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Acoustics and dynamic materials

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ABSTRACT

The governing equations for small-amplitude acoustic disturbances are derived by combining an admissible background flow or medium, balance laws, an equation of state and perturbation theory. This standard approach is used here when the background medium can vary in both space and time, thus defining a dynamic material. When the background velocity is zero, the background density cannot vary with time, assuming that the usual balance laws (such as conservation of mass) are obeyed. When mass conservation is discarded, perhaps replaced by a growth model, a time-dependent background density is permitted, leading to new equations governing acoustic disturbances in certain dynamic materials.

1. Introduction

A classical interpretation of the title of this paper is suggested by the title of a book: *Sound transmission through a fluctuating ocean* [1]. For such a problem, we would select an admissible ambient flow that is non-uniform in both space and time, and then we would consider a small-amplitude acoustic perturbation. This perturbation is governed by equations obtained from the usual balance laws (such as conservation of mass) and an equation of state.

However, the term ‘dynamic material’ has acquired a slightly different interpretation. In his recent book, Lurie defines dynamic materials ‘as formations assembled from ordinary materials distributed in space and time. When such formation is allotted with a microstructure, it becomes a dynamic (spatio-temporal) composite’ [2, p. 2]; his book describes many applications. For the beginnings of a general continuum theory of dynamic materials, see [3].

Yet another kind of dynamic material occurs in biology: living materials can grow, leading to continuum theories in which such growth is taken into account [4]. One feature of these theories is that mass conservation is no longer required.

We are interested in the equations governing small-amplitude acoustic disturbances in dynamic materials, materials in which the background medium can vary in both space and time. We assume that the background medium is an inviscid compressible fluid. In the first three sections, we recall standard theory for background media that do not vary with time; familiar equations, such as the wave equation and Bergmann's equation, are derived.

One observation from these derivations is pertinent: if the background medium is not moving, then the background density ρ_0 cannot vary with time. In order to overcome this restriction, we abandon conservation of mass. This leads to our study of dynamic materials in Section 5. We start with a simple growth model, as is used in some theories of biological growth. The result is a modified form of Bergmann's equation for the excess pressure p. More general growth models (with ρ_0 varying proportional to $\exp\{\eta(\boldsymbol{r}, t)\}$ with some chosen function η of position $\boldsymbol{r}$ and time t) are shown to lead to a very complicated equation for p. Although alternative growth models could be considered, we have not pursued this option. Instead, we considered a simpler approach, discarding growth models entirely, motivated by the thought: can we merely specify $\rho_0(\boldsymbol{r}, t)$, while retaining the other standard balance laws? It turns out that we cannot, at least if we want an isothermal model with a standard equation of state.

We conclude that *acoustics and dynamic materials* is a subject with great potential. Technological advances are likely to lead to dynamic materials for which well-founded acoustic models will be required. Further work is needed.

2. Governing equations: compressible inviscid fluid

As we are interested in acoustics, we start by recalling the exact governing equations for the motion of a compressible inviscid fluid [5, Section 3.6], [6, Section I], [7, Section 2.1.1]. Conservation of mass gives the continuity equation,

$$\frac{\mathrm{D}\rho_{\mathrm{ex}}}{\mathrm{D}t} + \rho_{\mathrm{ex}} \operatorname{div} \boldsymbol{v}_{\mathrm{ex}} = 0, \tag{1}$$

where ρ_{ex} is the mass density, $\boldsymbol{v}_{\mathrm{ex}}$ is the fluid velocity and t is time. (The subscript ‘ex’ denotes ‘exact’.) The material derivative is defined by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \boldsymbol{v}_{\mathrm{ex}} \cdot \operatorname{grad} f.$$

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In the absence of body forces, conservation of linear momentum gives

$$\rho_{\rm ex}\frac{{\rm D}\boldsymbol{v}_{\rm ex}}{{\rm D}t}+\operatorname{grad}p_{\rm ex}=\mathbf{0},\tag{2}$$

where $p_{\rm ex}$ is the pressure. For isentropic flows [5, p. 156], [8, Eq. (1-4.3)], the entropy per unit mass, $E_{\rm ex}$, satisfies

$$\frac{{\rm D}E_{\rm ex}}{{\rm D}t}=0.\tag{3}$$

There is also an equation of state which we take in the form [8, Sections 1–4]

$$p_{\rm ex}=p_{\rm ex}(\rho_{\rm ex},E_{\rm ex}).\tag{4}$$

Differentiating, we obtain

$$\operatorname{grad}p_{\rm ex}=c_{\rm ex}^2\operatorname{grad}\rho_{\rm ex}+h_{\rm ex}\operatorname{grad}E_{\rm ex}\tag{5}$$

and

$$\frac{{\rm D}p_{\rm ex}}{{\rm D}t}=c_{\rm ex}^2\frac{{\rm D}\rho_{\rm ex}}{{\rm D}t}+h_{\rm ex}\frac{{\rm D}E_{\rm ex}}{{\rm D}t}=-\rho_{\rm ex}c_{\rm ex}^2\operatorname{div}\boldsymbol{v}_{\rm ex},\tag{6}$$

using Eqs. (1) and (3), where

$$c_{\rm ex}^2(\rho_{\rm ex},E_{\rm ex})=\frac{\partial p_{\rm ex}}{\partial\rho_{\rm ex}}\quad\text{and}\quad h_{\rm ex}(\rho_{\rm ex},E_{\rm ex})=\frac{\partial p_{\rm ex}}{\partial E_{\rm ex}}.\tag{7}$$

Finally, the temperature $T_{\rm ex}$ satisfies [5, Eq. (3.6.6)]

$$\frac{1}{T_{\rm ex}}\frac{{\rm D}T_{\rm ex}}{{\rm D}t}=\frac{\varkappa}{\rho_{\rm ex}}\frac{{\rm D}p_{\rm ex}}{{\rm D}t}=-\varkappa\,c_{\rm ex}^2\operatorname{div}\boldsymbol{v}_{\rm ex},\tag{8}$$

using Eq. (6), where $\varkappa$ is the ratio of the coefficient of thermal expansion to the specific heat at constant pressure ($\varkappa=\beta/c_p$ in Batchelor's notation [5]).

3. Linearisation

3.1. Ambient flows

Consider an ambient flow in which $\boldsymbol{v}_{\rm ex}=\boldsymbol{U}$, a constant velocity. (The case $\boldsymbol{U}=\mathbf{0}$ will be of most interest to us.) For such a flow, let $\rho_{\rm ex}=\rho_0$, $p_{\rm ex}=p_0$, $E_{\rm ex}=E_0$, $T_{\rm ex}=T_0$, $c_{\rm ex}^2=c_0^2$ and $h_{\rm ex}=h_0$. We have $p_0=p_{\rm ex}(\rho_0,E_0)$, $c_0^2=c_{\rm ex}^2(\rho_0,E_0)$ and $h_0=h_{\rm ex}(\rho_0,E_0)$. Then Eqs. (1)–(6) and (8) give the following constraints on the ambient flow,

$$\frac{\mathcal{D}\rho_0}{\mathcal{D}t}=0,\quad\operatorname{grad}p_0=\mathbf{0},\quad\frac{\mathcal{D}E_0}{\mathcal{D}t}=0,\tag{9}$$

$$\frac{\mathcal{D}p_0}{\mathcal{D}t}=0\quad\text{and}\quad\frac{\mathcal{D}T_0}{\mathcal{D}t}=0,\tag{10}$$

where $\mathcal{D}f/\mathcal{D}t=\partial f/\partial t+\boldsymbol{U}\cdot\operatorname{grad}f$. Combining Eqs. $(9)_2$, and $(10)_1$ shows that p_0 is a constant, whereas Eq. (5) gives

$$\operatorname{grad}p_0=c_0^2\operatorname{grad}\rho_0+h_0\operatorname{grad}E_0=\mathbf{0}.\tag{11}$$

The easiest way to satisfy Eqs. $(9)_3$ and $(10)_2$ is to suppose that E_0 and T_0 are constants. Then Eqs. $(9)_1$ and (11) imply that ρ_0 is constant. In this situation, we say the fluid is *homogeneous*: its properties do not vary with position (or time). However, we are interested in more general situations.

3.2. Acoustics

For linear acoustics, we consider small perturbations about the ambient state, and write

$$\begin{aligned}
p_{\rm ex}&=p_0+\varepsilon p_1+\cdots, & \rho_{\rm ex}&=\rho_0+\varepsilon\rho_1+\cdots,\\
\boldsymbol{v}_{\rm ex}&=\boldsymbol{U}+\varepsilon\boldsymbol{v}_1+\cdots, & E_{\rm ex}&=E_0+\varepsilon E_1+\cdots,\\
c_{\rm ex}&=c_0+\varepsilon c_1+\cdots, & h_{\rm ex}&=h_0+\varepsilon h_1+\cdots,
\end{aligned}$$

where ε is a small parameter. Substitution in the equation of state, Eq. (4), gives

$$\begin{aligned}
p_{\rm ex}(\rho_{\rm ex},E_{\rm ex})&=p_{\rm ex}(\rho_0+\varepsilon\rho_1+\cdots,E_0+\varepsilon E_1+\cdots)\\
&=p_{\rm ex}(\rho_0,E_0)+\varepsilon\rho_1\frac{\partial p_{\rm ex}}{\partial\rho_{\rm ex}}(\rho_0,E_0)\\
&\quad+\varepsilon E_1\frac{\partial p_{\rm ex}}{\partial E_{\rm ex}}(\rho_0,E_0)+\cdots,
\end{aligned}$$

giving $p_0=p_{\rm ex}(\rho_0,E_0)$ and $p_1=c_0^2\rho_1+h_0E_1$ with

$$c_0^2=c_{\rm ex}^2(\rho_0,E_0)\quad\text{and}\quad h_0=h_{\rm ex}(\rho_0,E_0).$$

Substitution in Eqs. (1)–(3) and (8) gives, at first order in ε,

$$\frac{\mathcal{D}\rho_1}{\mathcal{D}t}+\operatorname{div}(\rho_0\boldsymbol{v}_1)=0,\quad\rho_0\frac{\mathcal{D}\boldsymbol{v}_1}{\mathcal{D}t}+\operatorname{grad}p_1=\mathbf{0},\tag{12}$$

$$\frac{\mathcal{D}E_1}{\mathcal{D}t}+\boldsymbol{v}_1\cdot\operatorname{grad}E_0=0,\tag{13}$$

$$\frac{\mathcal{D}T_1}{\mathcal{D}t}+\boldsymbol{v}_1\cdot\operatorname{grad}T_0=-\varkappa\,c_0^2T_0\operatorname{div}\boldsymbol{v}_1.\tag{14}$$

We are mainly interested in perturbations from the ambient state. Therefore we define the excess pressure p by $p_{\rm ex}=p_0+p$, and we accept the linear approximation, giving $p=\varepsilon p_1$. We make similar definitions for other relevant quantities. Thus

$$\begin{aligned}
&p=p_{\rm ex}-p_0=\varepsilon p_1,\quad\boldsymbol{v}=\boldsymbol{v}_{\rm ex}-\boldsymbol{U}=\varepsilon\boldsymbol{v}_1,\\
&\rho=\rho_{\rm ex}-\rho_0=\varepsilon\rho_1,\quad E=E_{\rm ex}-E_0=\varepsilon E_1,
\end{aligned}$$

and $T=T_{\rm ex}-T_0=\varepsilon T_1$. The equations relating these quantities are readily found, making use of Eqs. (9) and (10). They are

$$p=c_0^2\rho+h_0E,\quad\frac{\mathcal{D}\rho}{\mathcal{D}t}+\operatorname{div}(\rho_0\boldsymbol{v})=0,\tag{15}$$

$$\rho_0\frac{\mathcal{D}\boldsymbol{v}}{\mathcal{D}t}+\operatorname{grad}p=\mathbf{0},\quad\frac{\mathcal{D}E}{\mathcal{D}t}+\boldsymbol{v}\cdot\operatorname{grad}E_0=0,\tag{16}$$

$$\frac{\mathcal{D}T}{\mathcal{D}t}+\boldsymbol{v}\cdot\operatorname{grad}T_0=-\varkappa\,c_0^2T_0\operatorname{div}\boldsymbol{v}.\tag{17}$$

These are the basic equations for acoustic small-amplitude perturbations. We proceed to examine several special cases.

4. Some familiar special cases

4.1. Zero ambient velocity: Bergmann's equation

When $\boldsymbol{U}=\mathbf{0}$, Eqs. (9) and (10) imply that ρ_0, E_0 and T_0 do not depend on t, whereas p_0 is a constant. The constraint Eq. (11) permits us to have spatial variations in c_0^2 and ρ_0 within a stationary fluid (but not if E_0 is constant).

For the acoustic perturbation, Eqs. (15) and (16) give

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho_0\boldsymbol{v})=0,\quad\rho_0\frac{\partial\boldsymbol{v}}{\partial t}+\operatorname{grad}p=\mathbf{0},\tag{18}$$

$$\frac{\partial E}{\partial t}+\boldsymbol{v}\cdot\operatorname{grad}E_0=0.\tag{19}$$

As $p=c_0^2\rho+h_0E$ in which c_0^2 and h_0 do not depend on t, we can combine Eqs. $(18)_1$ and (19) to give

$$\frac{\partial p}{\partial t}+c_0^2\operatorname{div}(\rho_0\boldsymbol{v})+h_0\,\boldsymbol{v}\cdot\operatorname{grad}E_0=0.\tag{20}$$

Eliminating $h_0\operatorname{grad}E_0$ using Eq. (11), we obtain

$$\frac{\partial p}{\partial t}+\rho_0c_0^2\operatorname{div}\boldsymbol{v}=0.\tag{21}$$

Finally, eliminating $\boldsymbol{v}$, using Eq. (18)$_2$, gives

$$\rho_0 \operatorname{div}\left(\rho_0^{-1} \operatorname{grad} p\right) = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}, \tag{22}$$

in which $\rho_0(\boldsymbol{r})$ and $c_0^2(\boldsymbol{r})$ can be functions of position $\boldsymbol{r} = (x, y, z)$ (but not of t). This is *Bergmann's equation* for the (excess) pressure [9, Eq. (14)], [10, Eq. (5.15)].

Suppose that the motion is known to be irrotational, meaning that the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}$. Then we can write $\boldsymbol{v} = \operatorname{grad} \Phi$, where Φ is a velocity potential. It follows from Eq. (18)$_2$ that $p = -\rho_0(\boldsymbol{r})\, \partial\Phi/\partial t$ and then Eq. (21) yields

$$\nabla^2 \Phi = \frac{1}{c_0^2(\boldsymbol{r})}\frac{\partial^2 \Phi}{\partial t^2}. \tag{23}$$

4.2. Zero ambient velocity, constant ambient density

When $\boldsymbol{U} = \boldsymbol{0}$ and ρ_0 is a constant, Bergmann's equation, Eq. (22), reduces to

$$\nabla^2 p = \frac{1}{c_0^2(\boldsymbol{r})}\frac{\partial^2 p}{\partial t^2}. \tag{24}$$

As ρ_0 is constant, taking the curl of Eq. (18)$_2$ shows that the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}$ does not depend on t. Therefore if the motion starts from a state in which $\boldsymbol{v}$ is constant, then $\boldsymbol{\omega} = \boldsymbol{0}$: the motion is irrotational, and we can write $\boldsymbol{v} = \operatorname{grad} \Phi$. Then, as in Section 4.1, we have $p = -\rho_0\, \partial\Phi/\partial t$, whereas Eq. (24) shows that Φ satisfies the wave equation, Eq. (23).

Note that irrotationality was *assumed* in Section 4.1 in order to obtain Eq. (23), whereas it can be *proved* when ρ_0 is constant.

Eq. (23) often appears in the context of seismic inversion; see, for example, [10, Eq. (5.9)] and [11]. It also appears in other imaging contexts [12, Eq. (2.1)]. Stochastic versions of Eq. (24), in which $c_0^2(\boldsymbol{r})$ is a random function of position, have also been studied and used; see, for example, [12, Eq. (12.1)], [13,14] and [15, Eq. (3.17)].

4.3. Zero ambient velocity and homogeneous fluid

This is the textbook case, in which ρ_0 and c_0 are constants and $\boldsymbol{U} = \boldsymbol{0}$. The governing equations are the wave equation,

$$\nabla^2 \Phi = \frac{1}{c_0^2}\frac{\partial^2 \Phi}{\partial t^2}, \tag{25}$$

with $\boldsymbol{v} = \operatorname{grad} \Phi$ and $p = -\rho_0\, \partial\Phi/\partial t$. Evidently, p and any Cartesian component of $\boldsymbol{v}$ also solve (25).

4.4. Non-zero ambient velocity and homogeneous fluid

In this case, the governing equations are Eqs. (15) and (16), in which ρ_0, c_0, h_0 and E_0 are constants:

$$p = c_0^2 \rho + h_0 E, \quad \frac{\mathcal{D}\rho}{\mathcal{D}t} + \rho_0 \operatorname{div} \boldsymbol{v} = 0, \tag{26}$$

$$\frac{\mathcal{D}E}{\mathcal{D}t} = 0, \quad \rho_0 \frac{\mathcal{D}\boldsymbol{v}}{\mathcal{D}t} + \operatorname{grad} p = \boldsymbol{0}. \tag{27}$$

The first three of these give

$$\frac{\mathcal{D}p}{\mathcal{D}t} = -\rho_0 c_0^2 \operatorname{div} \boldsymbol{v}$$

from which we can eliminate $\boldsymbol{v}$ using Eq. (27)$_2$ to obtain

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\mathcal{D}^2 p}{\mathcal{D}t^2} = \frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\operatorname{grad}\right)^2 p. \tag{28}$$

This is the *convected wave equation* [6, Eq. (6)]. If the flow is irrotational, with $\boldsymbol{v} = \operatorname{grad} \Phi$, we find that Φ also satisfies Eq. (28) with $p = -\rho_0\,(\partial\Phi/\partial t + \boldsymbol{U}\cdot\operatorname{grad} \Phi)$.

Eq. (28) was used by Tatarski [16, Eq. (5.1)] with $\boldsymbol{U}$ replaced by $\boldsymbol{U}(\boldsymbol{r})$, the local ambient velocity at position $\boldsymbol{r}$; see also [6, Eq. (4)]. There are other versions of the convected wave equation that are intended for inhomogeneous fluids with a non-uniform ambient flow; see [6,7,17] and Section 5.

5. Non-uniform ambient flows, dynamic materials

Perhaps the simplest model of dynamic materials is obtained by considering the wave equation, Eq. (25), in which c_0 is a function of time, giving [18]

$$\nabla^2 w = \frac{1}{c_0^2(t)}\frac{\partial^2 w}{\partial t^2}. \tag{29}$$

More generally, models of the form $\operatorname{div}\{a(\boldsymbol{r}, t) \operatorname{grad} w\} = \partial^2 w/\partial t^2$ have been used [19]. In such models, including Eq. (29), no physical meaning is attributed to w. For some related one-dimensional studies, see [20–24].

5.1. Pierce's equation

For acoustic problems, Pierce [6] has derived a Bergmann-like wave equation, under certain assumptions about the dynamic medium: he assumes that it is 'slowly varying with position over distances comparable to a representative acoustic wavelength and that it is slowly varying with time over times comparable to a representative acoustic period' [6, p. 2293]. A stochastic version of Pierce's equation has been used recently [25].

For zero ambient velocity ($\boldsymbol{U} = \boldsymbol{0}$), Pierce's equation [6, Eq. (23)] reduces to

$$\frac{1}{\rho_0(\boldsymbol{r})} \operatorname{div}\{\rho_0(\boldsymbol{r}) \operatorname{grad} \Phi\} = \frac{\partial}{\partial t}\left(\frac{1}{c_0^2(\boldsymbol{r}, t)}\frac{\partial \Phi}{\partial t}\right), \tag{30}$$

where $\Phi(\boldsymbol{r}, t)$ is a velocity potential, with $\boldsymbol{v} = \operatorname{grad} \Phi$ and $p = -\rho_0\, \partial\Phi/\partial t$. Eq. (30) is W3 in the collection compiled by Campos [17]. Flatté [1, Eq. (5.1.11) with Eq. (6.1.1)] uses another equation for Φ,

$$\nabla^2 \Phi = \frac{1}{c_0^2(\boldsymbol{r}, t)}\frac{\partial^2 \Phi}{\partial t^2}, \tag{31}$$

which reduces to Bergmann's equation, Eq. (23), when c_0^2 does not depend on t.

Note that we have written $\rho_0(\boldsymbol{r})$ in Eq. (30), not $\rho_0(\boldsymbol{r}, t)$. This is because we showed in Section 4.1 that conservation of mass combined with $\boldsymbol{U} = \boldsymbol{0}$ implies that ρ_0 cannot depend on t. In other words, if we want to have $\rho_0(\boldsymbol{r}, t)$, then we must have a moving ambient flow or we must abandon conservation of mass.

Note also that if c_0^2 does not depend on t, then Eq. (30) does not reduce to Bergmann's equation, Eq. (23). Pierce [6, Eq. (30)] attributes the discrepancy to a second order effect that may be discarded.

In [26], the authors model a dynamic material by modifying Bergmann's equation, which we write as

$$\operatorname{div}\left(\frac{1}{\rho_0(\boldsymbol{r})} \operatorname{grad} p\right) = \kappa_0(\boldsymbol{r})\frac{\partial^2 p}{\partial t^2}, \tag{32}$$

in which $\kappa_0 = (\rho_0 c_0^2)^{-1}$ is the (adiabatic) compressibility [8, p. 30]. In [26, Eq. (4)], Eq. (32) is used but with $\rho_0(\boldsymbol{r}, t)$ in place of $\rho_0(\boldsymbol{r})$,

$$\operatorname{div}\left(\frac{1}{\rho_0(\boldsymbol{r}, t)} \operatorname{grad} p\right) = \kappa_0(\boldsymbol{r})\frac{\partial^2 p}{\partial t^2}. \tag{33}$$

We have already seen that such an equation is inconsistent with conservation of mass. This provides one motivation for relaxing the constraint of mass conservation. Another comes from continuum models of growing materials [4].

5.2. Exponential growth

Let us replace conservation of mass, Eq. (1), by

$$\frac{\mathrm{D}\rho_{\mathrm{ex}}}{\mathrm{D}t} + \rho_{\mathrm{ex}} \operatorname{div} \boldsymbol{v}_{\mathrm{ex}} = \rho_{\mathrm{ex}} \gamma, \tag{34}$$

where $\gamma(\boldsymbol{r})$ is a given function of position, the *growth rate function*; see [4, Eq. (13.5)]. We retain the other governing equations, namely Eqs. (2), (3) and (4).

Linearising about an ambient state in which $\boldsymbol{U} = \boldsymbol{0}$ (and ignoring any temperature dependence), we find that p_0 is constant, E_0 does not depend on t and

$$\frac{\partial \rho_0}{\partial t} = \rho_0 \gamma \quad \text{whence} \quad \rho_0(\boldsymbol{r}, t) = \rho_{00}(\boldsymbol{r})\, \mathrm{e}^{t\gamma(\boldsymbol{r})}, \tag{35}$$

where $\rho_{00}(\boldsymbol{r}) = \rho_0(\boldsymbol{r}, 0)$. As Eq. (11) also holds, we substitute ρ_0 and obtain

$$\mu\, \mathrm{e}^{\gamma t} (\operatorname{grad} \rho_{00} + t\rho_{00} \operatorname{grad} \gamma) + \operatorname{grad} E_0 = \boldsymbol{0}$$

where $\mu(\boldsymbol{r}, t) = c_0^2/h_0$. To eliminate the term containing $t\rho_{00}$, we are forced to take $\operatorname{grad} \gamma(\boldsymbol{r}) = \boldsymbol{0}$: γ is a constant, γ_0, say, giving

$$\rho_0(\boldsymbol{r}, t) = \rho_{00}(\boldsymbol{r})\, \mathrm{e}^{\gamma_0 t}. \tag{36}$$

Then, we infer that $\mu\, \mathrm{e}^{\gamma_0 t}$ cannot depend on t, whence

$$\mu(\boldsymbol{r}, t) = \mu_0(\boldsymbol{r})\, \mathrm{e}^{-\gamma_0 t} \tag{37}$$

and $\kappa_0^{-1} = \rho_0 \mu h_0 = \rho_{00}(\boldsymbol{r}) \mu_0(\boldsymbol{r}) h_{\mathrm{ex}}(\rho_0(\boldsymbol{r}, t), E_0(\boldsymbol{r}))$, which depends on t, in general.

For the acoustic perturbation, we obtain a slightly modified form of Eqs. (18) and (19):

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho_0 \boldsymbol{v}) = \rho \gamma_0, \quad \rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad} p = \boldsymbol{0}, \tag{38}$$

$$p = c_0^2 \rho + h_0 E, \quad \frac{\partial E}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} E_0 = 0. \tag{39}$$

As E_0 does not depend on t, differentiating Eq. $(39)_2$ gives

$$\frac{\partial^2 E}{\partial t^2} = -\frac{\partial \boldsymbol{v}}{\partial t} \cdot \operatorname{grad} E_0 = \frac{1}{\rho_0} (\operatorname{grad} p) \cdot (\operatorname{grad} E_0)$$

after use of Eq. $(38)_2$. Eliminating E using Eq. $(39)_1$ and $\operatorname{grad} E_0$ using Eq. (11), we arrive at

$$\frac{h_0}{c_0^2} \frac{\partial^2}{\partial t^2} \left(\frac{p - c_0^2 \rho}{h_0} \right) = -\frac{1}{\rho_0} (\operatorname{grad} p) \cdot (\operatorname{grad} \rho_0), \tag{40}$$

which is an equation relating p and ρ.

For a second equation, start by integrating Eq. $(38)_2$. Let $\boldsymbol{g}(\boldsymbol{r}, t) = \rho_0^{-1} \operatorname{grad} p$. Assuming that $\boldsymbol{v}(\boldsymbol{r}, 0) = \boldsymbol{0}$,

$$\boldsymbol{v}(\boldsymbol{r}, t) = -\int_0^t \boldsymbol{g}(\boldsymbol{r}, \tau)\, \mathrm{d}\tau. \tag{41}$$

We substitute this expression in Eq. $(38)_1$:

$$\frac{\partial \rho}{\partial t} - \rho \gamma_0 = \operatorname{div} \left(\rho_0(\boldsymbol{r}, t) \int_0^t \boldsymbol{g}(\boldsymbol{r}, \tau)\, \mathrm{d}\tau \right) = F(\boldsymbol{r}, t), \tag{42}$$

say. Assuming that $\rho(\boldsymbol{r}, 0) = 0$, we can solve for ρ:

$$\rho(\boldsymbol{r}, t) = \int_0^t \mathrm{e}^{\gamma_0 (t - \tau')} F(\boldsymbol{r}, \tau')\, \mathrm{d}\tau'. \tag{43}$$

Hence $\partial \rho / \partial t = F + \gamma_0 \rho$ and $\partial^2 \rho / \partial t^2 = \partial F / \partial t + \gamma_0 F + \gamma_0^2 \rho$. Using these relations, we substitute Eq. (43) in Eq. (40), recalling that $\mu(\boldsymbol{r}, t) = c_0^2/h_0$ is given by Eq. (37):

$$\frac{1}{\mu} \frac{\partial^2 (\mu \rho)}{\partial t^2} = \frac{\partial^2 \rho}{\partial t^2} + \frac{2}{\mu} \frac{\partial \mu}{\partial t} \frac{\partial \rho}{\partial t} + \frac{\rho}{\mu} \frac{\partial^2 \mu}{\partial t^2} = \frac{\partial F}{\partial t} - \gamma_0 F. \tag{44}$$

Next, let us evaluate F, defined by Eq. (42). Making use of Eq. $(35)_2$,

$$F(\boldsymbol{r}, t) = \operatorname{div} \int_0^t \mathrm{e}^{\gamma_0 (t - \tau)} \operatorname{grad} p(\boldsymbol{r}, \tau)\, \mathrm{d}\tau$$
$$= \int_0^t \mathrm{e}^{\gamma_0 (t - \tau)}\, \nabla^2 p(\boldsymbol{r}, \tau)\, \mathrm{d}\tau.$$

Hence

$$\frac{\partial F}{\partial t} = \nabla^2 p + \gamma_0 \int_0^t \mathrm{e}^{\gamma_0 (t - \tau)}\, \nabla^2 p\, \mathrm{d}\tau = \nabla^2 p + \gamma_0 F. \tag{45}$$

Using Eqs. (44) and (45), Eq. (40) becomes

$$\frac{h_0}{c_0^2} \frac{\partial^2}{\partial t^2} \left(\frac{p}{h_0} \right) = \frac{1}{\mu} \frac{\partial^2 (\mu \rho)}{\partial t^2} - \frac{1}{\rho_0} (\operatorname{grad} p) \cdot (\operatorname{grad} \rho_0)$$
$$= \rho_0 \operatorname{div} \left(\frac{\operatorname{grad} p}{\rho_0} \right). \tag{46}$$

Evidently, this is a generalisation of Bergmann's equation, Eq. (22).

Recall that $h_0 = h_{\mathrm{ex}}(\rho_0(\boldsymbol{r}, t), E_0(\boldsymbol{r}))$, so that, in general, h_0 depends on t. Exceptionally, when $\gamma_0 = 0$, ρ_0 no longer depends on t (see Eq. (36)), and then Eq. (46) reduces to Bergmann's equation, Eq. (22).

5.3. A more general growth model

We have seen that if we start from Eq. (34) with growth rate function $\gamma(\boldsymbol{r})$, then we are forced to take $\gamma = \gamma_0$, a constant, so that spatial variation of γ is lost.

For a more general model, let us replace Eq. (34) by

$$\frac{\mathrm{D}\rho_{\mathrm{ex}}}{\mathrm{D}t} + \rho_{\mathrm{ex}} \operatorname{div} \boldsymbol{v}_{\mathrm{ex}} = \rho_{\mathrm{ex}} \frac{\partial \eta}{\partial t}, \tag{47}$$

where $\eta(\boldsymbol{r}, t)$ is specified. Proceeding as in Section 5.2, we find that

$$\rho_0(\boldsymbol{r}, t) = \rho_{00}(\boldsymbol{r})\, \mathrm{e}^{\eta(\boldsymbol{r}, t)}. \tag{48}$$

The acoustic perturbation is governed by Eqs. $(38)_2$, (39) and

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho_0 \boldsymbol{v}) = \rho \frac{\partial \eta}{\partial t}. \tag{49}$$

As before, we obtain Eq. (40), an equation relating p and ρ. Then, eliminating $\boldsymbol{v}$ from Eq. (49) using Eq. (41), we find

$$\frac{\partial \rho}{\partial t} - \rho \frac{\partial \eta}{\partial t} = F(\boldsymbol{r}, t),$$

where F is defined by Eq. (42). Integrating,

$$\rho(\boldsymbol{r}, t) = \int_0^t \mathrm{e}^{\phi(t, \tau'; \boldsymbol{r})} F(\boldsymbol{r}, \tau')\, \mathrm{d}\tau', \tag{50}$$

where $\phi(t, \tau; \boldsymbol{r}) = \eta(\boldsymbol{r}, t) - \eta(\boldsymbol{r}, \tau)$. Hence

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial F}{\partial t} + F \frac{\partial \eta}{\partial t} + \rho \left\{ \frac{\partial^2 \eta}{\partial t^2} + \left(\frac{\partial \eta}{\partial t} \right)^2 \right\}. \tag{51}$$

For F itself, using Eqs. (42) and (48),

$$F(\boldsymbol{r}, t) = \operatorname{div} \int_0^t \mathrm{e}^{\phi(t, \tau; \boldsymbol{r})}\, \boldsymbol{q}(\boldsymbol{r}, \tau)\, \mathrm{d}\tau$$
$$= \int_0^t \mathrm{e}^{\phi(t, \tau; \boldsymbol{r})} \left\{ \nabla^2 p + \boldsymbol{q} \cdot \operatorname{grad} \phi \right\} \mathrm{d}\tau$$

where $\boldsymbol{q} = \operatorname{grad} p$. Hence

$$\frac{\partial F}{\partial t} = \nabla^2 p + F \frac{\partial \eta}{\partial t} + \left(\operatorname{grad} \frac{\partial \eta}{\partial t} \right) \cdot \int_0^t \mathrm{e}^{\phi} \boldsymbol{q}\, \mathrm{d}\tau \tag{52}$$

and

$$\rho(\boldsymbol{r}, t) = \int_0^t \mathrm{e}^{\phi(t, \tau'; \boldsymbol{r})} \int_0^{\tau'} \mathrm{e}^{\phi(\tau', \tau; \boldsymbol{r})} \left\{ \nabla^2 p + \boldsymbol{q} \cdot \operatorname{grad} \phi \right\} \mathrm{d}\tau\, \mathrm{d}\tau'$$

$$= \int_0^t \mathrm{e}^{\phi(t,\tau;\boldsymbol{r})} \int_\tau^t \left\{ \nabla^2 p + \boldsymbol{q} \cdot \operatorname{grad} \phi(\tau', \tau; \boldsymbol{r}) \right\} \mathrm{d}\tau' \, \mathrm{d}\tau$$
$$= \int_0^t \mathrm{e}^{\phi(t,\tau;\boldsymbol{r})} \left\{ (t-\tau)\nabla^2 p + \boldsymbol{q} \cdot \operatorname{grad} \Psi(t,\tau;\boldsymbol{r}) \right\} \mathrm{d}\tau \tag{53}$$

where

$$\Psi(t,\tau;\boldsymbol{r}) = \int_\tau^t \phi(\tau',\tau;\boldsymbol{r}) \, \mathrm{d}\tau'$$
$$= \int_\tau^t \eta(\boldsymbol{r},\tau') \, \mathrm{d}\tau' - (t-\tau)\eta(\boldsymbol{r},\tau). \tag{54}$$

We substitute for ρ from Eq. (53) in Eq. (40); we have

$$\frac{1}{\mu}\frac{\partial^2(\mu\rho)}{\partial t^2} = \frac{\partial^2 \rho}{\partial t^2} + \frac{2}{\mu}\frac{\partial \mu}{\partial t}\frac{\partial \rho}{\partial t} + \frac{\rho}{\mu}\frac{\partial^2 \mu}{\partial t^2}$$
$$= \frac{\partial F}{\partial t} + F\left(\frac{\partial \eta}{\partial t} + \frac{2}{\mu}\frac{\partial \mu}{\partial t}\right)$$
$$+ \rho\left\{\frac{1}{\mu}\frac{\partial^2 \mu}{\partial t^2} + \frac{2}{\mu}\frac{\partial \mu}{\partial t}\frac{\partial \eta}{\partial t} + \frac{\partial^2 \eta}{\partial t^2} + \left(\frac{\partial \eta}{\partial t}\right)^2\right\}$$
$$= \nabla^2 p + L, \tag{55}$$

say, where we have used Eqs. (51) and (52), and

$$L(\boldsymbol{r},t) = \left(\operatorname{grad}\frac{\partial \eta}{\partial t}\right) \cdot \int_0^t \mathrm{e}^{\phi} \boldsymbol{q} \, \mathrm{d}\tau$$
$$+ \left(2\frac{\partial \eta}{\partial t} + \frac{2}{\mu}\frac{\partial \mu}{\partial t}\right) \int_0^t \mathrm{e}^{\phi} \left\{ \nabla^2 p + \boldsymbol{q} \cdot \operatorname{grad} \phi \right\} \mathrm{d}\tau$$
$$+ \left\{\frac{1}{\mu}\frac{\partial^2 \mu}{\partial t^2} + \frac{2}{\mu}\frac{\partial \mu}{\partial t}\frac{\partial \eta}{\partial t} + \frac{\partial^2 \eta}{\partial t^2} + \left(\frac{\partial \eta}{\partial t}\right)^2\right\}$$
$$\times \int_0^t \mathrm{e}^{\phi} \left\{ (t-\tau)\nabla^2 p + \boldsymbol{q} \cdot \operatorname{grad} \Psi \right\} \mathrm{d}\tau. \tag{56}$$

In this formula, $\phi = \eta(\boldsymbol{r},t) - \eta(\boldsymbol{r},\tau)$, $\boldsymbol{q} = \operatorname{grad} p(\boldsymbol{r},\tau)$ and $\Psi(t,\ \tau;\ \boldsymbol{r})$ is defined by Eq. (54). Hence Eq. (40) becomes

$$\frac{h_0}{c_0^2}\frac{\partial^2}{\partial t^2}\left(\frac{p}{h_0}\right) = \rho_0 \operatorname{div}\left(\frac{\operatorname{grad} p}{\rho_0}\right) + L. \tag{57}$$

This is a complicated integrodifferential equation for $p(\boldsymbol{r},\ t)$.

As a simple check, suppose that $\eta = 0$: the material is static. Then both ρ_0 and E_0 do not depend on t, so that c_0 and h_0 also do not depend on t; here, we have used $c_0^2 = c_{\mathrm{ex}}^2(\rho_0, E_0)$ and $h_0 = h_{\mathrm{ex}}(\rho_0, E_0)$. Hence, as $\mu = c_0^2/h_0$ does not depend on t, $L = 0$ and so Eq. (57) reduces to Bergmann's equation, Eq. (22).

We have developed further examples (such as separable η, with $\eta(\boldsymbol{r},t) = \gamma(\boldsymbol{r})T(t)$), but we have not found any that lead to substantial simplifications of L, or to the model used in [26]. Of course, our growth model, Eq. (47), is simple (and linear in ρ_{ex}), so there is plenty of scope for alternative models.

5.4. No growth model at all: specify $\rho_0(\mathbf{r},\ t)$

Instead of replacing conservation of mass by a growth model, such as Eq. (34) or Eq. (47), let us simply specify $\rho_0(\boldsymbol{r},\ t)$, assuming that this specification is contrived by some external means. This is a plausible approach if we wish to create dynamic materials. As before, we take $\boldsymbol{U} = \boldsymbol{0}$, and we find that p_0 is constant and $\partial E_0/\partial t = 0$. Then, from Eq. (6), we obtain

$$0 = \frac{\partial p_0}{\partial t} = c_0^2 \frac{\partial \rho_0}{\partial t} + h_0 \frac{\partial E_0}{\partial t},$$

which reduces to $\partial \rho_0/\partial t = 0$. In other words, if we want $\partial \rho_0/\partial t \neq 0$, then we must modify Eq. (3), $\mathrm{D}E_{\mathrm{ex}}/\mathrm{D}t = 0$. This could be done, perhaps by retaining temperature effects [5, Eqs. (3.6.3)], [8, Eqs. (1-4.6)]. However, as far as we know, this option has not been contemplated.

6. Discussion

In this paper, we have investigated the possibility of building a linear theory of acoustics when the ambient flow varies in space and time, using the standard balance laws. We gave most attention to the special case in which the ambient velocity is zero ($\boldsymbol{U} = \boldsymbol{0}$); then, with a fairly general equation of state, we found that the ambient density ρ_0 cannot vary with time.

Motivated by this result, and by continuum theories of biological growth, we relaxed the requirement of mass conservation, starting with a simple model permitting exponential growth, Eq. (34). The result is an equation for the excess pressure, Eq. (46); this equation is similar to Bergmann's equation for inhomogeneous but static ambient conditions, Eq. (22). The new equation implies that $\rho_0(\boldsymbol{r},t) = \rho_{00}(\boldsymbol{r})\,\mathrm{e}^{\gamma_0 t}$, which is very special (and probably unrealistic). For a more general growth model, with $\gamma_0 t$ replaced by a function $\eta(\boldsymbol{r},\ t)$, we obtained a very complicated equation for the excess pressure, Eq. (57). We also considered discarding growth models entirely, opting instead for specification of $\rho_0(\boldsymbol{r},\ t)$. However, this was shown to be incompatible with standard equations of state under isothermal conditions.

The discussion above is essentially exact, within the limits of perturbation theory. We have not introduced additional approximations, such as those arising from relevant time scales. For example, time scales associated with acoustic disturbances are much shorter than those associated with biological growth [4, Section 13.1]. However, we should keep in mind that technological progress may lead to dynamic materials that can change rapidly, thereby making material and acoustic time scales comparable.

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