + 4\lambda_p\lambda_q\frac{\partial^2 W}{\partial I_1^2} + 4\lambda_p\lambda_q\left(I_1 - \lambda_p^2\right)\left(I_1 - \lambda_q^2\right)\frac{\partial^2 W}{\partial I_2^2} \\ &\quad + \frac{4I_3^2}{\lambda_p\lambda_q}\frac{\partial^2 W}{\partial I_3^2} + 4\lambda_p\lambda_q\left(2I_1 - \lambda_p^2 - \lambda_q^2\right)\frac{\partial^2 W}{\partial I_1\,\partial I_2} + 4I_3\left(\frac{\lambda_p}{\lambda_q} + \frac{\lambda_q}{\lambda_p}\right)\frac{\partial^2 W}{\partial I_1\,\partial I_3} \\ &\quad + 4I_3\left(\frac{\lambda_p}{\lambda_q}\left(I_1 - \lambda_p^2\right) + \frac{\lambda_q}{\lambda_p}\left(I_1 - \lambda_q^2\right)\right)\frac{\partial^2 W}{\partial I_2\,\partial I_3}. \end{aligned} \tag{3.30}$$

For an isotropic incompressible material, all the non-zero components of $\mathcal{A}_0$ are given by [16, Eqs. (6.3.15)–(6.3.17)] as follows. First, we have

$$\mathcal{A}_{ppqq} = \lambda_p\lambda_q W_{pq}, \tag{3.31}$$

$$\mathcal{A}_{pqqp} = \mathcal{A}_{qppq} = \mathcal{A}_{pqpq} - \lambda_p W_p, \quad p \neq q, \tag{3.32}$$

$$\mathcal{A}_{pqpq} = (\lambda_p W_p - \lambda_q W_q)\frac{\lambda_p^2}{\lambda_p^2 - \lambda_q^2}, \quad p \neq q,\ \lambda_p \neq \lambda_q, \tag{3.33}$$

where $W_p = \partial W/\partial\lambda_p$ and $W_{pq} = \partial^2 W/\partial\lambda_p\,\partial\lambda_q$. Combining (3.32) and (3.33),

$$\mathcal{A}_{pqqp} = \mathcal{A}_{qppq} = (\lambda_q W_p - \lambda_p W_q)\frac{\lambda_p\lambda_q}{\lambda_p^2 - \lambda_q^2}, \quad p \neq q,\ \lambda_p \neq \lambda_q. \tag{3.34}$$

Second, suppose that $\lambda_i = \lambda_j$ with $i \neq j$. (For our application, $\lambda_2 = \lambda_3$, see (2.5).) In this case, (3.29) gives $W_i = W_j$ whereas (3.30) gives $W_{ii} = W_{jj}$ and $W_{ik} = W_{jk}$. From (3.31),

$$\mathcal{A}_{iiii} = \mathcal{A}_{jjjj} = \lambda_i^2 W_{ii}, \qquad \mathcal{A}_{iijj} = \mathcal{A}_{jjii} = \lambda_i^2 W_{ij},$$
$$\mathcal{A}_{iikk} = \mathcal{A}_{jjkk} = \mathcal{A}_{kkii} = \mathcal{A}_{kkjj} = \lambda_i\lambda_k W_{ik}.$$

For $k \neq i \neq j \neq k$, (3.33) gives $\mathcal{A}_{ikik} = \mathcal{A}_{jkjk}$ and $\mathcal{A}_{kiki} = \mathcal{A}_{kjkj}$, and then (3.32) gives $\mathcal{A}_{ikki} = \mathcal{A}_{kiik} = \mathcal{A}_{jkkj} = \mathcal{A}_{kjjk}$. Finally, in place of (3.33), we have [16, Eq. (6.3.16)]

$$\mathcal{A}_{ijij} = \frac{1}{2}(\mathcal{A}_{iiii} - \mathcal{A}_{iijj} + \lambda_i W_i) = \mathcal{A}_{jiji}. \tag{3.35}$$

For our problem, $\lambda_2 = \lambda_3$, and so the non-zero components are as follows [16, Eq. (6.3.168)]:

$$\mathcal{A}_{1111}, \qquad \mathcal{A}_{1212} = \mathcal{A}_{1313}, \qquad \mathcal{A}_{1122} = \mathcal{A}_{2211} = \mathcal{A}_{1133} = \mathcal{A}_{3311},$$
$$\mathcal{A}_{1221} = \mathcal{A}_{1331} = \mathcal{A}_{2112} = \mathcal{A}_{3113}, \qquad \mathcal{A}_{2222} = \mathcal{A}_{3333}, \qquad \mathcal{A}_{2233} = \mathcal{A}_{3322},$$
$$\mathcal{A}_{2121} = \mathcal{A}_{3131}, \qquad \mathcal{A}_{2323} = \mathcal{A}_{3232}, \qquad \mathcal{A}_{2332} = \mathcal{A}_{3223}.$$

We see that there are 9 different components although they are related because they are given in terms of 6 partial derivatives of W, namely W_1, $W_2 = W_3$, W_{11}, $W_{12} = W_{21} = W_{13} = W_{31}$, $W_{22} = W_{33}$ and $W_{23} = W_{32}$. One useful relation follows by combining (3.32) and (3.35). They give

$$\mathcal{A}_{2332} = \mathcal{A}_{2323} - \lambda_2 W_2 \quad \text{and} \quad 2\mathcal{A}_{2323} = \mathcal{A}_{2222} - \mathcal{A}_{2233} + \lambda_2 W_2.$$

Eliminating $\lambda_2 W_2$, we obtain

$$\mathcal{A}_{2323} + \mathcal{A}_{2332} = \mathcal{A}_{2222} - \mathcal{A}_{2233}. \tag{3.36}$$

Using the non-trivial components of $\boldsymbol{\mathcal{A}}_0$, (3.24)–(3.26) simplify:

$$\mathbf{Q} = \begin{pmatrix} \mathcal{A}^p_{1111} & 0 & 0 \\ 0 & \mathcal{A}_{1212} & 0 \\ 0 & 0 & \mathcal{A}_{1212} \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} 0 & \mathcal{A}_{1122} & 0 \\ \mathcal{A}^p_{1221} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & \mathcal{A}_{1122} \\ 0 & 0 & 0 \\ \mathcal{A}^p_{1221} & 0 & 0 \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} \mathcal{A}_{2121} & 0 & 0 \\ 0 & \mathcal{A}^p_{2222} & 0 \\ 0 & 0 & \mathcal{A}_{2323} \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_{2233} \\ 0 & \mathcal{A}^p_{2332} & 0 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} \mathcal{A}_{2121} & 0 & 0 \\ 0 & \mathcal{A}_{2323} & 0 \\ 0 & 0 & \mathcal{A}^p_{2222} \end{pmatrix}.$$

Recalling (3.27), (3.36) gives

$$\mathcal{A}_{2323} + \mathcal{A}^p_{2332} = \mathcal{A}^p_{2222} - \mathcal{A}_{2233}. \tag{3.37}$$

4 Use of Vector Spherical Harmonics

In this section, we start by following Norris and Shuvalov [15]. As in [14, §3.17], define vector spherical harmonics by

$$\boldsymbol{P}_n^m(\hat{\boldsymbol{r}}) = \hat{\boldsymbol{r}} Y_n^m(\hat{\boldsymbol{r}}) = \hat{\boldsymbol{r}} Y_n^m(\theta,\phi), \tag{4.1}$$

$$\boldsymbol{B}_n^m(\hat{\boldsymbol{r}}) = \frac{1}{\lambda}\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{\sin\theta}\frac{\partial}{\partial\phi}\right) Y_n^m(\theta,\phi), \tag{4.2}$$

$$\boldsymbol{C}_n^m(\hat{\boldsymbol{r}}) = \frac{1}{\lambda}\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right) Y_n^m(\theta,\phi), \tag{4.3}$$

where $\lambda = [n(n+1)]^{-1/2}$ and Y_n^m are spherical harmonics [14, Definition 3.2]. Collecting the components of (4.1)–(4.3) into column vectors, we see the structure

$$\mathbf{P}_n^m = \left(Y_n^m, 0, 0\right)^T, \qquad \mathbf{B}_n^m = \left(0, B_n^m, C_n^m\right)^T, \qquad \mathbf{C}_n^m = \left(0, C_n^m, -B_n^m\right)^T$$

with

$$B_n^m = \lambda^{-1}\partial Y_n^m/\partial\theta \quad \text{and} \quad C_n^m = (\lambda\sin\theta)^{-1}\partial Y_n^m/\partial\phi. \tag{4.4}$$

Suppose

$$\mathbf{u} = \mathrm{Re}\sum_{n,m}\left\{U_{P,n}^m(r)\mathbf{P}_n^m(\hat{\boldsymbol{r}}) + U_{B,n}^m(r)\,\mathbf{B}_n^m(\hat{\boldsymbol{r}}) + U_{C,n}^m(r)\,\mathbf{C}_n^m(\hat{\boldsymbol{r}})\right\}\mathrm{e}^{-\mathrm{i}\omega t},$$

$$p^* = \mathrm{Re}\sum_{n,m}\mathcal{P}_n^m(r)Y_n^m(\hat{\boldsymbol{r}})\,\mathrm{e}^{-\mathrm{i}\omega t}.$$

Exploiting the orthogonality properties of the vector spherical harmonics, we can suppress all sums, subscripts and superscripts, and the harmonic time dependence, giving the structure

$$\mathbf{u} = U_P(r)\mathbf{P}(\hat{\boldsymbol{r}}) + U_B(r)\,\mathbf{B}(\hat{\boldsymbol{r}}) + U_C(r)\,\mathbf{C}(\hat{\boldsymbol{r}}), \quad p^* = \mathcal{P}(r)Y(\hat{\boldsymbol{r}}), \tag{4.5}$$

$$u_1 = U_P(r)Y(\hat{\boldsymbol{r}}), \quad u_2 = U_B(r)B(\hat{\boldsymbol{r}}) + U_C(r)C(\hat{\boldsymbol{r}}), \tag{4.6}$$

$$u_3 = U_B(r)C(\hat{\boldsymbol{r}}) - U_C(r)B(\hat{\boldsymbol{r}}). \tag{4.7}$$

The motion is governed by the equation of motion (3.14), the constitutive relation (3.23) and the incompressibility condition (3.12). We are going to write these equations in Stroh form. The equation of motion (3.14) starts with the radial derivative of $r^2\mathbf{t}_1$, where $\mathbf{t}_1$ is the radial traction column vector; see (3.13). Expanding $\mathbf{t}_1$ using vector spherical harmonics gives

$$\mathbf{t}_1 = T_P(r)\,\mathbf{P}(\hat{\boldsymbol{r}}) + T_B(r)\,\mathbf{B}(\hat{\boldsymbol{r}}) + T_C(r)\,\mathbf{C}(\hat{\boldsymbol{r}}). \tag{4.8}$$

Define column vectors $\mathbf{U} = (U_P, U_B, U_C)^T$ and $\mathbf{T} = (T_P, T_B, T_C)^T$. Then we will see that the governing equations can be cast in Stroh form (see [15, p. 473]) as

$$\boldsymbol{\eta}' = \frac{\mathrm{i}}{r^2}\mathbf{N}\boldsymbol{\eta} \quad \text{with} \quad \boldsymbol{\eta}(r) = \begin{pmatrix}\mathbf{U}\\ \mathrm{i}r^2\mathbf{T}\end{pmatrix}, \tag{4.9}$$

where $\mathbf{N}$ is a 6×6 matrix,

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_4 \end{pmatrix}. \tag{4.10}$$

The four constituent 3×3 matrices will be defined later; see (4.20) and (4.37). In particular, $\mathbf{N}_1$ is diagonal, $\mathbf{N}_2$ is singular, $\mathbf{N}_3 = \mathbf{N}_3^T$ and $\mathbf{N}_4 = -\mathbf{N}_1^T$.

Written out, (4.9) gives

$$\mathbf{U}' = \mathrm{i}r^{-2}\mathbf{N}_1\mathbf{U} - \mathbf{N}_2\mathbf{T}, \qquad \left(r^2\mathbf{T}\right)' = r^{-2}\mathbf{N}_3\mathbf{U} + \mathrm{i}\mathbf{N}_4\mathbf{T}, \tag{4.11}$$

where the first of these will come from the constitutive relation and the incompressibility condition, and the second of (4.11) will come from the equation of motion.

4.1 Incompressibility Condition

Substitution of (4.7) in the incompressibility condition (3.12) yields

$$\frac{Y}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 U_P\right) + \frac{U_B}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(B\sin\theta) + \frac{\partial C}{\partial\phi}\right) + \frac{U_C}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(C\sin\theta) - \frac{\partial B}{\partial\phi}\right) = 0. \tag{4.12}$$

From (4.4) and the partial differential equation satisfied by Y [15, Eq. (3.5)],

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\lambda^2 Y, \tag{4.13}$$

we obtain

$$\frac{\partial}{\partial\theta}(C\sin\theta) - \frac{\partial B}{\partial\phi} = 0, \qquad \frac{\partial}{\partial\theta}(B\sin\theta) + \frac{\partial C}{\partial\phi} = -\lambda Y\sin\theta. \tag{4.14}$$

Equation (4.12) reduces to $(r^2U_P)' = \lambda r U_B$ after use of (4.14). Thus the incompressibility condition is

$$rU_P' = \lambda U_B - 2U_P. \tag{4.15}$$

4.2 Constitutive Relation

From (3.16) and (3.17),

$$\begin{aligned}
\mathbf{e}_1 &= \left(U_P'Y, U_B'B + U_C'C, U_B'C - U_C'B\right)^T = U_P'\mathbf{P} + U_B'\mathbf{B} + U_C'\mathbf{C},\\
\mathbf{e}_2 &= \frac{U_P}{r}(\lambda B, Y, 0)^T + \frac{U_B}{r}\left(-B, \frac{\partial B}{\partial\theta}, \frac{\partial C}{\partial\theta}\right)^T + \frac{U_C}{r}\left(-C, \frac{\partial C}{\partial\theta}, -\frac{\partial B}{\partial\theta}\right)^T,\\
\mathbf{e}_3 &= \frac{U_P}{r}(\lambda C, 0, Y)^T + \frac{U_B}{r}\left(-C, \frac{\partial C}{\partial\theta}, -\lambda Y - \frac{\partial B}{\partial\theta}\right)^T\\
&\quad + \frac{U_C}{r}\left(B, -\lambda Y - \frac{\partial B}{\partial\theta}, -\frac{\partial C}{\partial\theta}\right)^T,
\end{aligned}$$

where we have used (4.4) and (4.14), which give

$$\frac{\partial B}{\partial\phi} - C\cos\theta = \frac{\partial C}{\partial\theta}\sin\theta \quad \text{and} \quad B\cot\theta + \frac{1}{\sin\theta}\frac{\partial C}{\partial\phi} = -\lambda Y - \frac{\partial B}{\partial\theta}.$$

From the first row of (3.23), $\mathbf{t}_1 = \mathbf{Q}\mathbf{e}_1 + \mathbf{R}\mathbf{e}_2 + \mathbf{L}\mathbf{e}_3 - p^*\mathbf{i}_1$. Calculating,

$$\mathbf{Q}\mathbf{e}_1 = \mathcal{A}^p_{1111} U'_P \mathbf{P} + \mathcal{A}_{1212}\big(U'_B \mathbf{B} + U'_C \mathbf{C}\big),$$

$$\mathbf{R}\mathbf{e}_2 + \mathbf{L}\mathbf{e}_3 = \frac{1}{r}\begin{pmatrix} \mathcal{A}_{1122}(2U_P Y - U_B \lambda Y) \\ \mathcal{A}^p_{1221}(U_P \lambda B - U_B B - U_C C) \\ \mathcal{A}^p_{1221}(U_P \lambda C - U_B C + U_C B) \end{pmatrix}.$$

Hence $\mathbf{t}_1 = T_P \mathbf{P} + T_B \mathbf{B} + T_C \mathbf{C}$ (see (4.8)) where, as $p^*\mathbf{i}_1 = \mathcal{P}\mathbf{P}$,

$$T_P = \mathcal{A}^p_{1111} U'_P + r^{-1}\mathcal{A}_{1122}(2U_P - U_B\lambda) - \mathcal{P}, \tag{4.16}$$

$$T_B = \mathcal{A}_{1212} U'_B + r^{-1}\mathcal{A}^p_{1221}(U_P\lambda - U_B), \tag{4.17}$$

$$T_C = \mathcal{A}_{1212} U'_C - r^{-1}\mathcal{A}^p_{1221} U_C. \tag{4.18}$$

Combining the first of these with (4.15) gives

$$\mathcal{P} = -T_P + r^{-1}\big(\mathcal{A}_{1122} - \mathcal{A}^p_{1111}\big)(2U_P - U_B\lambda). \tag{4.19}$$

This enables the determination of the incremental form of the hydrostatic pressure $p^* = \mathcal{P}Y$ once U_P, U_B and T_P have been determined.

Equations (4.15), (4.17) and (4.18) give $\mathbf{U}'$ in terms of $\mathbf{U}$ and $\mathbf{T}$. We rewrite these equations in Stroh form as $\mathbf{U}' = \mathrm{i}r^{-2}\mathbf{N}_1\mathbf{U} - \mathbf{N}_2\mathbf{T}$ (see (4.11)) and find

$$\mathbf{N}_1 = \frac{\mathrm{i}r}{\mathcal{A}_{1212}}\begin{pmatrix} 2\mathcal{A}_{1212} & -\lambda\mathcal{A}_{1212} & 0 \\ \lambda\mathcal{A}^p_{1221} & -\mathcal{A}^p_{1221} & 0 \\ 0 & 0 & -\mathcal{A}^p_{1221} \end{pmatrix}, \qquad \mathbf{N}_2 = \frac{1}{\mathcal{A}_{1212}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.20}$$

Next, evaluate $\mathbf{t}_2 = \mathbf{R}^T\mathbf{e}_1 + \mathbf{J}\mathbf{e}_2 + \mathbf{S}\mathbf{e}_3 - p^*\mathbf{i}_2$ and $\mathbf{t}_3 = \mathbf{L}^T\mathbf{e}_1 + \mathbf{S}^T\mathbf{e}_2 + \mathbf{M}\mathbf{e}_3 - p^*\mathbf{i}_3$:

$$(\mathbf{t}_2)_1 = \mathcal{A}^p_{1221}\big(U'_B B + U'_C C\big) + r^{-1}\mathcal{A}_{2121}\big\{(U_P\lambda - U_B)B - U_C C\big\}, \tag{4.21}$$

$$(\mathbf{t}_3)_1 = \mathcal{A}^p_{1221}\big(U'_B C - U'_C B\big) + r^{-1}\mathcal{A}_{2121}\big\{(U_P\lambda - U_B)C + U_C B\big\}, \tag{4.22}$$

$$\begin{aligned}(\mathbf{t}_2)_2 = {} & \mathcal{A}_{1122} U'_P Y + r^{-1}\big(\mathcal{A}^p_{2222} + \mathcal{A}_{2233}\big)U_P Y - r^{-1}\mathcal{A}_{2233} U_B \lambda Y \\ & + \frac{1}{r}\big(\mathcal{A}^p_{2222} - \mathcal{A}_{2233}\big)\left(U_B\frac{\partial B}{\partial\theta} + U_C\frac{\partial C}{\partial\theta}\right) - \mathcal{P}Y,\end{aligned} \tag{4.23}$$

$$(\mathbf{t}_3)_2 = -\frac{1}{r}\mathcal{A}_{2323} U_C \lambda Y + \frac{1}{r}\big(\mathcal{A}_{2323} + \mathcal{A}^p_{2332}\big)\left(U_B\frac{\partial C}{\partial\theta} - U_C\frac{\partial B}{\partial\theta}\right), \tag{4.24}$$

$$(\mathbf{t}_2)_3 = -\frac{1}{r}\mathcal{A}^p_{2332} U_C \lambda Y + \frac{1}{r}\big(\mathcal{A}_{2323} + \mathcal{A}^p_{2332}\big)\left(U_B\frac{\partial C}{\partial\theta} - U_C\frac{\partial B}{\partial\theta}\right), \tag{4.25}$$

$$\begin{aligned}(\mathbf{t}_3)_3 = {} & \mathcal{A}_{1122} U'_P Y + r^{-1}\big(\mathcal{A}^p_{2222} + \mathcal{A}_{2233}\big)U_P Y - r^{-1}\mathcal{A}^p_{2222} U_B \lambda Y \\ & - \frac{1}{r}\big(\mathcal{A}^p_{2222} - \mathcal{A}_{2233}\big)\left(U_B\frac{\partial B}{\partial\theta} + U_C\frac{\partial C}{\partial\theta}\right) - \mathcal{P}Y.\end{aligned} \tag{4.26}$$

These formulas will be used in Sect. 4.3.

4.3 The Equation of Motion

The equation of motion is (3.14), which we write as

$$\frac{\partial}{\partial r}\big(r^2\mathbf{t}_1\big)+\mathbf{z}=\rho r^2\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.27}$$

where $\mathbf{z}=(z_1,z_2,z_3)^T$ is defined by

$$\mathbf{z}=\frac{r}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\mathbf{t}_2\sin\theta)+\frac{\partial}{\partial\phi}\mathbf{t}_3+\sin\theta\,\mathbf{K}\mathbf{t}_2+\mathbf{H}\mathbf{t}_3\right). \tag{4.28}$$

We start by evaluating the components of $\mathbf{z}$ before returning to (4.27) in Sect. 4.3.4.

4.3.1 The Radial Component

For the radial component of (4.28), z_1, substitution of (4.21) and (4.22) gives

$$\frac{r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\big[(\mathbf{t}_2)_1\sin\theta\big]+\frac{\partial}{\partial\phi}(\mathbf{t}_3)_1\right\}=-\lambda\big\{\mathcal{A}^p_{1221}rU'_B+\mathcal{A}_{2121}(U_P\lambda-U_B)\big\}Y$$

after using (4.14). To this, we add (see (4.28))

$$\begin{aligned}r\big(\mathbf{K}\mathbf{t}_2+[\sin\theta]^{-1}\mathbf{H}\mathbf{t}_3\big)_1&=-r(\mathbf{t}_2)_2-r(\mathbf{t}_3)_3\\&=-\big\{2\mathcal{A}_{1122}rU'_P+\big(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\big)(2U_P-U_B\lambda)\big\}Y+2r\mathcal{P}Y,\end{aligned}$$

using (3.15) and (4.14). Hence

$$\begin{aligned}z_1=2r\big(\mathcal{P}-\mathcal{A}_{1122}U'_P\big)Y-\big(2\mathcal{A}^p_{2222}+2\mathcal{A}_{2233}-\mathcal{A}_{2121}\lambda^2\big)U_PY\\-\lambda\mathcal{A}^p_{1221}rU'_BY+\big(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}+\mathcal{A}_{2121}\big)U_B\lambda Y.\end{aligned} \tag{4.29}$$

4.3.2 The Polar Component

For the θ-component of $\mathbf{z}$, z_2, we require

$$\begin{aligned}&r\left\{\frac{\partial}{\partial\theta}\big[(\mathbf{t}_2)_2\sin\theta\big]+\frac{\partial}{\partial\phi}(\mathbf{t}_3)_2\right\}\\&=\big(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\big)\frac{\partial}{\partial\theta}\left\{\sin\theta\left(U_B\frac{\partial B}{\partial\theta}+U_C\frac{\partial C}{\partial\theta}\right)\right\}\\&\quad+\big\{\mathcal{A}_{1122}rU'_P+\big(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\big)U_P-\mathcal{A}_{2233}U_B\lambda-r\mathcal{P}\big\}\frac{\partial}{\partial\theta}(Y\sin\theta)\\&\quad-\mathcal{A}_{2323}U_C\lambda\frac{\partial Y}{\partial\phi}+\big(\mathcal{A}_{2323}+\mathcal{A}^p_{2332}\big)\frac{\partial}{\partial\phi}\left(U_B\frac{\partial C}{\partial\theta}-U_C\frac{\partial B}{\partial\theta}\right)\\&=\big\{\mathcal{A}_{1122}rU'_P+\big(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\big)U_P-\mathcal{A}_{2233}U_B\lambda-r\mathcal{P}\big\}(Y\cos\theta+\lambda B\sin\theta)\\&\quad-\mathcal{A}_{2323}U_C\lambda^2C\sin\theta+\big(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\big)U_B\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B}{\partial\theta}\right)+\frac{\partial^2C}{\partial\theta\,\partial\phi}\right\}\end{aligned}$$

$$+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)U_C\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial C}{\partial\theta}\right)-\frac{\partial^2 B}{\partial\theta\,\partial\phi}\right\}$$

where we have used the relation (3.37) and

$$\frac{\partial}{\partial\theta}(Y\sin\theta)=\lambda B\sin\theta+Y\cos\theta,\qquad \frac{\partial Y}{\partial\phi}=\lambda C\sin\theta.$$

Eliminating the ϕ-derivatives using (4.14), we obtain

$$\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial B}{\partial\theta}\right)+\frac{\partial^2 C}{\partial\theta\,\partial\phi}=-\lambda Y\cos\theta+\left(1-\lambda^2\right)B\sin\theta-\cos\theta\,\frac{\partial B}{\partial\theta},\tag{4.30}$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial C}{\partial\theta}\right)-\frac{\partial^2 B}{\partial\theta\,\partial\phi}=C\sin\theta-\cos\theta\,\frac{\partial C}{\partial\theta}.\tag{4.31}$$

Hence

$$\begin{aligned}
&\frac{r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[(\mathbf{t}_2)_2\sin\theta\right]+\frac{\partial}{\partial\phi}(\mathbf{t}_3)_2\right\}\\
&\quad=-r\mathcal{P}(Y\cot\theta+\lambda B)+\left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P-\mathcal{A}^p_{2222}U_B\lambda\right\}Y\cot\theta\\
&\qquad+\left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P-\mathcal{A}_{2233}U_B\lambda\right\}\lambda B\\
&\qquad+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\left(1-\lambda^2\right)U_B B+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2323}\lambda^2\right)U_C C\\
&\qquad-\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\left(U_B\frac{\partial B}{\partial\theta}+U_C\frac{\partial C}{\partial\theta}\right)\cot\theta.
\end{aligned}$$

To this, we add (see (4.28))

$$\begin{aligned}
r\left(\mathbf{K}\mathbf{t}_2+[\sin\theta]^{-1}\mathbf{H}\mathbf{t}_3\right)_2&=r(\mathbf{t}_2)_1-r(\mathbf{t}_3)_3\cot\theta\\
&=\mathcal{A}^p_{1221}r\left(U_B'B+U_C'C\right)+\mathcal{A}_{2121}\left\{(U_P\lambda-U_B)B-U_C C\right\}\\
&\quad-\left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P-\mathcal{A}^p_{2222}U_B\lambda\right\}Y\cot\theta\\
&\quad+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\left(U_B\frac{\partial B}{\partial\theta}+U_C\frac{\partial C}{\partial\theta}\right)\cot\theta+r\mathcal{P}Y\cot\theta
\end{aligned}$$

and obtain

$$\begin{aligned}
z_2&=\left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P-\mathcal{A}_{2233}U_B\lambda\right\}\lambda B\\
&\quad+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\left(1-\lambda^2\right)U_B B-r\mathcal{P}\lambda B+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2323}\lambda^2\right)U_C C\\
&\quad+\mathcal{A}^p_{1221}r\left(U_B'B+U_C'C\right)+\mathcal{A}_{2121}\left\{(U_P\lambda-U_B)B-U_C C\right\}\\
&=-r\left(\mathcal{P}-\mathcal{A}_{1122}U_P'\right)\lambda B+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}+\mathcal{A}_{2121}\right)U_P\lambda B\\
&\quad+\left\{\mathcal{A}^p_{1221}rU_B'+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2121}-\mathcal{A}^p_{2222}\lambda^2\right)U_B\right\}B\\
&\quad+\left\{\mathcal{A}^p_{1221}rU_C'+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2121}-\mathcal{A}_{2323}\lambda^2\right)U_C\right\}C.
\end{aligned}\tag{4.32}$$

4.3.3 The Azimuthal Component

For the ϕ-component of $\mathbf{z}$, z_3, we want

$$\begin{aligned}
& r\left\{\frac{\partial}{\partial\theta}\left[(\mathbf{t}_2)_3\sin\theta\right]+\frac{\partial}{\partial\phi}(\mathbf{t}_3)_3\right\} \\
&= \left(\mathcal{A}_{2323}+\mathcal{A}^p_{2332}\right)\frac{\partial}{\partial\theta}\left\{\sin\theta\left(U_B\frac{\partial C}{\partial\theta}-U_C\frac{\partial B}{\partial\theta}\right)\right\}-\mathcal{A}^p_{2332}U_C\lambda\frac{\partial}{\partial\theta}(Y\sin\theta) \\
&\quad + \left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P-\mathcal{A}^p_{2222}U_B\lambda\right\}\frac{\partial Y}{\partial\phi} \\
&\quad - \left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\frac{\partial}{\partial\phi}\left(U_B\frac{\partial B}{\partial\theta}+U_C\frac{\partial C}{\partial\theta}\right)-r\mathcal{P}\frac{\partial Y}{\partial\phi} \\
&= \left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)U_B\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial C}{\partial\theta}\right)-\frac{\partial^2 B}{\partial\theta\,\partial\phi}\right\}-r\mathcal{P}\lambda C\sin\theta \\
&\quad - \left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)U_C\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B}{\partial\theta}\right)+\frac{\partial^2 C}{\partial\theta\,\partial\phi}\right\}-\mathcal{A}^p_{2332}U_C\lambda(\lambda B\sin\theta+Y\cos\theta) \\
&\quad + \left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P-\mathcal{A}^p_{2222}U_B\lambda\right\}\lambda C\sin\theta.
\end{aligned}$$

Hence, using (4.30) and (4.31),

$$\begin{aligned}
& \frac{r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[(\mathbf{t}_2)_3\sin\theta\right]+\frac{\partial}{\partial\phi}(\mathbf{t}_3)_3\right\} \\
&= \mathcal{A}_{2323}U_C\lambda Y\cot\theta-r\mathcal{P}\lambda C+\left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P\right\}\lambda C \\
&\quad + \left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}^p_{2222}\lambda^2\right)U_BC-\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2323}\lambda^2\right)U_CB \\
&\quad - \left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\left(U_B\frac{\partial C}{\partial\theta}-U_C\frac{\partial B}{\partial\theta}\right)\cot\theta.
\end{aligned}$$

To this, we add

$$\begin{aligned}
& r\left(\mathbf{K}\mathbf{t}_2+[\sin\theta]^{-1}\mathbf{H}\mathbf{t}_3\right)_3 = r(\mathbf{t}_3)_1+r(\mathbf{t}_3)_2\cot\theta \\
&= \mathcal{A}^p_{1221}r\left(U_B'C-U_C'B\right)+\mathcal{A}_{2121}\left\{(U_P\lambda-U_B)C+U_CB\right\}-\mathcal{A}_{2323}U_C\lambda Y\cot\theta \\
&\quad + \left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}\right)\left(U_B\frac{\partial C}{\partial\theta}-U_C\frac{\partial B}{\partial\theta}\right)\cot\theta
\end{aligned}$$

and obtain

$$\begin{aligned}
z_3 &= \left\{\mathcal{A}_{1122}rU_P'+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}\right)U_P\right\}\lambda C+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}^p_{2222}\lambda^2\right)U_BC \\
&\quad - r\mathcal{P}\lambda C-\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2323}\lambda^2\right)U_CB+\mathcal{A}^p_{1221}r\left(U_B'C-U_C'B\right) \\
&\quad + \mathcal{A}_{2121}\left\{(U_P\lambda-U_B)C+U_CB\right\} \\
&= -r\left(\mathcal{P}-\mathcal{A}_{1122}U_P'\right)\lambda C+\left(\mathcal{A}^p_{2222}+\mathcal{A}_{2233}+\mathcal{A}_{2121}\right)U_P\lambda C \\
&\quad + \left\{\mathcal{A}^p_{1221}rU_B'+\left(\mathcal{A}^p_{2222}-\mathcal{A}_{2233}-\mathcal{A}_{2121}-\mathcal{A}^p_{2222}\lambda^2\right)U_B\right\}C
\end{aligned}$$

$$-\left\{\mathcal{A}^p_{1221} r U'_C + \left(\mathcal{A}^p_{2222} - \mathcal{A}_{2233} - \mathcal{A}_{2121} - \mathcal{A}_{2323}\lambda^2\right) U_C\right\} B. \tag{4.33}$$

4.3.4 Synthesis

Inspection of (4.29), (4.32) and (4.33) shows that

$$\mathbf{z} = Z_P \mathbf{P} + Z_B \mathbf{B} + Z_C \mathbf{C}, \tag{4.34}$$

where

$$\begin{aligned}
Z_P &= 2r\left(\mathcal{P} - \mathcal{A}_{1122} U'_P\right) - \left(2\mathcal{A}^p_{2222} + 2\mathcal{A}_{2233} - \mathcal{A}_{2121}\lambda^2\right) U_P \\
&\quad - \lambda \mathcal{A}^p_{1221} r U'_B + \left(\mathcal{A}^p_{2222} + \mathcal{A}_{2233} + \mathcal{A}_{2121}\right) U_B \lambda, \\
Z_B &= -r\left(\mathcal{P} - \mathcal{A}_{1122} U'_P\right)\lambda + \left(\mathcal{A}^p_{2222} + \mathcal{A}_{2233} + \mathcal{A}_{2121}\right) U_P \lambda \\
&\quad + \mathcal{A}^p_{1221} r U'_B + \left(\mathcal{A}^p_{2222} - \mathcal{A}_{2233} - \mathcal{A}_{2121} - \mathcal{A}^p_{2222}\lambda^2\right) U_B, \\
Z_C &= \mathcal{A}^p_{1221} r U'_C + \left(\mathcal{A}^p_{2222} - \mathcal{A}_{2233} - \mathcal{A}_{2121} - \mathcal{A}_{2323}\lambda^2\right) U_C.
\end{aligned}$$

We eliminate $r(\mathcal{P} - \mathcal{A}_{1122} U'_P)$, rU'_B and rU'_C in favour of $\mathbf{U}$ and $\mathbf{T}$. Thus, combining (4.15) and (4.19) gives

$$r\left(\mathcal{P} - \mathcal{A}_{1122} U'_P\right) = -rT_P + \left(2\mathcal{A}_{1122} - \mathcal{A}^p_{1111}\right)(2U_P - U_B\lambda). \tag{4.35}$$

Also (4.17) gives $\mathcal{A}_{1212} r U'_B = rT_B + \mathcal{A}^p_{1221}(U_B - U_P\lambda)$ and (4.18) gives $\mathcal{A}_{1212} r U'_C = rT_C + \mathcal{A}^p_{1221} U_C$. Substitution then gives

$$\mathbf{Z} = \frac{(\mathcal{A}^p_{1221})^2}{\mathcal{A}_{1212}} \begin{pmatrix} \lambda^2 & -\lambda & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{U} - \begin{pmatrix} \nu_1 & \nu_2 & 0 \\ \nu_2 & \nu_3 & 0 \\ 0 & 0 & \nu_4 \end{pmatrix} \mathbf{U} + \mathrm{i}\, \mathbf{N}_1^T \mathbf{T}, \tag{4.36}$$

where $\mathbf{Z} = (Z_P, Z_B, Z_C)^T$, $\mathbf{N}_1$ is defined by (4.20),

$$\begin{aligned}
\nu_1 &= 4\mathcal{A}^p_{1111} - 8\mathcal{A}_{1122} + 2\mathcal{A}^p_{2222} + 2\mathcal{A}_{2233} - \mathcal{A}_{2121}\lambda^2, \\
\nu_2 &= \lambda\left(4\mathcal{A}_{1122} - 2\mathcal{A}^p_{1111} - \mathcal{A}^p_{2222} - \mathcal{A}_{2233} - \mathcal{A}_{2121}\right), \\
\nu_3 &= \mathcal{A}_{2233} - \mathcal{A}^p_{2222} + \mathcal{A}_{2121} + \left(\mathcal{A}^p_{1111} - 2\mathcal{A}_{1122} + \mathcal{A}^p_{2222}\right)\lambda^2, \\
\nu_4 &= \mathcal{A}_{2233} - \mathcal{A}^p_{2222} + \mathcal{A}_{2121} + \mathcal{A}_{2323}\lambda^2.
\end{aligned}$$

Substituting (4.5), (4.8) and (4.34) in (4.27), we obtain

$$\left(r^2 \mathbf{T}\right)' = -\mathbf{Z} - \rho r^2 \omega^2 \mathbf{U}.$$

Comparison with (4.11), $(r^2\mathbf{T})' = r^{-2}\mathbf{N}_3\mathbf{U} + \mathrm{i}\,\mathbf{N}_4\mathbf{T}$, then gives $\mathbf{N}_4 = -\mathbf{N}_1^T$ and

$$\frac{1}{r^2}\mathbf{N}_3 = \begin{pmatrix} \nu_1 & \nu_2 & 0 \\ \nu_2 & \nu_3 & 0 \\ 0 & 0 & \nu_4 \end{pmatrix} - \frac{(\mathcal{A}^p_{1221})^2}{\mathcal{A}_{1212}} \begin{pmatrix} \lambda^2 & -\lambda & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \rho r^2 \omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.37}$$

We note that $\mathbf{N}_3$ is symmetric, $\mathbf{N}_3 = \mathbf{N}_3^T$.

4.4 Decoupling

The Stroh formulation gives a 6×6 system of coupled first-order ordinary differential equations (ODEs) for $(\mathbf{U}, \mathbf{T})$ (4.9), which we write as

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\mathbf{U}\\ \mathrm{i}r^2\mathbf{T}\end{pmatrix} = \frac{\mathrm{i}}{r^2}\begin{pmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & -\mathbf{N}_1^T\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \mathrm{i}r^2\mathbf{T}\end{pmatrix}. \tag{4.38}$$

Once solved, the incremental pressure $p^* = \mathcal{P}Y$ is given by (4.19).

As might be expected from the symmetry of the problem (and from the analysis in [15]) the system (4.38) can be decoupled into a 2×2 system for (U_C, T_C) and a 4×4 system for (U_P, U_B, T_P, T_B). No generality is lost by making this decoupling.

To obtain the simpler system, suppose that $\mathbf{U} = (0, 0, U_C)^T$. Then the first piece of the Stroh system (4.11), $\mathbf{U}' = \mathrm{i}r^{-2}\mathbf{N}_1\mathbf{U} - \mathbf{N}_2\mathbf{T}$, reduces to

$$\mathcal{A}_{1212}\big(0, 0, U_C'\big) = r^{-1}\mathcal{A}_{1221}^p(0, 0, U_C) + (0, T_B, T_C),$$

which gives $T_B = 0$ and (4.18). Using $T_B = 0$, the second piece of (4.11), $(r^2\mathbf{T})' = r^{-2}\mathbf{N}_3\mathbf{U} + \mathrm{i}\,\mathbf{N}_4\mathbf{T}$, becomes

$$\begin{aligned}\big(\big(r^2T_P\big)', 0, \big(r^2T_C\big)'\big) &= (0, 0, \mu_{33}U_C)\\ &\quad + (r/\mathcal{A}_{1212})\big(2\mathcal{A}_{1212}T_P, -\lambda\mathcal{A}_{1212}T_P, -\mathcal{A}_{1221}^p T_C\big),\end{aligned} \tag{4.39}$$

where $\mu_{33} = \nu_4 - (\mathcal{A}_{1221}^p)^2/\mathcal{A}_{1212} - \rho r^2\omega^2$. The second component of (4.39) gives $T_P = 0$ so that the first component of (4.39) is satisfied. The remaining component yields a second-order ODE for $U_C(r)$ after T_C is eliminated using (4.18). The ODE does not simplify further, in general, because it contains $\mathcal{A}_{1212}(r)$ and $\mathcal{A}_{1221}^p = \mathcal{A}_{1221}(r) + p_{\mathrm{hp}}(r)$. From (4.19), we obtain $\mathcal{P} = 0$; it is not surprising that $p^* = \mathcal{P}Y = 0$ because the incompressibility condition (4.15) is satisfied identically.

For the 4×4 system, suppose that $\mathbf{U} = (U_P, U_B, 0)^T$. The third component of the Stroh system (which is equivalent to (4.18)) then gives $T_C = 0$ and the sixth component is seen to be satisfied. This leaves a 4×4 system

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\widetilde{\mathbf{U}}\\ \mathrm{i}r^2\widetilde{\mathbf{T}}\end{pmatrix} = \frac{\mathrm{i}}{r^2}\begin{pmatrix}\widetilde{\mathbf{N}}_1 & \widetilde{\mathbf{N}}_2\\ \widetilde{\mathbf{N}}_3 & -\widetilde{\mathbf{N}}_1^T\end{pmatrix}\begin{pmatrix}\widetilde{\mathbf{U}}\\ \mathrm{i}r^2\widetilde{\mathbf{T}}\end{pmatrix} \tag{4.40}$$

where $\widetilde{\mathbf{U}} = (U_P, U_B)^T$, $\widetilde{\mathbf{T}} = (T_P, T_B)^T$,

$$\widetilde{\mathbf{N}}_1 = \frac{\mathrm{i}r}{\mathcal{A}_{1212}}\begin{pmatrix}2\mathcal{A}_{1212} & -\lambda\mathcal{A}_{1212}\\ \lambda\mathcal{A}_{1221}^p & -\mathcal{A}_{1221}^p\end{pmatrix}, \qquad \widetilde{\mathbf{N}}_2 = \frac{1}{\mathcal{A}_{1212}}\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}, \tag{4.41}$$

$$\frac{1}{r^2}\widetilde{\mathbf{N}}_3 = \begin{pmatrix}\nu_1 & \nu_2\\ \nu_2 & \nu_3\end{pmatrix} - \frac{(\mathcal{A}_{1221}^p)^2}{\mathcal{A}_{1212}}\begin{pmatrix}\lambda^2 & -\lambda\\ -\lambda & 1\end{pmatrix} - \rho r^2\omega^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.42}$$

The formula for $\mathcal{P}$, (4.19), is unchanged.

The system (4.40) is similar to one obtained by Ciarletta (see Sect. 1.3 in the Supplementary Material for [1]) for a simpler (static) problem, one in which $\mathcal{A}_{ijkl} = \mu\lambda_i^2\delta_{ik}\delta_{jl}$ (no sum over i, μ is a constant); see just below [1, Eq. (3)]. See also [17, Eq. (26)], where a residually-stressed solid sphere is studied.

5 Alternative Approaches

Recall the basic incremental equations, (3.1)–(3.3),

$$\operatorname{div}\boldsymbol{\Sigma} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_A + p_{\text{hp}}\boldsymbol{\Gamma} - p^*\mathbf{I}, \ \operatorname{div}\boldsymbol{u} = 0, \tag{5.1}$$

where $\boldsymbol{\Sigma}_A = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma} = \operatorname{grad}\boldsymbol{u}$. If we eliminate $\boldsymbol{\Sigma}$ between the first two of (5.1), we can use [16, p. 50] $\operatorname{div}(p_{\text{hp}}\boldsymbol{\Gamma}) = (\operatorname{grad} p_{\text{hp}})\boldsymbol{\Gamma} + p_{\text{hp}}\operatorname{div}\boldsymbol{\Gamma}$. However, incompressibility implies that $\operatorname{div}\boldsymbol{\Gamma} = \mathbf{0}$. (This is easily seen by reverting to Cartesian components.) Consequently, $\operatorname{grad} p_{\text{hp}}$ appears in the equation of motion but not p_{hp} itself:

$$\operatorname{div}\boldsymbol{\Sigma} = \operatorname{div}\boldsymbol{\Sigma}_A + \boldsymbol{\Gamma}^T\operatorname{grad} p_{\text{hp}} - \operatorname{grad} p^* = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}. \tag{5.2}$$

Eliminating the p_{hp} terms has been a common feature in previous analyses. Our analysis began with the equation of motion in the form (3.14). If we had started with (5.2), we would have found the following equivalent form,

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathbf{t}_1^A\right) + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\mathbf{t}_2^A\sin\theta\right) + \frac{\partial}{\partial\phi}\mathbf{t}_3^A + \sin\theta\,\mathbf{K}\mathbf{t}_2^A + \mathbf{H}\mathbf{t}_3^A\right\} \\ &\quad + \begin{pmatrix}\mathbf{p}^T\mathbf{e}_1\\ \mathbf{p}^T\mathbf{e}_2\\ \mathbf{p}^T\mathbf{e}_3\end{pmatrix} - \begin{pmatrix}\partial p^*/\partial r\\ r^{-1}\partial p^*/\partial\theta\\ (r\sin\theta)^{-1}\partial p^*/\partial\phi\end{pmatrix} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \end{aligned} \tag{5.3}$$

where $\mathbf{p} = (\partial p_{\text{hp}}/\partial r, r^{-1}\partial p_{\text{hp}}/\partial\theta, (r\sin\theta)^{-1}\partial p_{\text{hp}}/\partial\phi)^T$. When p_{hp} is spherically symmetric, $\mathbf{p}^T\mathbf{e}_m = p'_{\text{hp}}(r)\gamma_{1m}$, $m = 1, 2, 3$.

We have developed a Stroh-type formulation starting from (5.3). However, we do not give details here because it seems to be inferior to (4.38). One reason concerns the application of boundary conditions on spherical surfaces. Then we usually want $\mathbf{t}_1$, not $\mathbf{t}_1^A$. Of course, $\mathbf{t}_1$ and $\mathbf{t}_1^A$ are related by the first line of (3.28), but this leads to another difficulty: how to calculate p^*? A related question is: how is incompressibility to be enforced (and not just used)? With (4.38), incompressibility is enforced explicitly by the first equation of the 6×6 system. The system itself does not involve p^*, which is computed by a separate formula (4.19).

Most previous analyses have derived coupled second-order partial differential equations (PDEs) for the components of $\mathbf{u}$ prior to the introduction of spherical harmonics. Inevitably, these PDEs are complicated, and comparisons are not straightforward; one reason is that multiples of the incompressibility equation (3.12) can be added. Nevertheless, we have made comparisons; see the Online Resource (Supplementary Material) for details.

The most general formulation is that given by Haughton and Chen [10]. For the radial equation, we find complete agreement with [10, Eq. (3.9)]. For the polar equation, we agree with [10, Eq. (3.10)] apart from one discrepancy: in their coefficient multiplying $w_\phi \equiv \partial u_3/\partial\phi$ replace $(B_{2233} + B_{2332})$ by $(B_{2222} + B_{2323})$. For the azimuthal equation, we agree with [10, Eq. (3.11)] apart from one discrepancy: in their coefficient multiplying $u_\phi \equiv \partial u_1/\partial\phi$ replace B_{1122} by B_{1221}.

Several publications are concerned with axisymmetric solutions, implying that there is no dependence on the azimuthal angle ϕ and $u_3 = 0$. In this case, a single PDE for u_1 can be derived; we find agreement with [11, Eq. (92)], [16, Eq. (6.3.173)] and [2, Eq. (64)]. Another PDE relates u_1 and u_2; we find complete agreement with [11, Eq. (93)] and [16, Eq. (6.3.174)]. We also see that there is a typographical error in the second line of [2, Eq. (65)]: replace $\mathrm{d}A_{01221}/\mathrm{d}r$ by $\mathrm{d}A_{01212}/\mathrm{d}r$. For another discussion, see [5, §15.6].

It seems worthwhile to make these comparisons because the equations themselves are so complicated, especially when compared to the elegantly compact Stroh formalism.

6 Prospects

We have said little so far about the original problem, scattering of sound by an inflated spherical balloon. The Stroh formalism for the solid must be combined with the acoustic fields inside and outside the balloon; these scalar fields are represented in a standard way using separated solutions of the governing Helmholtz equation [14, §4.6]. The fields are then connected using transmission conditions across the spherical interfaces at $r = a$ and $r = b$. All this remains to be done.

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References

1. Ciarletta, P.: Buckling instability in growing tumor spheroids. Phys. Rev. Lett. **110**, 158102 (2013)
2. deBotton, G., Bustamante, R., Dorfmann, A.: Axisymmetric bifurcations of thick spherical shells under inflation and compression. Int. J. Solids Struct. **50**, 403–413 (2013)
3. Destrade, M., N Annaidh, A., Coman, C.D.: Bending instabilities of soft biological tissues. Int. J. Solids Struct. **46**, 4322–4330 (2009)
4. Fu, Y.B.: Perturbation methods and nonlinear stability analysis. In: Fu, Y.B., Ogden, R.W. (eds.) Nonlinear Elasticity, pp. 345–391. Cambridge University Press, Cambridge (2001)
5. Goriely, A.: The Mathematics and Mechanics of Biological Growth. Springer, New York (2017)
6. Goriely, A., Destrade, M., Ben Amar, M.: Instabilities in elastomers and in soft tissues. Q. J. Mech. Appl. Math. **59**, 615–630 (2006)
7. Green, A.E., Shield, R.T.: Finite elastic deformation of incompressible isotropic bodies. Proc. R. Soc. A **202**, 407–419 (1950)
8. Green, A.E., Zerna, W.: Theoretical Elasticity, 2nd edn. Dover, New York (1992). Reprint of 1968 original
9. Gunderson, A.M., Daniel, T.D., Marston, P.L., Isakson, M.J.: Observation and modeling of acoustic scattering from a rubber spherical shell. J. Acoust. Soc. Am. **143**, 3036–3046 (2018)
10. Haughton, D.M., Chen, Y-c.: On the eversion of incompressible elastic spherical shells. Z. Angew. Math. Phys. **50**, 312–326 (1999)
11. Haughton, D.M., Ogden, R.W.: On the incremental equations in non-linear elasticity—II. Bifurcation of pressurized spherical shells. J. Mech. Phys. Solids **26**, 111–138 (1978)
12. Horgan, C.O.: Equilibrium solutions for compressible nonlinearly elastic materials. In: Fu, Y.B., Ogden, R.W. (eds.) Nonlinear Elasticity, pp. 135–159. Cambridge University Press, Cambridge (2001)
13. Kuo, K.A., Hunt, H.E.M., Lister, J.R.: Small oscillations of a pressurized, elastic, spherical shell: model and experiments. J. Sound Vib. **359**, 168–178 (2015)
14. Martin, P.A.: Multiple Scattering. Cambridge University Press, Cambridge (2006)
15. Norris, A.N., Shuvalov, A.L.: Elastodynamics of radially inhomogeneous spherically anisotropic elastic materials in the Stroh formalism. Proc. R. Soc. A **468**, 467–484 (2012)
16. Ogden, R.W.: Non-Linear Elastic Deformations. Dover, New York (1997). Corrected reprint of 1984 original
17. Riccobelli, D., Ciarletta, P.: Shape transitions in a soft incompressible sphere with residual stresses. Math. Mech. Solids **23**, 1507–1524 (2018)

18. Shearer, T.: Waves in nonlinear elastic media with inhomogeneous pre-stress. PhD thesis, University of Manchester (2013)
19. Shearer, T., Parnell, W.J., Abrahams, I.D.: Antiplane wave scattering from a cylindrical cavity in pre-stressed nonlinear elastic media. Proc. R. Soc. A **471**, 20150450 (2015). Correction, 474, 20180268, 2018
20. Steigmann, D.J.: Finite Elasticity Theory. Oxford University Press, Oxford (2017)
21. Truesdell, C., Noll, W.: The non-linear field theories of mechanics. In: Fl gge, S. (ed.) Encyclopedia of Physics, vol. III/3, pp. 1–579. Springer, Berlin (1965)