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Acoustic scattering in a rarefied gas: Solving the R13 equations in spherical polar coordinates

P. A. Martin

Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado, USA

Correspondence
P. A. Martin, Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO, USA.
Email: pamartin@mines.edu

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In some circumstances, sound waves in a rarefied gas can be studied using a linearised form of the regularised 13-moment equations of Struchtrup and Torrilhon. We build solutions of those equations in spherical polar coordinates using vector spherical harmonics. We first solve a reduced system of equations (with 11 unknowns) after introduction of a force vector (divergence of the stress). We then show that the stresses themselves can be recovered by solving five additional equations. The results obtained are expected to be useful for problems such as acoustic scattering of a plane wave by a sphere in a rarefied gas.

KEYWORDS
kinetic theory, rarefied gas, regularised 13-moment equations

MSC CLASSIFICATION
35Q70; 74J05

1 | INTRODUCTION

Motions of a rarefied gas may be characterised using the Knudsen number $\mathbb{K}$, defined as the ratio of the mean free path in the gas to a typical macroscopic length scale $\hat{L}$. Classical continuum models are associated with the limit $\mathbb{K} \to 0$. More elaborate continuum models have been developed for small values of $\mathbb{K}$, starting with a famous paper by Grad[1]; see also Grad.[2, §28] A regularised form of Grad's 13 moment equations was derived by Struchtrup and Torrilhon,[3] and it is the linearised form of these equations (known as the R13 equations) that provides our starting point.

The linearised R13 equations are developed by Struchtrup and Torrilhon[3, §II.E] and in Struchtrup's book.[4, §9.4] For associated numerical methods, see Claydon et al.[5] and Westerkamp and Torrilhon.[6] We are interested in semianalytical methods where it is natural to introduce spherical polar coordinates. Indeed, some relatively simple problems involving spheres have been studied; see previous studies[7-9] and references therein.

The R13 equations relate 13 unknowns, including the five independent components of a symmetric trace-free stress tensor $\boldsymbol{\sigma}$. Our first observation is that it is simpler to dispense with the equation governing $\boldsymbol{\sigma}$ and to replace it with a new equation for a force vector $\mathbf{g} = \operatorname{div} \boldsymbol{\sigma}$. The resulting 11×11 system can be solved by separation of variables using vector spherical harmonics: if Y_n^m denotes a spherical harmonic, systems for each of the mode numbers n and m are obtained with no coupling with other mode numbers.

Of course, this is only part of the story: we have to show that we can recover the stresses themselves, that they are consistent with the solution for $\mathbf{g}$ and that the five equations governing the stress components are satisfied. We do all this by introducing suitable representations for the stresses and then checking that all relevant equations are satisfied. These representations do couple between mode numbers, even though the components of $\mathbf{g}$ do not. Exceptionally, this coupling does not occur for pulsation problems (spherical symmetry, $n = 0$) as studied in Ben-Ami and Manela[9] or for simple axisymmetric problems (steady flow past a rigid sphere, with $n = 1$ and $m = 0$) as studied in Torrilhon.[7]

For scattering of a plane wave by a sphere, more general solutions are required; constructing these solutions is the main purpose of this paper.

The governing linear equations are reviewed in Section 2, together with a reduced system involving the force vector $\boldsymbol{g} = \operatorname{div} \boldsymbol{\sigma}$. This system is solved in Section 3 using vector spherical harmonics. This approach has been used in other (simpler) situations including anisotropic linear elastodynamics[10] and small-on-large problems arising in nonlinear elastodynamics.[11]

The stresses are considered in Section 4. The calculations are fairly complicated; some details are relegated to Appendices B and C. Nevertheless, we show that the five independent stress components can be represented using five independent radial functions (see Equations 61, 65 and 67) and that these functions satisfy five equations (i.e., Equations 68–70, 81 and 88).

Special cases of our analysis are compared with Ben-Ami and Manela[9] and Torrilhon[7] in Section 5; reassuringly, precise agreement is found. Concluding remarks are in Section 6.

2 | THE GOVERNING LINEAR EQUATIONS

The basic R13 equations are given in Struchtrup's book[4]; see also Torrilhon's review.[12] We shall state linearised forms of these equations in Section 2.1. Using hats to denote dimensional quantities, the basic variables are as follows: $\hat{\rho}$ is the mass density, $\hat{\boldsymbol{v}}$ is the velocity and $\hat{\vartheta}$ is the temperature in energy units. The pressure $\hat{p}$ obeys the ideal gas law, $\hat{p} = \hat{\rho}\hat{\vartheta}$. We also need the stress $\hat{\sigma}_{ij}$, the heat flux vector $\hat{\boldsymbol{q}}$ and certain moments, $\hat{\Delta}$, $\hat{R}_{ij}$ and $\hat{m}_{ijk}$. The stress is symmetric ($\hat{\sigma}_{ij} = \hat{\sigma}_{ji}$) and trace-free ($\hat{\sigma}_{ii} = 0$), and so it is specified by five unknowns. Thus, when combined with the two scalars $\hat{\rho}$ and $\hat{\vartheta}$ and the two vectors $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{q}}$, we see that there are 13 unknowns, hence R13, with R indicating regularised.[3]

2.1 | Linearisation

In a linear approximation,[4, §9.4] we consider small perturbations about an equilibrium state in which $\hat{\rho} = \hat{\rho}_0$, $\hat{\boldsymbol{v}} = \mathbf{0}$, $\hat{\vartheta} = \hat{\vartheta}_0$ and $\hat{p} = \hat{p}_0 = \hat{\rho}_0\hat{\vartheta}_0$, where $\hat{\rho}_0$ and $\hat{\vartheta}_0$ are constants. Associated dimensionless quantities are defined by

$$\hat{\rho} = \hat{\rho}_0(1+\rho),\ \hat{\vartheta} = \hat{\vartheta}_0(1+\vartheta),\ \hat{p} = \hat{p}_0(1+p),\ \hat{\boldsymbol{v}} = \hat{\vartheta}_0^{1/2}\boldsymbol{v},$$

with $p = \rho + \vartheta$. Furthermore, write

$$\hat{\sigma}_{ij} = \hat{p}_0\sigma_{ij},\ \hat{q}_i = \hat{p}_0\hat{\vartheta}_0^{1/2}q_i,\ \hat{\Delta} = \hat{p}_0\hat{\vartheta}_0\Delta,\ \hat{R}_{ij} = \hat{p}_0\hat{\vartheta}_0R_{ij},\ \hat{m}_{ijk} = \hat{p}_0\hat{\vartheta}_0^{1/2}m_{ijk},\ \hat{x}_i = \hat{L}x_i,\ \hat{t} = \hat{L}\hat{\vartheta}_0^{-1/2}t,$$

where $\hat{L}$ is a length scale and $\hat{t}$ is time. The viscosity of the gas also plays a role and leads to a dimensionless quantity,

$$\mathbb{K} = \frac{\hat{\mu}_0\hat{\vartheta}_0^{1/2}}{\hat{p}_0\hat{L}},$$

the *Knudsen number*,[4, eq. 9.41] where $\hat{\mu}_0$ is the dynamic viscosity coefficient in the equilibrium state.

In the absence of body forces, the governing (linear) equations are as follows. There are three conservation equations[4, §9.4.1]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial v_j}{\partial x_j} = 0, \tag{1}$$

$$\frac{3}{2}\frac{\partial \vartheta}{\partial t} + \frac{\partial q_j}{\partial x_j} = -\frac{\partial v_j}{\partial x_j}, \tag{2}$$

$$\frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0,\ i = 1,2,3. \tag{3}$$

Let $\mathcal{V} = \operatorname{div} \boldsymbol{v} = \partial v_j/\partial x_j$ and $\mathcal{Q} = \operatorname{div} \boldsymbol{q} = \partial q_j/\partial x_j$. As $p = \rho + \vartheta$, eliminate ρ from Equation (1) using Equation (2) giving

$$\frac{3}{2}\frac{\partial p}{\partial t} + \mathcal{Q} + \frac{5}{2}\mathcal{V} = 0 \text{ and } \frac{3}{2}\frac{\partial \vartheta}{\partial t} + \mathcal{Q} + \mathcal{V} = 0. \tag{4}$$

For steady problems ($\partial/\partial t \equiv 0$), Equations (3) and (4) reduce to $\nabla p + \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$, $\mathcal{V} = 0$ and $\mathcal{Q} = 0$.[5, eq. 2.1]

Equations (3) and (4) are combined with equations for heat flux and stress[4, §9.4.3:]

$$\frac{\partial q_i}{\partial t} + \frac{5}{2}\frac{\partial \vartheta}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{1}{2}\frac{\partial R_{ij}}{\partial x_j} + \frac{1}{6}\frac{\partial \Delta}{\partial x_i} = -\frac{2}{3}\frac{q_i}{\mathbb{K}}, \; i = 1, 2, 3, \tag{5}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \frac{\partial m_{ijk}}{\partial x_k} = -\frac{\sigma_{ij}}{\mathbb{K}}, \; i, j = 1, 2, 3, \tag{6}$$

together with[4, eq. 9.49]

$$\Delta = -12\mathbb{K}\frac{\partial q_j}{\partial x_j}, \; R_{ij} = -\frac{24}{5}\mathbb{K}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}, \; m_{ijk} = -2\mathbb{K}\frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}}. \tag{7}$$

The angular brackets in Equations (6) and (7) have the following meaning. First, round brackets indicate symmetric part, so that[4, §A.2.1]

$$A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right), \; B_{(ijk)} = \frac{1}{6}\left(B_{ijk} + B_{ikj} + B_{jik} + B_{jki} + B_{kij} + B_{kji}\right).$$

Angular brackets give trace-free versions of these symmetric tensors[4, §A.2.2:]

$$S_{\langle ij\rangle} = S_{(ij)} - \frac{1}{3}S_{kk}\delta_{ij}, \; T_{\langle ijk\rangle} = T_{(ijk)} - \frac{1}{5}\left(T_{(ill)}\delta_{jk} + T_{(jll)}\delta_{ik} + T_{(kll)}\delta_{ij}\right). \tag{8}$$

Some calculation gives

$$\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) - \frac{1}{3}\mathcal{Q}\delta_{ij}, \tag{9}$$

$$\frac{\partial}{\partial x_j}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} = \frac{1}{2}\nabla^2 q_i + \frac{1}{6}\frac{\partial \mathcal{Q}}{\partial x_i} \tag{10}$$

and, using Equation (7),

$$\frac{1}{2}\frac{\partial R_{ij}}{\partial x_j} + \frac{1}{6}\frac{\partial \Delta}{\partial x_i} = -\frac{6}{5}\mathbb{K}\nabla^2 q_i - \frac{12}{5}\mathbb{K}\frac{\partial \mathcal{Q}}{\partial x_i}.$$

Substitution in Equation (5) then gives

$$q_i + \frac{3}{2}\mathbb{K}\frac{\partial q_i}{\partial t} = -\frac{15}{4}\mathbb{K}\frac{\partial \vartheta}{\partial x_i} - \frac{3}{2}\mathbb{K}\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{9}{5}\mathbb{K}^2\nabla^2 q_i + \frac{18}{5}\mathbb{K}^2\frac{\partial \mathcal{Q}}{\partial x_i}. \tag{11}$$

This agrees with Struchtrup and Torrilhon[13, eq. 2]; for steady problems, it agrees with Claydon et al.[5, eq. 2.2b]

Next, consider Equation (6) with Equation $(7)_3$, which we write as

$$\sigma_{ij} + \mathbb{K}\frac{\partial \sigma_{ij}}{\partial t} = -2\mathbb{K}\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - \frac{4}{5}\mathbb{K}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2\mathbb{K}^2\frac{\partial}{\partial x_k}\frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}}. \tag{12}$$

This agrees with Struchtrup and Torrilhon.[13, eq. 3] Let us evaluate the last term. As $\sigma_{ij} = \sigma_{ji}$,

$$\frac{\partial \sigma_{(ij}}{\partial x_{k)}} = \frac{1}{3}\left(\frac{\partial \sigma_{ij}}{\partial x_k} + \frac{\partial \sigma_{jk}}{\partial x_i} + \frac{\partial \sigma_{ki}}{\partial x_j}\right),$$

whence, using $\sigma_{ii} = 0$,

$$\frac{\partial \sigma_{(il}}{\partial x_{l)}} = \frac{1}{3}\left(\frac{\partial \sigma_{il}}{\partial x_l} + \frac{\partial \sigma_{ll}}{\partial x_i} + \frac{\partial \sigma_{li}}{\partial x_l}\right) = \frac{2}{3}\frac{\partial \sigma_{il}}{\partial x_l}$$

and, using Equation $(8)_2$,

$$\frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}} = \frac{1}{3}\left(\frac{\partial \sigma_{ij}}{\partial x_k} + \frac{\partial \sigma_{jk}}{\partial x_i} + \frac{\partial \sigma_{ki}}{\partial x_j}\right) - \frac{2}{15}\left(\frac{\partial \sigma_{il}}{\partial x_l}\delta_{jk} + \frac{\partial \sigma_{jl}}{\partial x_l}\delta_{ki} + \frac{\partial \sigma_{kl}}{\partial x_l}\delta_{ij}\right). \tag{13}$$

From Equation $(7)_3$, we verify that $m_{ijj} = 0$. Differentiating Equation (13),

$$\frac{\partial}{\partial x_k}\frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}} = \frac{1}{3}\nabla^2\sigma_{ij} + \frac{1}{5}\left(\frac{\partial^2\sigma_{ik}}{\partial x_j\,\partial x_k} + \frac{\partial^2\sigma_{jk}}{\partial x_i\,\partial x_k}\right) - \frac{2}{15}\frac{\partial^2\sigma_{kl}}{\partial x_k\,\partial x_l}\delta_{ij}. \tag{14}$$

Now, using Equation $(8)_1$, we have

$$\frac{\partial}{\partial x_{\langle i}}\frac{\partial \sigma_{j\rangle k}}{\partial x_k} = \frac{1}{2}\left(\frac{\partial^2\sigma_{ik}}{\partial x_j\,\partial x_k} + \frac{\partial^2\sigma_{jk}}{\partial x_i\,\partial x_k}\right) - \frac{1}{3}\frac{\partial^2\sigma_{kl}}{\partial x_k\,\partial x_l}\delta_{ij}.$$

Comparison with Equation (14) then gives (see above[13, eq. 101])

$$\frac{\partial}{\partial x_k}\frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}} = \frac{1}{3}\nabla^2\sigma_{ij} + \frac{2}{5}\frac{\partial}{\partial x_{\langle i}}\frac{\partial \sigma_{j\rangle k}}{\partial x_k}, \tag{15}$$

which is a simplification (because the left-hand side involves a rank 3 computation whereas the right-hand side involves a rank 2 computation). When Equation (15) is substituted in Equation (12), we obtain

$$\sigma_{ij} + \mathbb{K}\frac{\partial \sigma_{ij}}{\partial t} = -2\mathbb{K}\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - \frac{4}{5}\mathbb{K}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \frac{2}{3}\mathbb{K}^2\,\nabla^2\sigma_{ij} + \frac{4}{5}\mathbb{K}^2\,\frac{\partial}{\partial x_{\langle i}}\frac{\partial \sigma_{j\rangle k}}{\partial x_k}. \tag{16}$$

The steady form of this equation agrees with Claydon et al.[5, eq. 2.2a]

Inspection of Equations (3) and (11) suggests introducing a vector $\boldsymbol{g}$ with components

$$g_i = \frac{\partial \sigma_{ij}}{\partial x_j},\ \ i = 1, 2, 3.$$

This *force vector* satisfies an equation obtained by taking the divergence of the stress equation (16). Thus, making use of Equation (10), we find that $\boldsymbol{g}$ satisfies

$$g_i + \mathbb{K}\,\frac{\partial g_i}{\partial t} = -\mathbb{K}\,\nabla^2 v_i - \frac{1}{3}\mathbb{K}\,\frac{\partial \mathcal{V}}{\partial x_i} - \frac{2}{5}\mathbb{K}\,\nabla^2 q_i - \frac{2}{15}\mathbb{K}\,\frac{\partial \mathcal{Q}}{\partial x_i} + \frac{16}{15}\mathbb{K}^2\,\nabla^2 g_i + \frac{2}{15}\mathbb{K}^2\,\frac{\partial \mathcal{G}}{\partial x_i} \tag{17}$$

with $\mathcal{G} = \operatorname{div}\boldsymbol{g} = \partial g_j/\partial x_j$.

2.2 | Summary of governing equations

Let us collect the governing partial differential equations, writing them in vector–tensor notation. From Equations (3), (4), (11) and (17), we have

$$\frac{3}{2}\,\frac{\partial p}{\partial t} + \operatorname{div}\boldsymbol{q} + \frac{5}{2}\operatorname{div}\boldsymbol{v} = 0, \tag{18}$$

$$\frac{3}{2}\,\frac{\partial \vartheta}{\partial t} + \operatorname{div}\boldsymbol{q} + \operatorname{div}\boldsymbol{v} = 0, \tag{19}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad} p + \boldsymbol{g} = \boldsymbol{0}, \tag{20}$$

$$\boldsymbol{q} + \frac{3}{2}\mathbb{K}\frac{\partial \boldsymbol{q}}{\partial t} = -\frac{15}{4}\mathbb{K}\,\operatorname{grad}\vartheta - \frac{3}{2}\mathbb{K}\,\boldsymbol{g} + \frac{9}{5}\mathbb{K}^2\,\nabla^2\boldsymbol{q} + \frac{18}{5}\mathbb{K}^2\,\operatorname{grad}(\operatorname{div}\boldsymbol{q}), \tag{21}$$

$$\boldsymbol{g} + \mathbb{K}\frac{\partial \boldsymbol{g}}{\partial t} = -\mathbb{K}\,\nabla^2\boldsymbol{v} - \frac{1}{3}\mathbb{K}\,\operatorname{grad}(\operatorname{div}\boldsymbol{v}) - \frac{2}{5}\mathbb{K}\,\nabla^2\boldsymbol{q} - \frac{2}{15}\mathbb{K}\,\operatorname{grad}(\operatorname{div}\boldsymbol{q}) + \frac{16}{15}\mathbb{K}^2\,\nabla^2\boldsymbol{g} + \frac{2}{15}\mathbb{K}^2\,\operatorname{grad}(\operatorname{div}\boldsymbol{g}). \tag{22}$$

This system of equations comprises two scalar equations and three vector equations for two scalar unknowns (p and ϑ) and three vector unknowns ($\boldsymbol{v}$, $\boldsymbol{q}$ and $\boldsymbol{g}$). Recall that $\boldsymbol{g} = \operatorname{div}\boldsymbol{\sigma}$ where the stresses are given by Equation (16), which we write as

$$\boldsymbol{\sigma} + \mathbb{K}\frac{\partial \boldsymbol{\sigma}}{\partial t} = -2\mathbb{K}\,\langle\nabla\boldsymbol{v}\rangle - \frac{4}{5}\mathbb{K}\,\langle\nabla\boldsymbol{q}\rangle + \frac{2}{3}\mathbb{K}^2\,\nabla^2\boldsymbol{\sigma} + \frac{4}{5}\mathbb{K}^2\,\langle\nabla\boldsymbol{g}\rangle\,. \tag{23}$$

Here, $\langle\nabla\boldsymbol{q}\rangle$ denotes the symmetrised trace-free version of $\nabla\boldsymbol{q}$; see Equation (9).

In what follows, we assume that the Knudsen number $\mathbb{K} > 0$. Indeed, the problem becomes very simple when $\mathbb{K} = 0$. In that case, Equations (21) and (22) give $\boldsymbol{q} = \mathbf{0}$ and $\boldsymbol{g} = \mathbf{0}$. Then, as $\operatorname{div} \boldsymbol{q} = 0$, Equations (18) and (20) reduce to

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{5}{2}\operatorname{div}\boldsymbol{v} = 0 \text{ and } \frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad} p = \mathbf{0}.$$

Eliminating $\boldsymbol{v}$ between these equations gives $\nabla^2 p = (3/5)\,\partial^2 p/\partial t^2$, the wave equation with dimensionless speed $\sqrt{5/3}$.[4, §10.1.3]

Plane-wave solutions of the governing equations can be constructed. Such solutions are investigated in Appendix A. Both compressional waves and transverse waves are found. However, our main interest in this paper is in the construction of solutions using spherical polar coordinates.

3 | USE OF SPHERICAL POLAR COORDINATES

We are interested in solving the equations of Section 2.2 using spherical polar coordinates, (r, θ, Φ), where θ is the polar angle and Φ is the azimuthal angle. Let $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Phi}}$ be the spherical polar unit vectors. We write $\boldsymbol{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\Phi\hat{\boldsymbol{\Phi}}$ and then put the components in a column vector $\mathbf{v} = (v_r, v_\theta, v_\Phi)^T$. We use similar notation for $\boldsymbol{q}$ and $\boldsymbol{g}$.

We are going to use vector spherical harmonics, defined as in,[14, §3.17]

$$\boldsymbol{P}_n^m(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_n^m(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_n^m(\theta, \Phi), \tag{24}$$

$$\boldsymbol{B}_n^m(\hat{\mathbf{r}}) = \frac{1}{\lambda}\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\Phi}}}{\sin\theta}\frac{\partial}{\partial\Phi}\right) Y_n^m(\theta, \Phi), \tag{25}$$

$$\boldsymbol{C}_n^m(\hat{\mathbf{r}}) = \frac{1}{\lambda}\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\Phi} - \hat{\boldsymbol{\Phi}}\frac{\partial}{\partial\theta}\right) Y_n^m(\theta, \Phi), \tag{26}$$

where $\lambda = [n(n+1)]^{-1/2}$ and Y_n^m are spherical harmonics.[14, Definition 3.2] Collecting the components of Equations (24)–(26) into column vectors, we see the structure

$$\mathbf{P}_n^m = (Y_n^m, 0, 0)^T,\ \mathbf{B}_n^m = (0, B_n^m, C_n^m)^T,\ \mathbf{C}_n^m = (0, C_n^m, -B_n^m)^T$$

with

$$B_n^m = \lambda^{-1}\partial Y_n^m/\partial\theta \text{ and } C_n^m = (\lambda\sin\theta)^{-1}\partial Y_n^m/\partial\Phi. \tag{27}$$

We start by seeking separated time-harmonic solutions of Equations (18) and (19) in the form

$$\mathbf{v} = \operatorname{Re}\sum_{n,m}\left\{V_{P,n}^m(r)\mathbf{P}_n^m(\hat{\mathbf{r}}) + V_{B,n}^m(r)\ \mathbf{B}_n^m(\hat{\mathbf{r}}) + V_{C,n}^m(r)\ \mathbf{C}_n^m(\hat{\mathbf{r}})\right\}\mathrm{e}^{-\mathrm{i}\omega t},\ p = \operatorname{Re}\sum_{n,m}\mathcal{P}_n^m(r) Y_n^m(\hat{\mathbf{r}})\ \mathrm{e}^{-\mathrm{i}\omega t},$$

with similar expansions for $\mathbf{q}$ and ϑ. Exploiting the orthogonality properties of the vector spherical harmonics, we can suppress all sums, subscripts and superscripts, and the harmonic time dependence, giving the structure

$$\mathbf{v} = V_P(r)\ \mathbf{P}(\hat{\mathbf{r}}) + V_B(r)\ \mathbf{B}(\hat{\mathbf{r}}) + V_C(r)\ \mathbf{C}(\hat{\mathbf{r}}),\ p = \mathcal{P}(r) Y(\hat{\mathbf{r}}), \tag{28}$$

$$v_r = V_P Y,\ v_\theta = V_B B + V_C C,\ v_\Phi = V_B CC - V_C B. \tag{29}$$

Then we shall see from Equation (20) that $\mathbf{g}$ has a similar structure, with

$$\mathbf{g} = G_P(r)\ \mathbf{P}(\hat{\mathbf{r}}) + G_B(r)\ \mathbf{B}(\hat{\mathbf{r}}) + G_C(r)\ \mathbf{C}(\hat{\mathbf{r}}). \tag{30}$$

Finally, the heat flux and force vector equations, Equations (21) and (22), will lead to an 11×11 system for 11 unknown scalar functions of r.

3.1 | The three conservation equations

For Equations (18) and (19), we require div $\boldsymbol{v}$ and div $\boldsymbol{q}$. We have

$$\operatorname{div} \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\Phi}{\partial\Phi} \tag{31}$$

$$= \frac{Y}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 V_P) + \frac{V_B}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(B\sin\theta) + \frac{\partial C}{\partial\Phi}\right) + \frac{V_C}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(C\sin\theta) - \frac{\partial B}{\partial\Phi}\right). \tag{32}$$

But, from Equation (27) and the partial differential equation satisfied by Y,[11, eq. 4.13]

$$\mathcal{D}Y \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\Phi^2} = -\lambda^2 Y, \tag{33}$$

we obtain

$$\frac{\partial}{\partial\theta}(C\sin\theta) - \frac{\partial B}{\partial\Phi} = 0, \quad \frac{\partial}{\partial\theta}(B\sin\theta) + \frac{\partial C}{\partial\Phi} = -\lambda Y\sin\theta. \tag{34}$$

Hence,

$$\operatorname{div} \boldsymbol{v} = \left(\overset{\bullet}{V}_P - (\lambda/r)V_B\right) Y, \tag{35}$$

where we have introduced the shorthand notation

$$\overset{\bullet}{f} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 f) = f'(r) + \frac{2}{r}\,f(r). \tag{36}$$

If we expand $\mathbf{q}$ and ϑ as Equation (28) with $V_A(r)$ replaced by $Q_A(r)$ and $\mathcal{P}(r)$ replaced by $\Theta(r)$, Equations (18) and (19) give

$$-3\mathrm{i}\omega\mathcal{P} + 2\overset{\bullet}{Q}_P - 2(\lambda/r)Q_B + 5\overset{\bullet}{V}_P - 5(\lambda/r)V_B = 0, \tag{37}$$

$$-3\mathrm{i}\omega\Theta + 2\overset{\bullet}{Q}_P - 2(\lambda/r)Q_B + 2\overset{\bullet}{V}_P - 2(\lambda/r)V_B = 0. \tag{38}$$

Next, consider Equation (20). One ingredient is

$$\operatorname{grad} p = \frac{\partial p}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial p}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial p}{\partial\Phi}\,\hat{\boldsymbol{\Phi}}. \tag{39}$$

Using $p = \mathcal{P}Y$ and Equation (27), we obtain

$$\operatorname{grad} p = \left(\mathcal{P}'Y,\ \mathcal{P}r^{-1}\partial Y/\partial\theta,\ \mathcal{P}(r\sin\theta)^{-1}\partial Y/\partial\Phi\right) = \left(\mathcal{P}'Y,\ (\lambda/r)\mathcal{P}B,\ (\lambda/r)\mathcal{P}C\right) = \mathcal{P}'\mathbf{P}^T + (\lambda/r)\mathcal{P}\mathbf{B}^T. \tag{40}$$

Hence, Equations (20) and (28) show that $\mathbf{g}$ has the expansion (30) in which

$$G_P = \mathrm{i}\omega V_P - \mathcal{P}',\ G_B = \mathrm{i}\omega V_B - (\lambda/r)\mathcal{P},\ G_C = \mathrm{i}\omega V_C. \tag{41}$$

3.2 | The heat flux equation

For Equation (21), we need expressions for $\nabla^2\boldsymbol{v}$ and grad (div $\boldsymbol{v}$). Combining Equations (35) and (40) gives

$$\begin{aligned}\operatorname{grad}\,(\operatorname{div} \boldsymbol{v}) &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\overset{\bullet}{V}_P - \frac{\lambda}{r}V_B\right)\mathbf{P}^T + \frac{\lambda}{r}\left(\overset{\bullet}{V}_P - \frac{\lambda}{r}V_B\right)\mathbf{B}^T \\ &= \left\{\nabla^2 V_P - \frac{2}{r^2}V_P - \frac{\lambda}{r}\left(\overset{\bullet}{V}_B - \frac{3}{r}V_B\right)\right\}\mathbf{P}^T + \frac{\lambda}{r}\left(\overset{\bullet}{V}_P - \frac{\lambda}{r}V_B\right)\mathbf{B}^T,\end{aligned}$$

where we have used Equation (36) and

$$\nabla^2 f(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f}{\mathrm{d}r}\right) = f''(r) + \frac{2}{r}f'. \tag{42}$$

For $\nabla^2\boldsymbol{v}$, we can use Dahlen and Tromp,[15, eq. A.143] but it is more convenient to use

$$\nabla^2\{f(r)\boldsymbol{P}(\hat{\mathbf{r}})\} = (\nabla^2 f)\boldsymbol{P} + r^{-2}\{2\lambda\boldsymbol{B} - (\lambda^2+2)\boldsymbol{P}\}f,$$
$$\nabla^2\{f(r)\boldsymbol{B}(\hat{\mathbf{r}})\} = (\nabla^2 f)\boldsymbol{B} + r^{-2}(2\lambda\boldsymbol{P} - \lambda^2\boldsymbol{B})f,\ \nabla^2\{f(r)\boldsymbol{C}(\hat{\mathbf{r}})\} = (\nabla^2 f)\boldsymbol{C} - (\lambda/r)^2\boldsymbol{C}f;$$

see Norris and Shuvalov.[10, eq. 3.9] Hence,

$$\nabla^2\boldsymbol{v} = \left\{\nabla^2 V_P - r^{-2}(\lambda^2+2)V_P + 2\lambda r^{-2}V_B\right\}\mathbf{P}^T + \left\{\nabla^2 V_B - (\lambda/r)^2 V_B + 2\lambda r^{-2}V_P\right\}\mathbf{B}^T + \left\{\nabla^2 V_C - (\lambda/r)^2 V_C\right\}\mathbf{C}^T.$$

Using these results in Equation (21), we obtain

$$Q_P - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_P = -\frac{15}{4}\mathbb{K}\Theta' - \frac{3}{2}\mathbb{K}\ G_P + \frac{9}{5}\mathbb{K}^2\left\{3\nabla^2 Q_P - r^{-2}(\lambda^2+6)Q_P - 2(\lambda/r)\overset{\bullet}{Q}_B + 8\lambda r^{-2}Q_B\right\}, \tag{43}$$

$$Q_B - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_B = -\frac{15}{4}\mathbb{K}\frac{\lambda}{r}\Theta - \frac{3}{2}\mathbb{K}\ G_B + \frac{9}{5}\mathbb{K}^2\left\{2(\lambda/r)\overset{\bullet}{Q}_P + 2\lambda r^{-2}Q_P + \nabla^2 Q_B - 3(\lambda/r)^2 Q_B\right\}, \tag{44}$$

$$Q_C - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_C = -\frac{3}{2}\mathbb{K}\ G_C + \frac{9}{5}\mathbb{K}^2\left\{\nabla^2 Q_C - (\lambda/r)^2 Q_C\right\}. \tag{45}$$

3.3 | The force vector equation

Similar calculations with Equation (22) yield

$$\begin{aligned}(1-\mathrm{i}\omega\mathbb{K})G_P = &-\frac{1}{3}\mathbb{K}\left\{4\nabla^2 V_P - (3\lambda^2+8)r^{-2}V_P - (\lambda/r)\overset{\bullet}{V}_B + 9\lambda r^{-2}V_B\right\}\\ &-\frac{2}{15}\mathbb{K}\left\{4\nabla^2 Q_P - (3\lambda^2+8)r^{-2}Q_P - (\lambda/r)\overset{\bullet}{Q}_B + 9\lambda r^{-2}Q_B\right\}\\ &+\frac{2}{15}\mathbb{K}^2\left\{9\nabla^2 G_P - (8\lambda^2+18)r^{-2}G_P - (\lambda/r)\overset{\bullet}{G}_B + 19\lambda r^{-2}G_B\right\},\end{aligned} \tag{46}$$

$$\begin{aligned}(1-\mathrm{i}\omega\mathbb{K})G_B = &-\frac{1}{3}\mathbb{K}\left\{(\lambda/r)\overset{\bullet}{V}_P + 6\lambda r^{-2}V_P + 3\nabla^2 V_B - 4(\lambda/r)^2 V_B\right\}\\ &-\frac{2}{15}\mathbb{K}\left\{(\lambda/r)\overset{\bullet}{Q}_P + 6\lambda r^{-2}Q_P + 3\nabla^2 Q_B - 4(\lambda/r)^2 Q_B\right\}\\ &+\frac{2}{15}\mathbb{K}^2\left\{(\lambda/r)\overset{\bullet}{G}_P + 16\lambda r^{-2}G_P + 8\nabla^2 G_B - 9(\lambda/r)^2 G_B\right\},\end{aligned} \tag{47}$$

$$(1-\mathrm{i}\omega\mathbb{K})G_C = -\mathbb{K}\left\{\nabla^2 V_C - (\lambda/r)^2 V_C\right\} - \frac{2}{5}\mathbb{K}\left\{\nabla^2 Q_C - (\lambda/r)^2 Q_C\right\} + \frac{16}{15}\mathbb{K}^2\left\{\nabla^2 G_C - (\lambda/r)^2 G_C\right\}. \tag{48}$$

3.4 | Summary

At this stage, we have a system of 11 equations for 11 unknown (scalar) functions of r, namely, $\mathcal{P}$, Θ, V_A, Q_A and G_A with $A = P, B, C$. This 11×11 system decouples into a 3×3 system and an 8×8 system. The smaller system comprises Equations $(41)_3$, (45) and (48). We call it the C-system because the three unknowns are V_C, Q_C and G_C. The larger system governs the other eight unknowns and comprises Equations (37), (38), $(41)_{1,2}$, (43), (44), (46) and (47); we call this the *PB*-system. However, before further investigation, we must consider the associated stresses $\boldsymbol{\sigma}$; we do that next.

4 | STRESSES

Recall that the stresses are given by Equation (23), which we repeat here:

$$\boldsymbol{\sigma} + \mathbb{K}\frac{\partial \boldsymbol{\sigma}}{\partial t} = -2\mathbb{K}\,\langle \nabla \boldsymbol{v}\rangle - \frac{4}{5}\mathbb{K}\,\langle \nabla \boldsymbol{q}\rangle + \frac{4}{5}\mathbb{K}^2\,\langle \nabla \boldsymbol{g}\rangle + \frac{2}{3}\mathbb{K}^2\,\nabla^2\boldsymbol{\sigma}. \tag{49}$$

In this equation, $\boldsymbol{g} = \operatorname{div}\boldsymbol{\sigma}$ and $\langle \nabla \boldsymbol{q}\rangle$ is the symmetrised trace-free version of $\nabla \boldsymbol{q}$. In fact, every term in Equation (49) is symmetric and trace-free, so there are five independent components.

The components of $\boldsymbol{v}$, $\boldsymbol{q}$ and $\boldsymbol{g}$ all have similar forms:

$$v_r = V_P Y,\ v_\theta = V_B B + V_C C,\ v_\Phi = V_B C - V_C B, \tag{50}$$

$$q_r = Q_P Y,\ q_\theta = Q_B B + Q_C C,\ q_\Phi = Q_B C - Q_C B, \tag{51}$$

$$g_r = G_P Y,\ g_\theta = G_B B + G_C C,\ g_\Phi = G_B C - G_C B. \tag{52}$$

With these, we can compute $\langle \nabla \boldsymbol{v}\rangle$, $\langle \nabla \boldsymbol{q}\rangle$ and $\langle \nabla \boldsymbol{g}\rangle$ in Equation (49); see Section 4.1. The forms of the stresses themselves are unclear. We have to determine them so that Equation (49) is satisfied, which entails computing $\nabla^2\boldsymbol{\sigma}$; see Section 4.3. In addition, the components of $\boldsymbol{g}$ are given by Equations (B7)–(B9), and these must be consistent with Equation (52).

4.1 | Computation of $\langle \nabla \mathbf{v}\rangle$

Let us compute $\langle \nabla \boldsymbol{v}\rangle$ ($\langle \nabla \boldsymbol{q}\rangle$ and $\langle \nabla \boldsymbol{g}\rangle$ are similar) using formulas collected in Appendix C. From Equations (B2)–(B6), we find

$$\langle \nabla \boldsymbol{v}\rangle_{rr} = \left(\frac{2}{3}\overset{\bullet}{V}_P - \frac{2}{r}V_P + \frac{\lambda}{3r}V_B\right) Y, \tag{53}$$

$$\langle \nabla \boldsymbol{v}\rangle_{r\theta} = \frac{1}{2}\left(\frac{\lambda}{r}V_P + \overset{\bullet}{V}_B - \frac{3}{r}V_B\right) B + \frac{1}{2}\left(\overset{\bullet}{V}_C - \frac{3}{r}V_C\right) C, \tag{54}$$

$$\langle \nabla \boldsymbol{v}\rangle_{r\Phi} = -\frac{1}{2}\left(\overset{\bullet}{V}_C - \frac{3}{r}V_C\right) B + \frac{1}{2}\left(\frac{\lambda}{r}V_P + \overset{\bullet}{V}_B - \frac{3}{r}V_B\right) C, \tag{55}$$

$$\langle \nabla \boldsymbol{v}\rangle_{\theta\theta} = \left(\frac{1}{r}V_P - \frac{1}{3}\overset{\bullet}{V}_P + \frac{\lambda}{3r}V_B\right) Y + \frac{1}{r}\left(V_B\frac{\partial B}{\partial \theta} + V_C\frac{\partial C}{\partial \theta}\right), \tag{56}$$

$$\langle \nabla \boldsymbol{v}\rangle_{\Phi\Phi} = \left(\frac{1}{r}V_P - \frac{1}{3}\overset{\bullet}{V}_P - \frac{2\lambda}{3r}V_B\right) Y - \frac{1}{r}\left(V_B\frac{\partial B}{\partial \theta} + V_C\frac{\partial C}{\partial \theta}\right), \tag{57}$$

$$\langle \nabla \boldsymbol{v}\rangle_{\theta\Phi} = -\frac{\lambda}{2r}V_C Y + \frac{1}{r}\left(V_B\frac{\partial C}{\partial \theta} - V_C\frac{\partial B}{\partial \theta}\right). \tag{58}$$

Here, we have used Equation (35) for $\operatorname{div}\boldsymbol{v}$ and, from Equation (36), $f'(r) = \overset{\bullet}{f} - (2/r)f$. We also confirm that $\langle \nabla \boldsymbol{v}\rangle_{rr} + \langle \nabla \boldsymbol{v}\rangle_{\theta\theta} + \langle \nabla \boldsymbol{v}\rangle_{\Phi\Phi} = 0$.

Note the presence of $\partial B/\partial\theta$ and $\partial C/\partial\theta$ on the right-hand sides of Equations (56)–(58). This implies some interaction with other modes: for example, $\partial B_n^m/\partial\theta$ cannot be written solely in terms of Y_n^m, B_n^m and C_n^m.

4.2 | The force vector

The components of the force vector $\mathbf{g} = \operatorname{div}\boldsymbol{\sigma}$ are given by Equations (B7)–(B9). They involve all components of $\boldsymbol{\sigma}$, and they must be consistent with Equation (52).

Let us start with the radial component, g_r, and the rr-component of Equation (49),

$$\sigma_{rr} + \mathbb{K}\,\frac{\partial \sigma_{rr}}{\partial t} = -2\mathbb{K}\langle \nabla \boldsymbol{v}\rangle_{rr} - \frac{4}{5}\mathbb{K}\langle \nabla \boldsymbol{q}\rangle_{rr} + \frac{4}{5}\mathbb{K}^2\langle \nabla \boldsymbol{g}\rangle_{rr} + \frac{2}{3}\mathbb{K}^2\left(\nabla^2\boldsymbol{\sigma}\right)_{rr}. \tag{59}$$

From Equation (53), the first three terms on the right-hand side of Equation (59) are multiples of Y. From Equation (B10), $\left(\nabla^2\boldsymbol{\sigma}\right)_{rr}$ involves σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\Phi}$ and $\sigma_{\theta\theta} + \sigma_{\Phi\Phi} = -\sigma_{rr}$ (because $\boldsymbol{\sigma}$ is trace-free). The same stress components are present in $g_r = (\operatorname{div}\,\boldsymbol{\sigma})_r$; see Equation (B7). These observations suggest proposing the form

$$\mathbf{t} = T_P(r)\ \mathbf{P}(\hat{\mathbf{r}}) + T_B(r)\ \mathbf{B}(\hat{\mathbf{r}}) + T_C(r)\ \mathbf{C}(\hat{\mathbf{r}}), \tag{60}$$

where $\mathbf{t} = (\sigma_{rr},\ \sigma_{r\theta},\ \sigma_{r\Phi})^T$. Thus,

$$\sigma_{rr} = T_P Y,\ \sigma_{r\theta} = T_B B + T_C C,\ \sigma_{r\Phi} = T_B C - T_C B. \tag{61}$$

From Equation (B7), we have

$$\begin{aligned} g_r = (\mathrm{div}\,\boldsymbol{\sigma})_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r}\sigma_{rr} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{r\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{r\Phi}}{\partial\Phi} \\ &= \left(\overset{\bullet}{T}_P + \frac{T_P}{r}\right)Y + \frac{T_B}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(B\sin\theta) + \frac{\partial C}{\partial\Phi}\right) + \frac{T_C}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(C\sin\theta) - \frac{\partial B}{\partial\Phi}\right) = \left(\overset{\bullet}{T}_P + \frac{T_P}{r} - \frac{\lambda}{r}T_B\right)Y = G_P Y, \end{aligned} \tag{62}$$

where we have used Equation (34). This relates $G_P(r)$ in Equation (52) to $T_P(r)$ and $T_B(r)$ in Equation (61).

For g_θ and g_Φ, we require tentative forms for $\sigma_{\theta\theta}$, $\sigma_{\Phi\Phi}$ and $\sigma_{\theta\Phi}$; these will be combined with the proposed forms for σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{r\Phi}$, Equation (61). Inspection of Equations (56)–(58) suggests trying

$$\sigma_{\theta\theta} = A_\theta Y + \left(L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right),\ \sigma_{\Phi\Phi} = A_\Phi Y - \left(L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right),\ \sigma_{\theta\Phi} = A_{\theta\Phi}Y + L\frac{\partial C}{\partial\theta} - M\frac{\partial B}{\partial\theta},$$

where the trace-free condition on $\boldsymbol{\sigma}$ gives $T_P + A_\theta + A_\Phi = 0$. There are four arbitrary functions of r in these proposed forms, so we are expecting constraints: we shall see that A_θ, A_Φ and $A_{\theta\Phi}$ can be written in terms of T_P, L and M.

From Equation (B8), we have an expression for $g_\theta = (\mathrm{div}\,\boldsymbol{\sigma})_\theta$. It gives

$$\begin{aligned} g_\theta &= B\left(\overset{\bullet}{T}_B + \frac{T_B}{r} + \frac{\lambda}{r}A_\theta\right) + C\left(\overset{\bullet}{T}_C + \frac{T_C}{r} + \frac{\lambda}{r}A_{\theta\Phi}\right) + \frac{L}{r}\left(\frac{\partial^2 B}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi}\right) \\ &\quad + \frac{M}{r}\left(\frac{\partial^2 C}{\partial\theta^2} - \frac{1}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi}\right) + \frac{2}{r}\left(L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right)\cot\theta + \frac{1}{r}(A_\theta - A_\Phi)\,Y\cot\theta \\ &= B\left(\overset{\bullet}{T}_B + \frac{T_B}{r} + \frac{\lambda}{r}A_\theta + \frac{L}{r}(1-\lambda^2)\right) + C\left(\overset{\bullet}{T}_C + \frac{T_C}{r} + \frac{\lambda}{r}A_{\theta\Phi} + \frac{M}{r}\right) + \frac{1}{r}(A_\theta - A_\Phi - \lambda L)\,Y\cot\theta. \end{aligned}$$

Here, we have used Equations (27) and (34) to show that

$$\frac{\partial^2 B}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi} = -2\frac{\partial B}{\partial\theta}\cot\theta + (1-\lambda^2)B - \lambda Y\cot\theta, \tag{63}$$

$$\frac{\partial^2 C}{\partial\theta^2} - \frac{1}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi} = -2\frac{\partial C}{\partial\theta}\cot\theta + C. \tag{64}$$

Similarly, from Equation (B9), we find that $g_\Phi = (\mathrm{div}\,\boldsymbol{\sigma})_\Phi$ is given by

$$g_\Phi = C\left(\overset{\bullet}{T}_B + \frac{T_B}{r} + \frac{\lambda}{r}A_\Phi + \frac{L}{r}\right) - B\left(\overset{\bullet}{T}_C + \frac{T_C}{r} - \frac{\lambda}{r}A_{\theta\Phi} + \frac{M}{r}(1-\lambda^2)\right) + \frac{1}{r}(2A_{\theta\Phi} + \lambda M)\,Y\cot\theta.$$

We want the structure $g_\theta = G_B B + G_C C$ and $g_\Phi = G_B C - G_C B$, and we do not want the terms in $Y\cot\theta$. Enforcing these constraints leads to $A_\theta - A_\Phi = \lambda L$ and $2A_{\theta\Phi} = -\lambda M$. As $A_\theta + A_\Phi = -T_P$, we obtain $2A_\theta = -T_P + \lambda L$ and $2A_\Phi = -T_P - \lambda L$. Hence,

$$\sigma_{\theta\theta} = -\frac{1}{2}(T_P - \lambda L)Y + \left(L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right), \tag{65}$$

$$\sigma_{\Phi\Phi} = -\frac{1}{2}(T_P + \lambda L)Y - \left(L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right), \tag{66}$$

$$\sigma_{\theta\Phi} = -\frac{1}{2}\lambda M Y + L\frac{\partial C}{\partial\theta} - M\frac{\partial B}{\partial\theta}, \tag{67}$$

where $L(r)$ and $M(r)$ are to be found.

Summarising, we have written the five independent stress components in terms of five independent functions of the radial coordinate, namely, T_P, T_B, T_C, L and M. Moreover, when combined with Equation (62), we have the following relations:

$$G_P = \overset{\bullet}{T}_P + \frac{1}{r}\,T_P - \frac{\lambda}{r}\,T_B = T_P' + \frac{3}{r}\,T_P - \frac{\lambda}{r}\,T_B, \tag{68}$$

$$G_B = \overset{\bullet}{T}_B + \frac{1}{r}\,T_B - \frac{1}{2r}(\lambda T_P + \mu L) = T_B' + \frac{3}{r}\,T_B - \frac{\lambda}{2r}\,T_P - \frac{\mu L}{2r}, \tag{69}$$

$$G_C = \overset{\bullet}{T}_C + \frac{1}{r}T_C - \frac{\mu M}{2r} = T_C' + \frac{3}{r}\,T_C - \frac{\mu M}{2r}, \tag{70}$$

where the parameter μ is defined by $\mu = \lambda^2 - 2 = (n-1)(n+2)$, indicating that the case $n = 1$ ($\mu = 0$) is special.

It remains to confirm that our forms for the stresses are consistent with the stress equation (49). We shall do that in Sections 4.4 and 4.5.

4.3 | Computation of $\nabla^2\boldsymbol{\sigma}$

Inspection of Equation (49) shows that we require $\nabla^2\boldsymbol{\sigma}$. In principle, this is a straightforward computation: substitute the proposed forms for the stresses, Equations (61) and (65)–(67), in the formulas for $\nabla^2\boldsymbol{\sigma}$, Equations (B10)–(B15), and then simplify; for details, see Appendix C.

The following results are obtained for the three radial components:

$$\left(\nabla^2\boldsymbol{\sigma}\right)_{rr} = \left(\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P + \frac{4\lambda}{r^2}T_B\right)Y, \tag{71}$$

$$\left(\nabla^2\boldsymbol{\sigma}\right)_{r\theta} = \left(\nabla^2 T_B - \frac{\lambda^2+4}{r^2}T_B + \frac{3\lambda}{r^2}T_P + \frac{\mu L}{r^2}\right)B + \left(\nabla^2 T_C - \frac{\lambda^2+4}{r^2}T_C + \frac{\mu M}{r^2}\right)C, \tag{72}$$

$$\left(\nabla^2\boldsymbol{\sigma}\right)_{r\Phi} = \left(\nabla^2 T_B - \frac{\lambda^2+4}{r^2}T_B + \frac{3\lambda}{r^2}T_P + \frac{\mu L}{r^2}\right)C - \left(\nabla^2 T_C - \frac{\lambda^2+4}{r^2}T_C + \frac{\mu M}{r^2}\right)B. \tag{73}$$

Note the BC-structure in Equations (72) and (73). For the other components, we find

$$\begin{aligned}\left(\nabla^2\boldsymbol{\sigma}\right)_{\theta\theta} = &-\frac{1}{2}\left(\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P\right)Y + \frac{\lambda}{2}\left(\nabla^2 L - \frac{\mu L}{r^2}\right)Y \\ &+ \left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right)\frac{\partial B}{\partial\theta} + \left(\nabla^2 M - \frac{\mu M}{r^2} + \frac{4}{r^2}T_C\right)\frac{\partial C}{\partial\theta},\end{aligned} \tag{74}$$

$$\left(\nabla^2\boldsymbol{\sigma}\right)_{\theta\Phi} = \left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right)\frac{\partial C}{\partial\theta} - \left(\nabla^2 M - \frac{\mu M}{r^2} + \frac{4}{r^2}T_C\right)\left(\frac{\lambda}{2}Y + \frac{\partial B}{\partial\theta}\right). \tag{75}$$

We also have $\left(\nabla^2\boldsymbol{\sigma}\right)_{rr} + \left(\nabla^2\boldsymbol{\sigma}\right)_{\theta\theta} + \left(\nabla^2\boldsymbol{\sigma}\right)_{\Phi\Phi} = 0$.

4.4 | The stress equation: radial components

Returning to Equation (59), we use $\sigma_{rr} = T_P Y$, Equations (53) and (71). Then, as each term in Equation (59) is a multiple of Y, we obtain

$$\begin{aligned}(1 - \mathrm{i}\omega\mathbb{K})T_P = &-2\mathbb{K}\left(\frac{2}{3}\overset{\bullet}{V}_P - \frac{2}{r}V_P + \frac{\lambda}{3r}V_B\right) - \frac{4}{5}\mathbb{K}\left(\frac{2}{3}\overset{\bullet}{Q}_P - \frac{2}{r}Q_P + \frac{\lambda}{3r}Q_B\right) \\ &+ \frac{4}{5}\mathbb{K}^2\left(\frac{2}{3}\overset{\bullet}{G}_P - \frac{2}{r}G_P + \frac{\lambda}{3r}G_B\right) + \frac{2}{3}\mathbb{K}^2\left(\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P + \frac{4\lambda}{r^2}T_B\right).\end{aligned} \tag{76}$$

Next, consider the $r\theta$-component of Equation (49). From Equations (54) and (72), each term on the right-hand side is a linear combinations of B and C. From Equation (61), the left-hand side has the same form. Matching the coefficients of the B-terms and the C-terms gives two equations:

$$(1 - \mathrm{i}\omega\mathbb{K})T_B = -\mathbb{K}\left(\frac{\lambda}{r}V_P + \overset{\bullet}{V}_B - \frac{3}{r}V_B\right) - \frac{2}{5}\mathbb{K}\left(\frac{\lambda}{r}Q_P + \overset{\bullet}{Q}_B - \frac{3}{r}Q_B\right) + \frac{2}{5}\mathbb{K}^2\left(\frac{\lambda}{r}G_P + \overset{\bullet}{G}_B - \frac{3}{r}G_B\right) + \frac{2}{3}\mathbb{K}^2\left(\nabla^2 T_B - \frac{\lambda^2+4}{r^2}T_B + \frac{3\lambda}{r^2}T_P + \frac{\mu L}{r^2}\right), \tag{77}$$

$$(1 - \mathrm{i}\omega\mathbb{K})T_C = -\mathbb{K}\left(\overset{\bullet}{V}_C - \frac{3}{r}V_C\right) - \frac{2}{5}\mathbb{K}\left(\overset{\bullet}{Q}_C - \frac{3}{r}Q_C\right) + \frac{2}{5}\mathbb{K}^2\left(\overset{\bullet}{G}_C - \frac{3}{r}G_C\right) + \frac{2}{3}\mathbb{K}^2\left(\nabla^2 T_C - \frac{\lambda^2+4}{r^2}T_C + \frac{\mu M}{r^2}\right). \tag{78}$$

As expected, the $r\Phi$-component of Equation (49) reproduces Equations (77) and (78).

4.4.1 | Simplification of Equation (78)

Let us examine Equation (78) in more detail. Suppose we have solved the 3×3 C-system from Section 3.4, so that V_C, Q_C and G_C are known. Thus, Equation (78) contains two unknown functions, $T_C(r)$ and $M(r)$. These two functions are also related to $G_C(r)$ by Equation (70): we have two equations in two unknowns.

To simplify Equation (78), start by writing it in the form

$$(1 - \mathrm{i}\omega\mathbb{K})T_C = -\mathbb{K}\mathcal{V}_C - \frac{2}{5}\mathbb{K}\mathcal{Q}_C + \frac{16}{15}\mathbb{K}^2\mathcal{G}_C, \tag{79}$$

where the fractional prefactors have been chosen to match those in Equation (48),

$$\mathcal{V}_C = V_C' - \frac{1}{r}\,V_C,\ \mathcal{Q}_C = Q_C' - \frac{1}{r}\,Q_C,\ \mathcal{G}_C = \frac{3}{8}\left(G_C' - \frac{1}{r}\,G_C\right) + \frac{5}{8}\left(\nabla^2 T_C - \frac{\lambda^2+4}{r^2}\,T_C + \frac{\mu M}{r^2}\right),$$

and we also have Equation (70), $G_C = T_C' + (3/r)T_C - \frac{1}{2}\mu M/r$. This leads to consideration of the combinations

$$\mathcal{V}_C' + (3/r)\mathcal{V}_C = \nabla^2 V_C - 2r^{-2}V_C,\ \mathcal{Q}_C' + (3/r)\mathcal{Q}_C = \nabla^2 Q_C - 2r^{-2}Q_C. \tag{80}$$

Differentiating Equation (70) gives

$$G_C' = \nabla^2 T_C + \frac{1}{r}\,G_C - \frac{6}{r^2}\,T_C - \frac{\mu}{2r}M' + \frac{\mu}{r^2}M.$$

Use this formula to eliminate $\nabla^2 T_C$ from $\mathcal{G}_C$, giving

$$\mathcal{G}_C = G_C' - \frac{1}{r}\,G_C - \frac{5\mu}{8r^2}\,T_C + \frac{5\mu}{16r}M'$$

whence

$$\mathcal{G}_C' + \frac{3}{r}\,\mathcal{G}_C = \nabla^2 G_C - \frac{6+5\lambda^2}{8r^2}\,G_C + \frac{5\mu}{4r^3}\,T_C + \frac{5\mu}{16r}\left(\nabla^2 M - \frac{\mu M}{r^2}\right).$$

Combining this result with Equation (79) and (80), we obtain

$$(1 - \mathrm{i}\omega\mathbb{K})\left(T_C' + \frac{3}{r}T_C\right) = -\mathbb{K}\left(\nabla^2 V_C - 2r^{-2}V_C\right) - \frac{2}{5}\mathbb{K}\left(\nabla^2 Q_C - 2r^{-2}Q_C\right) + \frac{16}{15}\mathbb{K}^2\left\{\nabla^2 G_C - \frac{6+5\lambda^2}{8r^2}\,G_C + \frac{5\mu}{4r^3}\,T_C + \frac{5\mu}{16r}\left(\nabla^2 M - \frac{\mu M}{r^2}\right)\right\}.$$

The left-hand side of this equation is $(1 - \mathrm{i}\omega\mathbb{K})(G_C + \frac{1}{2}\mu M/r)$, by Equation (70), whereas $(1 - \mathrm{i}\omega\mathbb{K})G_C$ is given by Equation (48). Hence,

$$(1 - \mathrm{i}\omega\mathbb{K})M = -2\mathbb{K}\frac{V_C}{r} - \frac{4}{5}\mathbb{K}\frac{Q_C}{r} + \frac{4}{5}\mathbb{K}^2\frac{G_C}{r} + \frac{2}{3}\mathbb{K}^2\left(\nabla^2 M - \frac{\mu M}{r^2} + \frac{4}{r^2}T_C\right). \tag{81}$$

This is a simplified relation between M and T_C. These two functions are also related by Equation (70). Thus, in principle, we can solve for M and T_C.

4.4.2 | Simplification of Equations (76) and (77)

Suppose we have solved the 8×8 *PB*-system from Section 3.4, so that $\mathcal{P}$, Θ, V_P, V_B, Q_P, Q_B, G_P and G_B are known. Then, inspection of Equations (76) and (77) shows that these two equations are coupled and they contain three unknown functions, $T_P(r)$, $T_B(r)$ and $L(r)$. However, these functions are also related to $G_P(r)$ and $G_B(r)$ by Equation (68) and (69). Apparently, we have four equations for three unknowns.

We proceed as in Section 4.4.1. Write Equations (76) and (77) as

$$(1 - \mathrm{i}\omega\mathbb{K})T_P = -\frac{1}{3}\mathbb{K}\mathcal{V}_P - \frac{2}{15}\mathbb{K}\mathcal{Q}_P + \frac{2}{15}\mathbb{K}^2\mathcal{G}_P, \tag{82}$$

$$(1 - \mathrm{i}\omega\mathbb{K})T_B = -\frac{1}{3}\mathbb{K}\mathcal{V}_B - \frac{2}{15}\mathbb{K}\mathcal{Q}_B + \frac{2}{15}\mathbb{K}^2\mathcal{G}_B, \tag{83}$$

where the prefactors have been chosen to match those in Equations (46) and (47) and

$$\mathcal{V}_P = 4\left(V_P' - \frac{V_P}{r} + \frac{\lambda}{2r}V_B\right), \quad \mathcal{V}_B = 3\left(\frac{\lambda}{r}V_P + V_B' - \frac{V_B}{r}\right),$$
$$\mathcal{Q}_P = 4\left(Q_P' - \frac{Q_P}{r} + \frac{\lambda}{2r}Q_B\right), \quad \mathcal{Q}_B = 3\left(\frac{\lambda}{r}Q_P + Q_B' - \frac{Q_B}{r}\right),$$
$$\mathcal{G}_P = 4\left(G_P' - \frac{G_P}{r} + \frac{\lambda}{2r}G_B\right) + 5\left(\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P + \frac{4\lambda}{r^2}T_B\right),$$
$$\mathcal{G}_B = 3\left(\frac{\lambda}{r}G_P + G_B' - \frac{G_B}{r}\right) + 5\left(\nabla^2 T_B - \frac{\lambda^2+4}{r^2}T_B + \frac{3\lambda}{r^2}T_P + \frac{\mu}{r^2}L\right).$$

From Equations (68) and (69), we require

$$\mathcal{V}_P' + (3/r)\mathcal{V}_P - (\lambda/r)\mathcal{V}_B = 4\nabla^2 V_P - (3\lambda^2+8)r^{-2}V_P - (\lambda/r)\overset{\bullet}{V}_B + 9\lambda r^{-2}V_B, \tag{84}$$

$$\mathcal{V}_B' + (3/r)\mathcal{V}_B - \{\lambda/(2r)\}\mathcal{V}_P = 3\nabla^2 V_B - (\lambda^2+6)r^{-2}V_B + (\lambda/r)\overset{\bullet}{V}_P + 6\lambda r^{-2}V_P. \tag{85}$$

The corresponding $\mathcal{Q}$ equations are the same as the $\mathcal{V}$ equations but with V replaced by Q.

Differentiating Equations (68) and (69) gives

$$G_P' = \nabla^2 T_P - \frac{\lambda^2+12}{2r^2}T_P + \frac{5\lambda}{r^2}T_B + \frac{G_P}{r} - \frac{\lambda G_B}{r} - \frac{\lambda\mu}{2r^2}L, \tag{86}$$

$$G_B' = \nabla^2 T_B - \frac{\lambda^2+12}{2r^2}T_B + \frac{5\lambda}{2r^2}T_P + \frac{G_B}{r} - \frac{\lambda G_P}{2r} - \frac{\mu}{2r}L' + \frac{\mu}{r^2}L. \tag{87}$$

Use the first of these to eliminate $\nabla^2 T_P$ from $\mathcal{G}_P$, giving

$$\mathcal{G}_P = 9\left(G_P' - \frac{G_P}{r}\right) + \frac{7\lambda}{r}G_B - \frac{5\lambda^2}{2r^2}T_P - \frac{5\lambda}{r^2}T_B + \frac{5\lambda\mu}{2r^2}L.$$

Similarly, eliminating $\nabla^2 T_B$ from $\mathcal{G}_B$ gives

$$\mathcal{G}_B = 8\left(G_B' - \frac{G_B}{r}\right) + \frac{11\lambda}{2r}G_P + \frac{20-5\lambda^2}{2r^2}T_B + \frac{5\lambda}{2r^2}T_P + \frac{5\mu}{2r}L'.$$

From Equation (68), we require

$$\mathcal{G}_P' + (3/r)\mathcal{G}_P - (\lambda/r)\mathcal{G}_B = 9\nabla^2 G_P - (8\lambda^2+18)r^{-2}G_P - (\lambda/r)\overset{\bullet}{G}_B + 19\lambda r^{-2}G_B;$$

surprisingly, T_P, T_B and L are absent. Then, forming $(1 - \mathrm{i}\omega\mathbb{K})\mathcal{G}_P$ from Equations (82) and (83), using Equations (68) and (84), we reproduce Equation (46): in other words, Equation (82) is redundant from the point of view of constraining the determination of T_P, T_B and L.

Next, consider Equation (83). From Equation (69), we require

$$\mathcal{G}_B' + \frac{3}{r}\,\mathcal{G}_B - \frac{\lambda}{2r}\,\mathcal{G}_P = 8\nabla^2 G_B - \frac{6}{r^2}(\lambda^2+1)G_B + \frac{\lambda}{r}\overset{\bullet}{G}_P + \frac{16\lambda}{r^2}G_P + \frac{10\mu}{r^3}T_B + \frac{5\mu}{2r}\left(\nabla^2 L - \frac{\mu L}{r^2}\right).$$

Then, forming $(1 - \mathrm{i}\omega\mathbb{K})\{G_B + \mu L/(2r)\}$, using Equations (69), (82), (83) and (85), followed by comparison with Equation (47), we obtain

$$(1-\mathrm{i}\omega\mathbb{K})L = -2\mathbb{K}\frac{V_B}{r} - \frac{4}{5}\mathbb{K}\frac{Q_B}{r} + \frac{4}{5}\mathbb{K}^2\frac{G_B}{r} + \frac{2}{3}\mathbb{K}^2\left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right), \tag{88}$$

which has a pleasing similarity to Equation (81).

At this stage, we have three equations for T_P, T_B and L, namely, Equations (68), (69) and (88). We also have two equations for T_C and M, namely, Equations (70) and (81). However, we still have to check that the angular components of the stress equation do not impose any further constraints; this does not seem to be obvious because we no longer have orthogonality of the angular functions occurring.

4.5 | The stress equation: angular components

For the $\theta\Phi$-component of Equation (49), we use Equations (67), (58) and (75) and obtain

$$\begin{aligned}(1-\mathrm{i}\omega\mathbb{K})\left\{L\frac{\partial C}{\partial\theta} - M\left(\frac{\lambda}{2}Y + \frac{\partial B}{\partial\theta}\right)\right\} &= -2\mathbb{K}\left\{\frac{V_B}{r}\frac{\partial C}{\partial\theta} - \frac{V_C}{r}\left(\frac{\lambda}{2}Y + \frac{\partial B}{\partial\theta}\right)\right\} - \frac{4}{5}\mathbb{K}\left\{\frac{Q_B}{r}\frac{\partial C}{\partial\theta} - \frac{Q_C}{r}\left(\frac{\lambda}{2}Y + \frac{\partial B}{\partial\theta}\right)\right\}\\ &+ \frac{4}{5}\mathbb{K}^2\left\{\frac{B_B}{r}\frac{\partial C}{\partial\theta} - \frac{G_C}{r}\left(\frac{\lambda}{2}Y + \frac{\partial B}{\partial\theta}\right)\right\} + \frac{2}{3}\mathbb{K}^2\left\{\left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right)\frac{\partial C}{\partial\theta} - \left(\nabla^2 M - \frac{\mu M}{r^2} + \frac{4}{r^2}T_C\right)\left(\frac{\lambda}{2}Y + \frac{\partial B}{\partial\theta}\right)\right\}.\end{aligned}$$

Write this equation as

$$\mathcal{A}_{\theta\Phi}\left(\frac{\lambda Y}{2} + \frac{\partial B}{\partial\theta}\right) - \mathcal{B}_{\theta\Phi}\frac{\partial C}{\partial\theta} = 0,$$

where

$$\begin{aligned}\mathcal{A}_{\theta\Phi} &= (1-\mathrm{i}\omega\mathbb{K})M + 2\mathbb{K}\frac{V_C}{r} + \frac{4}{5}\mathbb{K}\frac{Q_C}{r} - \frac{4}{5}\mathbb{K}^2\frac{G_C}{r} - \frac{2}{3}\mathbb{K}^2\left(\nabla^2 M - \frac{\mu M}{r^2} + \frac{4}{r^2}T_C\right),\\ \mathcal{B}_{\theta\Phi} &= (1-\mathrm{i}\omega\mathbb{K})L + 2\mathbb{K}\frac{V_B}{r} + \frac{4}{5}\mathbb{K}\frac{Q_B}{r} - \frac{4}{5}\mathbb{K}^2\frac{G_B}{r} - \frac{2}{3}\mathbb{K}^2\left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right).\end{aligned}$$

Both of these vanish by Equations (81) and (88), and so the $\theta\Phi$-component of the stress equation (49) is satisfied.

For the $\theta\theta$-component of Equation (49), we use Equations (65), (56) and (74), giving

$$\begin{aligned}(1-\mathrm{i}\omega\mathbb{K})\left\{-\frac{1}{2}(T_P - \lambda L)Y + L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right\} &= -2\mathbb{K}\left\{\left(\frac{1}{r}V_P - \frac{1}{3}\overset{\bullet}{V}_P + \frac{\lambda}{3r}V_B\right)Y + \frac{1}{r}\left(V_B\frac{\partial B}{\partial\theta} + V_C\frac{\partial C}{\partial\theta}\right)\right\}\\ &- \frac{4}{5}\mathbb{K}\left\{\left(\frac{1}{r}Q_P - \frac{1}{3}\overset{\bullet}{Q}_P + \frac{\lambda}{3r}Q_B\right)Y + \frac{1}{r}\left(Q_B\frac{\partial B}{\partial\theta} + Q_C\frac{\partial C}{\partial\theta}\right)\right\} + \frac{4}{5}\mathbb{K}^2\left\{\left(\frac{1}{r}G_P - \frac{1}{3}\overset{\bullet}{G}_P + \frac{\lambda}{3r}G_B\right)Y + \frac{1}{r}\left(G_B\frac{\partial B}{\partial\theta} + G_C\frac{\partial C}{\partial\theta}\right)\right\}\\ &+ \frac{2}{3}\mathbb{K}^2\left\{-\frac{1}{2}\left(\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P\right)Y + \frac{\lambda}{2}\left(\nabla^2 L - \frac{\mu L}{r^2}\right)Y + \left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right)\frac{\partial B}{\partial\theta} + \left(\nabla^2 M - \frac{\mu M}{r^2} + \frac{4}{r^2}T_C\right)\frac{\partial C}{\partial\theta}\right\}.\end{aligned} \tag{89}$$

Write this equation concisely as

$$\frac{1}{2}\mathcal{A}_\theta Y + \mathcal{B}_\theta\frac{\partial B}{\partial\theta} + \mathcal{C}_\theta\frac{\partial C}{\partial\theta} = 0,$$

where

$$\begin{aligned}\mathcal{A}_\theta = (1-\mathrm{i}\omega\mathbb{K})T_P - (1-\mathrm{i}\omega\mathbb{K})\lambda L &+ 2\mathbb{K}\left(\frac{2}{3}\overset{\bullet}{V}_P - \frac{2}{r}V_P - \frac{2\lambda}{3r}V_B\right) + \frac{4}{5}\mathbb{K}\left(\frac{2}{3}\overset{\bullet}{Q}_P - \frac{2}{r}Q_P - \frac{2\lambda}{3r}Q_B\right)\\ &- \frac{4}{5}\mathbb{K}^2\left(\frac{2}{3}\overset{\bullet}{G}_P - \frac{2}{r}G_P - \frac{2\lambda}{3r}G_B\right) - \frac{2}{3}\mathbb{K}^2\left\{\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P - \lambda\left(\nabla^2 L - \frac{\mu L}{r^2}\right)\right\};\end{aligned}$$

formulas from $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ can be read off from Equation (89). From Equation (76), we have an expression for $(1-\mathrm{i}\omega\mathbb{K})T_P$; using this in $\mathcal{A}_\theta$, we obtain

$$\mathcal{A}_\theta = -(1-\mathrm{i}\omega\mathbb{K})\lambda L - 2\mathbb{K}\lambda\frac{V_B}{r} - \frac{4}{5}\mathbb{K}\lambda\frac{Q_B}{r} + \frac{4}{5}\mathbb{K}^2\lambda\frac{G_B}{r} + \frac{2}{3}\mathbb{K}^2\lambda\left(\nabla^2 L - \frac{\mu L}{r^2} + \frac{4}{r^2}T_B\right),$$

and this vanishes by Equation (88). Equation (88) also shows that $\mathcal{B}_\theta = 0$, whereas Equation (81) shows that $\mathcal{C}_\theta = 0$. Hence, the $\theta\theta$-component of the stress equation (49) is satisfied.

4.6 | The stress equation: summary

We have represented the five independent stress components using five independent radial functions, T_P, T_B, T_C, L and M; see Equations (61), (65) and (67). These functions satisfy five equations, namely, Equation (68)–(70), (81) and (88). If these equations are satisfied, then the stress equation (49) is satisfied.

5 | TWO SPECIAL CASES

5.1 | Spherical symmetry

For problems with spherical symmetry (such as that for a pulsating sphere), we have $n = 0$. Thus, $\lambda = 0$, and the 8×8 *PB*-system of Section 3.4 reduces to a 5×5 system. We can replace the spherical harmonic Y_0^0 by 1, and the corresponding vector spherical harmonics $\boldsymbol{B}$ and $\boldsymbol{C}$ are zero, as are the six functions V_B, V_C, Q_B, Q_C, G_B and G_C. The remaining equations come from Equations (37), (38), (41), (43) and (46) and are

$$3\mathrm{i}\omega\mathcal{P} = 2\overset{\bullet}{Q}_P + 5\overset{\bullet}{V}_P,\ \ 3\mathrm{i}\omega\Theta = 2\overset{\bullet}{Q}_P + 2\overset{\bullet}{V}_P,\ \ G_P = \mathrm{i}\omega V_P - \mathcal{P}', \tag{90}$$

$$Q_P - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_P = -\frac{15}{4}\mathbb{K}\Theta' - \frac{3}{2}\mathbb{K}\,G_P + \frac{27}{5}\mathbb{K}^2\,\mathcal{L}Q_P, \tag{91}$$

$$(1-\mathrm{i}\omega\mathbb{K})G_P = -\frac{4}{3}\mathbb{K}\,\mathcal{L}V_P - \frac{8}{15}\mathbb{K}\,\mathcal{L}Q_P + \frac{6}{5}\mathbb{K}^2\,\mathcal{L}G_P, \tag{92}$$

where $\mathcal{L}$ is a second-order differential operator:

$$(\mathcal{L}f)(r) = \nabla^2 f - \frac{2}{r^2}f = f'' + \frac{2}{r}f' - \frac{2}{r^2}f.$$

We could rewrite the equations as a first-order 8×8 system, with unknowns $\mathcal{P}$, Θ, V_P, V_P', Q_P, Q_P', G_P and G_P'. Alternatively, we could reduce to a smaller system. To do this, start by eliminating $\mathcal{P}$ and Θ, noting that, for example, $(\mathrm{d}/\mathrm{d}r)\overset{\bullet}{V}_P = \mathcal{L}V_P$ (see Equations 36 and 42), whence $3\mathrm{i}\omega\mathcal{P}' = \mathcal{L}\{2Q_P + 5V_P\}$ and $3\mathrm{i}\omega\Theta' = \mathcal{L}\{2Q_P + 2V_P\}$. The 3×3 system for V_P, Q_P and G_P then comprises

$$3\mathrm{i}\omega G_P = -3\omega^2 V_P - \mathcal{L}\{2Q_P + 5V_P\}, \qquad \mathrm{i}\omega\left(Q_P - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_P\right) = -\frac{5}{2}\mathbb{K}\,\mathcal{L}\{Q_P + V_P\} - \frac{3}{2}\mathrm{i}\omega\mathbb{K}\,G_P + \frac{27}{5}\mathrm{i}\omega\mathbb{K}^2\,\mathcal{L}Q_P$$

and Equation (92). Evidently, further reductions are possible.

Once V_P, Q_P and G_P have been determined, $\mathcal{P}$ and Θ are given by Equation (90), $\boldsymbol{v} = V_P\,\hat{\mathbf{r}}$, $\boldsymbol{q} = Q_P\,\hat{\mathbf{r}}$ and $\mathbf{g} = G_P\,\hat{\mathbf{r}}$. For the stresses, the non-trivial components are $\sigma_{rr} = T_P$ and $\sigma_{\theta\theta} = \sigma_{\Phi\Phi} = -\frac{1}{2}T_P$, where $T_P(r)$ is determined by Equation (68):

$$T_P' + (3/r)T_P = r^{-3}\,(\mathrm{d}/\mathrm{d}r)(r^3 T_P) = G_P,$$

which is a simple first-order differential equation.

Ben-Ami and Manela[9] have treated the spherically symmetric R13 equations. They found a single sixth-order differential equation for the temperature.[9, eq. 3.12] We have derived a similar equation for $\Theta(r)$; it agrees precisely with their equation. See Appendix D for details.

5.2 | Axial symmetry with $n = 1$

For simple problems with axial symmetry (such as that for oscillations of a rigid sphere), we can take $n = 1$ and $m = 0$. Thus, $\lambda = \sqrt{2}$ and $\mu = 0$. For Y_1^0, we can take $Y = P_1(\cos\theta) = \cos\theta$ (P_1 is a Legendre polynomial), whence $B = -2^{-1/2}\sin\theta$ and $C = 0$ (see Equation 27). Consequently, we can take $V_C = Q_C = G_C = T_C = 0$. However, the 8×8 *PB*-system defined in Section 3.4 does not simplify, although reductions can be made (such as elimination of $\mathcal{P}$ and Θ).

In any case, the formulas for the stresses simplify. The radial stresses are

$$\sigma_{rr} = T_P\cos\theta, \;\; \sigma_{r\theta} = -2^{-1/2}T_B\sin\theta, \;\; \sigma_{r\Phi} = 0.$$

The angular stresses are given by Equations (65)–(67). At first sight, these formulas involve L and M, but those functions are multiplied by $\frac{1}{2}\lambda Y + \partial B/\partial\theta$, and this combination vanishes when $n = 1$. Hence, the angular stresses are simply

$$\sigma_{\theta\theta} = \sigma_{\Phi\Phi} = -2^{-1}T_P\cos\theta, \;\; \sigma_{\theta\Phi} = 0.$$

For T_P and T_B, we have Equations (68) and (69), a coupled 2×2 system:

$$T_P' + \frac{3}{r}\,T_P - \frac{\lambda}{r}\,T_B = G_P, \;\; T_B' + \frac{3}{r}\,T_B - \frac{\lambda}{2r}\,T_P = G_B. \tag{93}$$

Let us compare with Torrilhon's steady analysis.[7] In order to eliminate many factors of $\lambda = \sqrt{2}$, write $G_B = \lambda\tilde{G}$, $Q_B = \lambda\tilde{Q}$, $T_B = -\lambda\tilde{T}$ and $V_B = \lambda\tilde{V}$. Then, as $\omega = 0$, Equations (37) and (38) reduce to $\overset{\bullet}{V}_P = (2/r)\tilde{V}$ and $\overset{\bullet}{Q}_P = (2/r)\tilde{Q}$; these are Torrilhon.[7, eqs. 25 and 26] From Equation $(41)_{1,2}$, we obtain $\tilde{G} = -r^{-1}\mathcal{P}$ and $G_P = -\mathcal{P}'$. Combining these with Equation (93), we obtain

$$\mathcal{P}' + T_P' + \frac{3}{r}\,T_P + \frac{2}{r}\,\tilde{T} = 0, \;\; \tilde{T}' + \frac{3}{r}\,\tilde{T} + \frac{1}{2r}\,T_P - \frac{1}{r}\mathcal{P} = 0; \tag{94}$$

these are Torrilhon.[7, eqs. 27 and 28] From Equation (43), we have

$$Q_P = -\frac{15}{4}\mathbb{K}\Theta' - \frac{3}{2}\mathbb{K}\,G_P + \frac{9}{5}\mathbb{K}^2\left\{3\nabla^2 Q_P - 8r^{-2}Q_P - (4/r)\tilde{Q}' + 8r^{-2}\tilde{Q}\right\}. \tag{95}$$

Differentiating $2\tilde{Q} = r\overset{\bullet}{Q}_P = rQ_P' + 2Q_P$ gives

$$2\tilde{Q}' = r\nabla^2 Q_P + (2/r)(\tilde{Q} - Q_P). \tag{96}$$

Then, using $G_P = -\mathcal{P}'$ and dividing by $\frac{3}{2}\mathbb{K}$, Equation (95) becomes

$$\frac{2}{3\mathbb{K}}Q_P = -\frac{5}{2}\Theta' + \mathcal{P}' + \frac{6}{5}\mathbb{K}\left\{\nabla^2 Q_P - 4r^{-2}Q_P + 4r^{-2}\tilde{Q}\right\};$$

this is Torrilhon.[7, eq. 30] A similar calculation shows that Equation (44) is equivalent to Torrilhon.[7, eq. 31] For the stresses, we have Equations (76) and (77). The first of these becomes

$$T_P = -2\mathbb{K}V_P' - \frac{4}{5}\mathbb{K}Q_P' + \mathbb{K}^2\Lambda_P, \tag{97}$$

where

$$\Lambda_P = \frac{8}{15}\left(G_P' - \frac{1}{r}G_P + \frac{1}{r}\tilde{G}\right) + \frac{2}{3}\left(\nabla^2 T_P - \frac{8}{r^2}T_P - \frac{8}{r^2}\tilde{T}\right),$$

and we have used

$$\frac{2}{3}\overset{\bullet}{V}_P - \frac{2}{r}V_P + \frac{2}{3r}\tilde{V} = \overset{\bullet}{V}_P - \frac{2}{r}V_P = V_P'$$

with a similar equation for Q_P. But, from Equation (86), we have

$$G_P' - \frac{1}{r}G_P + \frac{1}{r}\tilde{G} = \nabla^2 T_P - \frac{7}{r^2}T_P - \frac{10}{r^2}\tilde{T} - \frac{1}{r}\tilde{G}$$

whence

$$\Lambda_P = \frac{6}{5}\left(\nabla^2 T_P - \frac{22}{3r^2}T_P + \frac{4}{9r}\tilde{T}' - \frac{68}{9r^2}\tilde{T}\right)$$

after using $\tilde{G} = -r^{-1}\mathcal{P}$ and Equation $(94)_2$. Thus, Equation (97) agrees with Torrilhon.[7, eq. 32] Finally, consider Equation (77),

$$\tilde{T} = \mathbb{K}\left(\tilde{V}' - \frac{1}{2}V_P'\right) + \frac{2}{5}\mathbb{K}\left(\tilde{Q}' - \frac{1}{2}Q_P'\right) + \mathbb{K}^2\Lambda_B, \tag{98}$$

where

$$\Lambda_B = -\frac{2}{5}\left(\tilde{G}' - \frac{1}{r}\tilde{G} + \frac{1}{r}G_P\right) + \frac{2}{3}\left(\nabla^2\tilde{T} - \frac{6}{r^2}\tilde{T} - \frac{3}{r^2}T_P\right),$$

and we have used

$$\frac{1}{r}V_P + \dot{\tilde{V}} - \frac{3}{r}\tilde{V} = \frac{1}{r}V_P + \tilde{V}' - \frac{1}{r}\tilde{V} = \tilde{V}' - \frac{1}{2}V_P',$$

with a similar equation for Q_P. From Equation $(93)_2$, $\tilde{G} = -\tilde{T}' - (3/r)\tilde{T} - (2r)^{-1}T_P$ from which we can calculate $\tilde{G}'$. From Equation $(93)_1$, we also have $G_P = T_P' + (3/r)T_P + (2/r)\tilde{T}$. Hence,

$$\tilde{G}' - \frac{1}{r}\tilde{G} + \frac{1}{r}G_P = -\nabla^2\tilde{T} + \frac{8}{r^2}\tilde{T} + \frac{1}{2r}T_P' + \frac{4}{r^2}T_P$$

and

$$\Lambda_B = \frac{16}{15}\left(\nabla^2\tilde{T} - \frac{27}{4r^2}\tilde{T} - \frac{3}{16r}T_P' - \frac{27}{8r^2}T_P\right).$$

Then we see that Equation (98) agrees with Torrilhon[7, eq. 33] (after use of Torrilhon[7, eq. 26] to simplify the right-hand side of Torrilhon[7, eq. 33]).

6 | DISCUSSION AND PROSPECTS

We have shown how to solve the linearised form of the R13 equations using a two-step process. Five of the 13 unknowns are components of the symmetric trace-free stress tensor $\boldsymbol{\sigma}$. In the first step, we introduce the force vector $\boldsymbol{g} = \operatorname{div}\boldsymbol{\sigma}$, leading to an 11×11 system for two unknown scalars and the components of three vectors. This system is solved in spherical polar coordinates using vector spherical harmonics. The result is a coupled system of 11 ordinary differential equations for the radial variation of the 11 unknown functions. In fact, the system decouples into a 3×3 system and an 8×8 system. These systems depend on m and n, where Y_n^m is a typical spherical harmonic, but they do not couple to systems with different values of m and n: this property is familiar when solving simpler problems, such as when the scalar wave equation is solved by separation of variables.

In the second step, we have shown how to recover the stress components. This required guessing the form of the stresses and then showing that the proposed form is consistent with the known form of the force vector $\boldsymbol{g}$ and with the five of the R13 equations that involve the stresses. The five stress components are written in terms of five radial functions, but just two of these (denoted by L and M) are required to solve new equations. Interestingly, the stress components associated with m and n *do* couple with solutions at other values of m and n (in general), even though the force vector $\boldsymbol{g} = \operatorname{div} \boldsymbol{\sigma}$ is decoupled.

Evidently, much remains to be done. The systems obtained involve second derivatives in the radial direction. We could build a system of coupled first-order differential equations, and this could be done in a variety of ways. For example, there may be merit in developing a Stroh-like formulation, as has been done in anisotropic elastodynamics[10] and for small-on-large problems arising in nonlinear elasticity.[11]

Specific choices will involve the boundary conditions that are to be imposed, and these are complicated. For example, the boundary conditions on a rigid impermeable surface have been developed by Torrilhon[12, eq. 24] and linearised in Claydon et al.[5, eq. 2.5]; for steady flows ($\omega = 0$), see Torrilhon.[7, eqs. 8 and 10] In addition, for unbounded flow regions, we also expect to impose some kind of far-field or radiation conditions. These will be especially relevant for scattering problems, such as when a plane wave (see Appendix A) interacts with a rigid sphere. All this awaits further investigations.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

P. A. Martin https://orcid.org/0000-0002-0384-3885

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APPENDIX A: PLANE WAVES

Look for waves propagating in the x_1-direction: put $p = P\mathcal{E}$, $\vartheta = \Theta\mathcal{E}$, $v_i = V_i\mathcal{E}$, $q_i = Q_i\mathcal{E}$ and $\sigma_{ij} = S_{ij}\mathcal{E}$, where P, Θ, V_i, Q_i and S_{ij} are constants and $\mathcal{E} = \exp\{\mathrm{i}(kx_1 - \omega t)\}$. As $\mathcal{Q} = \mathrm{i}kQ_1\mathcal{E}$ and $\mathcal{V} = \mathrm{i}kV_1\mathcal{E}$, Equation (4) gives

$$-3\omega P + 2kQ_1 + 5kV_1 = 0 \text{ and } -3\omega\Theta + 2kQ_1 + 2kV_1 = 0. \tag{A1}$$

From Equation (3), $-\omega V_i + kP\delta_{i1} + kS_{i1} = 0$, $i = 1, 2, 3$; written out, we have

$$-\omega V_1 + kP + kS_{11} = 0, -\omega V_2 + kS_{12} = 0, -\omega V_3 + kS_{13} = 0. \tag{A2}$$

From Equation (11),

$$Q_i - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_i = -\frac{15}{4}\mathrm{i}k\mathbb{K}\Theta\delta_{i1} - \frac{3}{2}\mathrm{i}k\mathbb{K}S_{i1} - \frac{9}{5}k^2\mathbb{K}^2Q_i - \frac{18}{5}k^2\mathbb{K}^2Q_1\delta_{i1},\ i = 1, 2, 3; \tag{A3}$$

written out,

$$Q_1 - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_1 = -\frac{15}{4}\mathrm{i}k\mathbb{K}\Theta - \frac{3}{2}\mathrm{i}k\mathbb{K}S_{11} - \frac{27}{5}k^2\mathbb{K}^2Q_1, \tag{A4}$$

$$Q_2 - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_2 = -\frac{3}{2}\mathrm{i}k\mathbb{K}S_{12} - \frac{9}{5}k^2\mathbb{K}^2Q_2, \tag{A5}$$

$$Q_3 - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q_3 = -\frac{3}{2}\mathrm{i}k\mathbb{K}S_{13} - \frac{9}{5}k^2\mathbb{K}^2Q_3. \tag{A6}$$

Finally, consider Equation (12). From Equation (10), we have

$$\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} = \mathrm{i}k\left(\frac{1}{2}\ Q_i\delta_{j1} + \frac{1}{2}\ Q_j\delta_{i1} - \frac{1}{3}\ Q_1\delta_{ij}\right)\mathcal{E}$$

with a similar equation for $\partial v_{\langle i}/\partial x_{j\rangle}$, whereas Equation (14) gives

$$\frac{\partial}{\partial x_k}\frac{\partial\sigma_{\langle ij}}{\partial x_{k\rangle}} = -k^2\left(\frac{1}{3}S_{ij} + \frac{1}{5}S_{i1}\delta_{j1} + \frac{1}{5}S_{j1}\delta_{i1} - \frac{2}{15}S_{11}\delta_{ij}\right)\mathcal{E}.$$

Hence,

$$\begin{aligned}(1 - \mathrm{i}\omega\mathbb{K})S_{ij} = {} & -2\mathrm{i}k\mathbb{K}\left(\frac{1}{2}\ V_i\delta_{j1} + \frac{1}{2}\ V_j\delta_{i1} - \frac{1}{3}\ V_1\delta_{ij}\right) - \frac{4}{5}\mathrm{i}k\mathbb{K}\left(\frac{1}{2}\ Q_i\delta_{j1} + \frac{1}{2}\ Q_j\delta_{i1} - \frac{1}{3}\ Q_1\delta_{ij}\right) \\ & - 2k^2\mathbb{K}^2\left(\frac{1}{3}S_{ij} + \frac{1}{5}S_{i1}\delta_{j1} + \frac{1}{5}S_{j1}\delta_{i1} - \frac{2}{15}S_{11}\delta_{ij}\right).\end{aligned} \tag{A7}$$

Five non-trivial equations are contained in Equation (A7):

$$(1 - \mathrm{i}\omega\mathbb{K})S_{11} = -\frac{4}{3}\mathrm{i}k\mathbb{K}V_1 - \frac{8}{15}\mathrm{i}\mathbb{K}Q_1 - \frac{6}{5}k^2\mathbb{K}^2S_{11}, \tag{A8}$$

$$(1 - \mathrm{i}\omega\mathbb{K})S_{12} = -\mathrm{i}k\mathbb{K}V_2 - \frac{2}{5}\mathrm{i}k\mathbb{K}Q_2 - \frac{16}{15}k^2\mathbb{K}^2S_{12}, \tag{A9}$$

$$(1 - \mathrm{i}\omega\mathbb{K})S_{13} = -\mathrm{i}k\mathbb{K}V_3 - \frac{2}{5}\mathrm{i}k\mathbb{K}Q_3 - \frac{16}{15}k^2\mathbb{K}^2S_{13}, \tag{A10}$$

$$(1 - \mathrm{i}\omega\mathbb{K})S_{22} = \frac{2}{3}\mathrm{i}k\mathbb{K}V_1 + \frac{4}{15}\mathrm{i}k\mathbb{K}Q_1 - \frac{2}{3}k^2\mathbb{K}^2S_{22} + \frac{4}{15}k^2\mathbb{K}^2S_{11}, \tag{A11}$$

$$(1 - \mathrm{i}\omega\mathbb{K})S_{23} = -\frac{2}{3}k^2\mathbb{K}^2S_{23}. \tag{A12}$$

The equation obtained from Equation (A7) with $i = j = 3$ is redundant: it is a linear combination of Equations (A8) and (A11) because $S_{ii} = 0$.

There are 13 homogeneous equations and 13 unknowns. However, they decouple. Equation (A12) gives $S_{23} = 0$ unless $1 - \mathrm{i}\omega\mathbb{K} + \frac{2}{3}k^2\mathbb{K}^2 = 0$. Equations $(\mathrm{A2})_2$, (A5) and (A9) give a system of three equations for V_2, Q_2 and S_{12}. Equations $(\mathrm{A2})_3$, (A6) and (A10) give a similar system for V_3, Q_3 and S_{13}. These two 3×3 systems determine transverse wave motions. Longitudinal motions are obtained by solving Equations (A1), $(\mathrm{A2})_1$, (A4), (A8) and (A11), a 6×6 system for P, Θ, V_1, Q_1, S_{11} and S_{22}.

A.1 | Transverse motions

Consider Equations $(\mathrm{A2})_2$, (A5) and (A9) for V_2, Q_2 and S_{12}. The first equation gives $\omega V_2 = kS_{12}$. Eliminating S_{12} from the other two equations gives

$$\left(1 - \frac{3}{2}\mathrm{i}\omega\mathbb{K} + \frac{9}{5}k^2\mathbb{K}^2\right) Q_2 + \frac{3}{2}\mathrm{i}\omega\mathbb{K} V_2 = 0 \text{ and } \frac{2}{5}\mathrm{i}k^2\mathbb{K}Q_2 + \left((1 - \mathrm{i}\omega\mathbb{K})\omega + \mathrm{i}k^2\mathbb{K} + \frac{16}{15}k^2\mathbb{K}^2\omega\right) V_2 = 0.$$

Setting the 2×2 determinant to zero gives

$$\left(1 - \frac{3}{2}\mathrm{i}\omega\mathbb{K} + \frac{9}{5}k^2\mathbb{K}^2\right)\left((1 - \mathrm{i}\omega\mathbb{K})\omega + \mathrm{i}k^2\mathbb{K} + \frac{16}{15}k^2\mathbb{K}^2\omega\right) + \frac{3}{5}\omega k^2\mathbb{K}^2 = 0. \tag{A13}$$

Given a real frequency ω, Equation (A13) is a quadratic equation for k^2. This is the dispersion relation for transverse waves. Exactly the same relation is obtained when solving Equations $(\mathrm{A2})_3$, (A6) and (A10) for V_3, Q_3 and S_{13}.

We note that we cannot put $\mathbb{K} = 0$ in Equation (A13). This is not surprising because the governing equations do not support transverse motions when $\mathbb{K} = 0$; see Section 2.2. We are not aware of previous studies of transverse motions.

A.2 | Longitudinal motions

Consider Equations (A1), $(\mathrm{A2})_1$, (A4), (A8) and (A11) for P, Θ, V_1, Q_1, S_{11} and S_{22}. Eliminate P and Θ from Equations $(\mathrm{A2})_1$ and (A4), using Equation (A1). When combined with Equation (A8), we obtain a 3×3 system for V_1, Q_1 and S_{11},

$$2k^2Q_1 + (5k^2 - 3\omega^2)V_1 + 3\omega k S_{11} = 0, \quad \frac{8}{15}\mathrm{i}\mathbb{K}Q_1 + \frac{4}{3}\mathrm{i}k\mathbb{K}V_1 + \left(1 - \mathrm{i}\omega\mathbb{K} + \frac{6}{5}k^2\mathbb{K}^2\right) S_{11} = 0,$$
$$\left(\omega + \frac{1}{2}\mathrm{i}\mathbb{K}(5k^2 - 3\omega^2) + \frac{27}{5}\omega k^2\mathbb{K}^2\right) Q_1 + \frac{5}{2}\mathrm{i}k^2\mathbb{K}V_1 + \frac{3}{2}\mathrm{i}\omega k\mathbb{K}S_{11} = 0.$$

This system is equivalent to the system studied in Struchtrup and Torrilhon[3, §IV.A]; see also Struchtrup.[4, §10.1] Note that Equation (A11) gives a formula for S_{22},

$$\left(1 - \mathrm{i}\omega\mathbb{K} + \frac{2}{3}k^2\mathbb{K}^2\right) S_{22} = \frac{2}{3}\mathrm{i}k\mathbb{K}\left(V_1 + \frac{2}{5}Q_1 - \frac{2}{5}\mathrm{i}k\mathbb{K}S_{11}\right),$$

and recall that $S_{ii} = 0$.

APPENDIX B: FORMULAS IN SPHERICAL POLAR COORDINATES

Formulas for grad, div and ∇^2 in spherical polar coordinates are familiar; see Equations (31), (39) and

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial p}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\Phi^2}. \tag{B1}$$

In Cartesians, $(\nabla\boldsymbol{v})_{ij} = \partial v_j/\partial x_i$. In spherical polars, we have Dahlen and Tromp[15, eq. A.138]

$$(\nabla\boldsymbol{v})_{rr} = \frac{\partial v_r}{\partial r}, \quad (\nabla\boldsymbol{v})_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}, \quad (\nabla\boldsymbol{v})_{\Phi\Phi} = \frac{1}{r\sin\theta}\frac{\partial v_\Phi}{\partial\Phi} + \frac{v_r}{r} + \frac{\cot\theta}{r}v_\theta, \quad (\nabla\boldsymbol{v})_{r\theta} = \frac{\partial v_\theta}{\partial r}, \quad (\nabla\boldsymbol{v})_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r},$$
$$(\nabla\boldsymbol{v})_{r\Phi} = \frac{\partial v_\Phi}{\partial r}, \quad (\nabla\boldsymbol{v})_{\Phi r} = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\Phi} - \frac{v_\Phi}{r}, \quad (\nabla\boldsymbol{v})_{\theta\Phi} = \frac{1}{r}\frac{\partial v_\Phi}{\partial\theta}, \quad (\nabla\boldsymbol{v})_{\Phi\theta} = \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\Phi} - \frac{\cot\theta}{r}v_\Phi.$$

Note that the trace of $\nabla \boldsymbol{v}$ is $\operatorname{div} \boldsymbol{v}$. Denote the symmetrised trace-free version of $\nabla \boldsymbol{v}$ by $\langle \nabla \boldsymbol{v} \rangle$. Then

$$\langle \nabla \boldsymbol{v} \rangle_{rr} = (\nabla \boldsymbol{v})_{\langle rr \rangle} = \frac{\partial v_r}{\partial r} - \frac{1}{3} \operatorname{div} \boldsymbol{v}, \tag{B2}$$

$$\langle \nabla \boldsymbol{v} \rangle_{\theta\theta} = (\nabla \boldsymbol{v})_{\langle \theta\theta \rangle} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} - \frac{1}{3} \operatorname{div} \boldsymbol{v}, \tag{B3}$$

$$\langle \nabla \boldsymbol{v} \rangle_{r\theta} = (\nabla \boldsymbol{v})_{\langle r\theta \rangle} = \frac{1}{2} \frac{\partial v_\theta}{\partial r} + \frac{1}{2r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{2r}, \tag{B4}$$

$$\langle \nabla \boldsymbol{v} \rangle_{r\Phi} = (\nabla \boldsymbol{v})_{\langle r\Phi \rangle} = \frac{1}{2} \frac{\partial v_\Phi}{\partial r} + \frac{1}{2r \sin\theta} \frac{\partial v_r}{\partial \Phi} - \frac{v_\Phi}{2r}, \tag{B5}$$

$$\langle \nabla \boldsymbol{v} \rangle_{\theta\Phi} = (\nabla \boldsymbol{v})_{\langle \theta\Phi \rangle} = \frac{1}{2r} \frac{\partial v_\Phi}{\partial \theta} + \frac{1}{2r \sin\theta} \frac{\partial v_\theta}{\partial \Phi} - \frac{\cot\theta}{2r} v_\Phi, \tag{B6}$$

with $\langle \nabla \boldsymbol{v} \rangle_{rr} + \langle \nabla \boldsymbol{v} \rangle_{\theta\theta} + \langle \nabla \boldsymbol{v} \rangle_{\Phi\Phi} = 0$.

Let $\boldsymbol{\sigma}$ be a symmetric tensor. The vector $\operatorname{div} \boldsymbol{\sigma}$ has the following entries[15, eq. A.144]:

$$(\operatorname{div} \boldsymbol{\sigma})_r = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \sigma_{rr} \right) + \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \theta} (\sigma_{r\theta} \sin\theta) + \frac{\partial \sigma_{r\Phi}}{\partial \Phi} \right) - \frac{1}{r} (\sigma_{\theta\theta} + \sigma_{\Phi\Phi}), \tag{B7}$$

$$(\operatorname{div} \boldsymbol{\sigma})_\theta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \sigma_{r\theta} \right) + \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \theta} (\sigma_{\theta\theta} \sin\theta) + \frac{\partial \sigma_{\theta\Phi}}{\partial \Phi} \right) + \frac{1}{r} (\sigma_{r\theta} - \sigma_{\Phi\Phi} \cot\theta), \tag{B8}$$

$$(\operatorname{div} \boldsymbol{\sigma})_\Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \sigma_{r\Phi} \right) + \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \theta} (\sigma_{\theta\Phi} \sin\theta) + \frac{\partial \sigma_{\Phi\Phi}}{\partial \Phi} \right) + \frac{1}{r} (\sigma_{r\Phi} + \sigma_{\theta\Phi} \cot\theta). \tag{B9}$$

The diagonal elements of $\nabla^2 \boldsymbol{\sigma}$ are[7, Appendix 1]

$$\left(\nabla^2 \boldsymbol{\sigma} \right)_{rr} = \nabla^2 \sigma_{rr} - \frac{2}{r^2} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\Phi\Phi}) - \frac{4}{r^2 \sin\theta} \left(\frac{\partial}{\partial \theta} (\sigma_{r\theta} \sin\theta) + \frac{\partial \sigma_{r\Phi}}{\partial \Phi} \right), \tag{B10}$$

$$\left(\nabla^2 \boldsymbol{\sigma} \right)_{\theta\theta} = \nabla^2 \sigma_{\theta\theta} + \frac{4}{r^2} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{2}{r^2 \sin^2\theta} \left(\sigma_{rr} \sin^2\theta - \sigma_{\theta\theta} + \sigma_{\Phi\Phi} \cos^2\theta \right) - \frac{4 \cot\theta}{r^2 \sin\theta} \frac{\partial \sigma_{\theta\Phi}}{\partial \Phi}, \tag{B11}$$

$$\left(\nabla^2 \boldsymbol{\sigma} \right)_{\Phi\Phi} = \nabla^2 \sigma_{\Phi\Phi} + \frac{4 \cot\theta}{r^2} \sigma_{r\theta} + \frac{4}{r^2 \sin\theta} \frac{\partial \sigma_{r\Phi}}{\partial \Phi} + \frac{4 \cot\theta}{r^2 \sin\theta} \frac{\partial \sigma_{\theta\Phi}}{\partial \Phi} + \frac{2}{r^2 \sin^2\theta} \left(\sigma_{rr} \sin^2\theta + \sigma_{\theta\theta} \cos^2\theta - \sigma_{\Phi\Phi} \right). \tag{B12}$$

Note the trace $\left(\nabla^2 \boldsymbol{\sigma} \right)_{rr} + \left(\nabla^2 \boldsymbol{\sigma} \right)_{\theta\theta} + \left(\nabla^2 \boldsymbol{\sigma} \right)_{\Phi\Phi} = 0$ when $\boldsymbol{\sigma}$ is trace-free, $\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{\Phi\Phi} = 0$.

The off-diagonal terms are[7, Appendix 1]

$$\left(\nabla^2 \boldsymbol{\sigma} \right)_{r\theta} = \nabla^2 \sigma_{r\theta} - \frac{2 \cot\theta}{r^2 \sin\theta} \frac{\partial \sigma_{r\Phi}}{\partial \Phi} - \frac{1 + 4\sin^2\theta}{r^2 \sin^2\theta} \sigma_{r\theta} + \frac{2}{r^2} \frac{\partial}{\partial \theta} (\sigma_{rr} - \sigma_{\theta\theta}) - \frac{2}{r^2 \sin\theta} \frac{\partial \sigma_{\theta\Phi}}{\partial \Phi} + \frac{2 \cot\theta}{r^2} (\sigma_{\Phi\Phi} - \sigma_{\theta\theta}), \tag{B13}$$

$$\left(\nabla^2 \boldsymbol{\sigma} \right)_{r\Phi} = \nabla^2 \sigma_{r\Phi} + \frac{2 \cot\theta}{r^2 \sin\theta} \frac{\partial \sigma_{r\theta}}{\partial \Phi} - \frac{1 + 4\sin^2\theta}{r^2 \sin^2\theta} \sigma_{r\Phi} + \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \Phi} (\sigma_{rr} - \sigma_{\Phi\Phi}) - \frac{2}{r^2} \frac{\partial \sigma_{\theta\Phi}}{\partial \theta} - \frac{4 \cot\theta}{r^2} \sigma_{\theta\Phi}, \tag{B14}$$

$$\left(\nabla^2 \boldsymbol{\sigma} \right)_{\theta\Phi} = \nabla^2 \sigma_{\theta\Phi} + \frac{2}{r^2 \sin\theta} \frac{\partial \sigma_{r\theta}}{\partial \Phi} + \frac{2}{r^2} \frac{\partial \sigma_{r\Phi}}{\partial \theta} - \frac{2 \cot\theta}{r^2} \sigma_{r\Phi} + \frac{2 \cot\theta}{r^2 \sin\theta} \frac{\partial}{\partial \Phi} (\sigma_{\theta\theta} - \sigma_{\Phi\Phi}) - \frac{2(1 + \cos^2\theta)}{r^2 \sin^2\theta} \sigma_{\theta\Phi}. \tag{B15}$$

APPENDIX C: COMPUTATION OF $\nabla^2 \boldsymbol{\sigma}$

We start with $\left(\nabla^2 \boldsymbol{\sigma} \right)_{rr}$. Substituting Equation (61) and $\sigma_{\theta\theta} + \sigma_{\Phi\Phi} = -\sigma_{rr}$ in Equation (B10), we readily obtain Equation (71) after use of Equations (33), (34) and (B1).

Next, from Equation (B13), $\left(\nabla^2 \boldsymbol{\sigma} \right)_{r\theta} = \nabla^2 \sigma_{r\theta} + r^{-2} \Omega_1 + 2\, r^{-2} \Omega_2$, with

$$\Omega_1 = -\frac{2 \cot\theta}{\sin\theta} \frac{\partial \sigma_{r\Phi}}{\partial \Phi} - \frac{1 + 4\sin^2\theta}{\sin^2\theta} \sigma_{r\theta}, \ \Omega_2 = \frac{\partial}{\partial \theta} (\sigma_{rr} - \sigma_{\theta\theta}) - \frac{1}{\sin\theta} \frac{\partial \sigma_{\theta\Phi}}{\partial \Phi} + (\sigma_{\Phi\Phi} - \sigma_{\theta\theta}) \cot\theta.$$

For $\nabla^2\sigma_{r\theta}$, we use Equation (B1) and the following relations, obtained by differentiating Equation (33):

$$\mathcal{D}B \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial B}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 B}{\partial\Phi^2} = -2\frac{\partial B}{\partial\theta}\cot\theta - B\frac{\cos 2\theta}{\sin^2\theta} - \lambda^2 B - 2\lambda Y\cot\theta, \tag{C1}$$

$$\mathcal{D}C \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial C}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 C}{\partial\Phi^2} = -2\frac{\partial C}{\partial\theta}\cot\theta - C\frac{\cos 2\theta}{\sin^2\theta} - \lambda^2 C. \tag{C2}$$

Then $\nabla^2\sigma_{r\theta} = B\nabla^2 T_B + C\nabla^2 T_C + (T_B\,\mathcal{D}B + T_C\,\mathcal{D}C)/r^2$. We also have

$$\Omega_1 = T_C\left(2\frac{\partial C}{\partial\theta}\cot\theta + \frac{C}{\sin^2\theta}\left[\cos 2\theta - 4\sin^2\theta\right]\right) + T_B\left(2\frac{\partial B}{\partial\theta}\cot\theta + \frac{B}{\sin^2\theta}\left[\cos 2\theta - 4\sin^2\theta\right] + 2\lambda Y\cot\theta\right).$$

Hence, combining this with the expression for $\nabla^2\sigma_{r\theta}$, we see that the terms containing $\partial B/\partial\theta$, $\partial C/\partial\theta$ and $Y\cot\theta$ all cancel, leaving

$$\nabla^2\sigma_{r\theta} + \frac{\Omega_1}{r^2} = \left(\nabla^2 T_B - \frac{\lambda^2+4}{r^2}T_B\right)B + \left(\nabla^2 T_C - \frac{\lambda^2+4}{r^2}T_C\right)C. \tag{C3}$$

For Ω_2, we use Equations (63) and (64) and obtain

$$\begin{aligned}
\Omega_2 &= \frac{\partial}{\partial\theta}\left(\frac{1}{2}(3T_P - \lambda L)Y - L\frac{\partial B}{\partial\theta} - M\frac{\partial C}{\partial\theta}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\Phi}\left(-\frac{1}{2}\lambda MY + L\frac{\partial C}{\partial\theta} - M\frac{\partial B}{\partial\theta}\right) - \left(\lambda LY + 2L\frac{\partial B}{\partial\theta} + 2M\frac{\partial C}{\partial\theta}\right)\cot\theta \\
&= \frac{1}{2}(3T_P - \lambda L)\lambda B + \frac{1}{2}\lambda^2 MC - \lambda LY\cot\theta - L\left(\frac{\partial^2 B}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi}\right) - M\left(\frac{\partial^2 C}{\partial\theta^2} - \frac{1}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi}\right) - 2\left(L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right)\cot\theta \\
&= \frac{1}{2}(3\lambda T_P + \mu L)B + \frac{1}{2}\mu MC,
\end{aligned}$$

where $\mu = \lambda^2 - 2$. Combining this formula with Equation (C3), we obtain Equation (72).

Next, from Equation (B14), $\left(\nabla^2\boldsymbol{\sigma}\right)_{r\Phi} = \nabla^2\sigma_{r\Phi} + r^{-2}\Omega_3 + 2\,r^{-2}\Omega_4$, with

$$\Omega_3 = \frac{2\cot\theta}{\sin\theta}\frac{\partial\sigma_{r\theta}}{\partial\Phi} - \frac{1+4\sin^2\theta}{\sin^2\theta}\sigma_{r\Phi},\quad \Omega_4 = \frac{1}{\sin\theta}\frac{\partial}{\partial\Phi}\left(\sigma_{rr} - \sigma_{\Phi\Phi}\right) - \frac{\partial\sigma_{\theta\Phi}}{\partial\theta} - 2\sigma_{\theta\Phi}\cot\theta.$$

Then $\nabla^2\sigma_{r\Phi} = C\nabla^2 T_B - B\nabla^2 T_C + (T_B\,\mathcal{D}C - T_C\,\mathcal{D}B)/r^2$. We also have

$$\Omega_3 = T_B\left(2\frac{\partial C}{\partial\theta}\cot\theta + \frac{C}{\sin^2\theta}[\cos 2\theta - 4\sin^2\theta]\right) - T_C\left(2\frac{\partial B}{\partial\theta}\cot\theta + \frac{B}{\sin^2\theta}[\cos 2\theta - 4\sin^2\theta] + 2\lambda Y\cot\theta\right).$$

Hence, combining this with the expression for $\nabla^2\sigma_{r\Phi}$, we obtain

$$\nabla^2\sigma_{r\Phi} + \frac{\Omega_3}{r^2} = \left(\nabla^2 T_B - \frac{\lambda^2+4}{r^2}T_B\right)C - \left(\nabla^2 T_C - \frac{\lambda^2+4}{r^2}T_B\right)B. \tag{C4}$$

For Ω_4,

$$\begin{aligned}
\Omega_4 &= \frac{1}{\sin\theta}\frac{\partial}{\partial\Phi}\left(\frac{1}{2}(3T_P + \lambda L)Y + L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right) - \frac{\partial}{\partial\theta}\left(-\frac{1}{2}\lambda MY + L\frac{\partial C}{\partial\theta} - M\frac{\partial B}{\partial\theta}\right) - 2\left(-\frac{1}{2}\lambda MY + L\frac{\partial C}{\partial\theta} - M\frac{\partial B}{\partial\theta}\right)\cot\theta \\
&= \frac{1}{2}(3T_P + \lambda L)\lambda C + \frac{1}{2}\lambda^2 MB + \lambda MY\cot\theta - L\left(\frac{\partial^2 C}{\partial\theta^2} - \frac{1}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi}\right) + M\left(\frac{\partial^2 B}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi}\right) - 2L\frac{\partial C}{\partial\theta}\cot\theta + 2M\frac{\partial B}{\partial\theta}\cot\theta \\
&= \frac{1}{2}(3\lambda T_P + \mu L)C - \frac{1}{2}\mu MB.
\end{aligned}$$

Combining this formula with Equation (C4) gives Equation (73).

For the $\theta\theta$-component, we have Equation (B11), in which

$$
\begin{aligned}
\nabla^2\sigma_{\theta\theta} &= -\frac{1}{2}\left(\nabla^2 T_P - \frac{\lambda^2}{r^2}T_P\right)Y + \frac{\lambda}{2}\left(\nabla^2 L - \frac{\lambda^2}{r^2}L\right)Y + (\nabla^2 L)\frac{\partial B}{\partial\theta} + (\nabla^2 M)\frac{\partial C}{\partial\theta} + \frac{L}{r^2}\,\mathcal{D}\frac{\partial B}{\partial\theta} + \frac{M}{r^2}\,\mathcal{D}\frac{\partial C}{\partial\theta},\\
\frac{\partial\sigma_{r\theta}}{\partial\theta} &= T_B\frac{\partial B}{\partial\theta} + T_C\frac{\partial C}{\partial\theta}, \qquad \frac{\partial\sigma_{\theta\Phi}}{\partial\Phi} = -\frac{1}{2}\lambda^2 MC\sin\theta + L\frac{\partial^2 C}{\partial\theta\,\partial\Phi} - M\frac{\partial^2 B}{\partial\theta\,\partial\Phi},\\
\sigma_{rr}\sin^2\theta - \sigma_{\theta\theta} + \sigma_{\Phi\Phi}\cos^2\theta &= \frac{3}{2}T_P Y\sin^2\theta - (1+\cos^2\theta)\left(\frac{1}{2}\lambda LY + L\frac{\partial B}{\partial\theta} + M\frac{\partial C}{\partial\theta}\right).
\end{aligned}
$$

Substituting in Equation (B11) gives

$$
\begin{aligned}
(\nabla^2\boldsymbol{\sigma})_{\theta\theta} = &-\frac{1}{2}\left(\nabla^2 T_P - \frac{\lambda^2+6}{r^2}T_P\right)Y + \frac{\lambda}{2}\left(\nabla^2 L - \frac{\lambda^2}{r^2}L\right)Y\\
&+\left(\nabla^2 L + \frac{4}{r^2}T_B\right)\frac{\partial B}{\partial\theta} + \left(\nabla^2 M + \frac{4}{r^2}T_C\right)\frac{\partial C}{\partial\theta} + \frac{L}{r^2}\Omega_5 + \frac{M}{r^2}\Omega_6,
\end{aligned}
\tag{C5}
$$

where

$$
\Omega_5 = \mathcal{D}\frac{\partial B}{\partial\theta} - \frac{4\cot\theta}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi} - \frac{1+\cos^2\theta}{\sin^2\theta}\left(\lambda Y + 2\frac{\partial B}{\partial\theta}\right),
\tag{C6}
$$

$$
\Omega_6 = \mathcal{D}\frac{\partial C}{\partial\theta} + \frac{4\cot\theta}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi} + 2\lambda^2 C\cot\theta - 2\,\frac{1+\cos^2\theta}{\sin^2\theta}\,\frac{\partial C}{\partial\theta}.
\tag{C7}
$$

From the definition of the operator $\mathcal{D}$, Equation (33), we have

$$
\mathcal{D}\frac{\partial B}{\partial\theta} = \frac{\partial^3 B}{\partial\theta^3} + \frac{1}{\sin^2\theta}\frac{\partial^3 B}{\partial\theta\,\partial\Phi^2} + \frac{\partial^2 B}{\partial\theta^2}\cot\theta
$$

whereas, from Equation (C1), we have

$$
\frac{\partial^2 B}{\partial\Phi^2} = -\frac{\partial^2 B}{\partial\theta^2}\sin^2\theta - \frac{3}{2}\frac{\partial B}{\partial\theta}\sin 2\theta - B(\cos 2\theta + \lambda^2\sin^2\theta) - \lambda Y\sin 2\theta
$$

whence

$$
\frac{\partial^3 B}{\partial\theta\,\partial\Phi^2} = -\frac{\partial^3 B}{\partial\theta^3}\sin^2\theta - \frac{5}{2}\frac{\partial^2 B}{\partial\theta^2}\sin 2\theta - \frac{\partial B}{\partial\theta}(4\cos 2\theta + \lambda^2\sin^2\theta) + 2(1-\lambda^2)B\sin 2\theta - 2\lambda Y\cos 2\theta
$$

and

$$
\mathcal{D}\frac{\partial B}{\partial\theta} = -4\frac{\partial^2 B}{\partial\theta^2}\cot\theta - \frac{4\cos 2\theta + \lambda^2\sin^2\theta}{\sin^2\theta}\,\frac{\partial B}{\partial\theta} + 4(1-\lambda^2)B\cot\theta - \frac{2\cos 2\theta}{\sin^2\theta}\lambda Y.
$$

From Equation (63), we have

$$
-\frac{4\cot\theta}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi} = 4\frac{\partial^2 B}{\partial\theta^2}\cot\theta + 8\frac{\partial B}{\partial\theta}\cot^2\theta - 4(1-\lambda^2)B\cot\theta + 4\lambda Y\cot^2\theta
$$

whence

$$
\mathcal{D}\frac{\partial B}{\partial\theta} - \frac{4\cot\theta}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi} = \frac{4-\lambda^2\sin^2\theta}{\sin^2\theta}\,\frac{\partial B}{\partial\theta} + \frac{2\lambda Y}{\sin^2\theta},
\tag{C8}
$$

and then substitution in Equation (C6) gives $\Omega_5 = \lambda Y - \mu\,\partial B/\partial\theta$.

For Ω_6, we use Equation (C2) to give

$$
\frac{\partial^2 C}{\partial\Phi^2} = -\frac{\partial^2 C}{\partial\theta^2}\sin^2\theta - \frac{3}{2}\frac{\partial C}{\partial\theta}\sin 2\theta - C(\cos 2\theta + \lambda^2\sin^2\theta)
$$

whence

$$
\frac{\partial^3 C}{\partial\theta\,\partial\Phi^2} = -\frac{\partial^3 C}{\partial\theta^3}\sin^2\theta - \frac{5}{2}\frac{\partial^2 C}{\partial\theta^2}\sin 2\theta - \frac{\partial C}{\partial\theta}(4\cos 2\theta + \lambda^2\sin^2\theta) + (2-\lambda^2)C\sin 2\theta
$$

and

$$\mathcal{D}\frac{\partial C}{\partial \theta} = -4\frac{\partial^2 C}{\partial \theta^2}\cot\theta - \frac{4\cos 2\theta + \lambda^2\sin^2\theta}{\sin^2\theta}\frac{\partial C}{\partial \theta} + 2(2-\lambda^2)C\cot\theta.$$

From Equation (64)

$$\frac{4\cot\theta}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi} = 4\frac{\partial^2 C}{\partial\theta^2}\cot\theta + 8\frac{\partial C}{\partial\theta}\cot^2\theta - 4C\cot\theta$$

whence

$$\mathcal{D}\frac{\partial C}{\partial \theta} + \frac{4\cot\theta}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi} = \frac{4-\lambda^2\sin^2\theta}{\sin^2\theta}\,\frac{\partial C}{\partial\theta} - 2\lambda^2 C\cot\theta \tag{C9}$$

and then substitution in Equation (C7) gives $\Omega_6 = -\mu\,\partial C/\partial\theta$. Finally, substitution in Equation (C5) gives Equation (74).

For the $\theta\Phi$-component, we have Equation (B15), which we write as $\left(\nabla^2\boldsymbol{\sigma}\right)_{\theta\Phi} = \nabla^2\sigma_{\theta\Phi} + 2r^{-2}(\Omega_7+\Omega_8)$, where

$$\Omega_7 = \frac{\cot\theta}{\sin\theta}\frac{\partial}{\partial\Phi}(\sigma_{\theta\theta}-\sigma_{\Phi\Phi}) - \frac{1+\cos^2\theta}{\sin^2\theta}\sigma_{\theta\Phi},\ \Omega_8 = \frac{1}{\sin\theta}\frac{\partial\sigma_{r\theta}}{\partial\Phi} + \frac{\partial\sigma_{r\Phi}}{\partial\theta} - \sigma_{r\Phi}\cot\theta.$$

Proceeding as before,

$$\begin{aligned}
\nabla^2\sigma_{\theta\Phi} &= -\frac{\lambda}{2}\left(\nabla^2 M - \frac{\lambda^2}{r^2}M\right)Y + (\nabla^2 L)\frac{\partial C}{\partial\theta} + \frac{L}{r^2}\,\mathcal{D}\frac{\partial C}{\partial\theta} - (\nabla^2 M)\frac{\partial B}{\partial\theta} - \frac{M}{r^2}\,\mathcal{D}\frac{\partial B}{\partial\theta},\\
\Omega_7 &= \lambda^2 LC\cot\theta + \frac{2\cot\theta}{\sin\theta}\left(L\frac{\partial^2 B}{\partial\theta\,\partial\Phi} + M\frac{\partial^2 C}{\partial\theta\,\partial\Phi}\right) + \frac{1+\cos^2\theta}{\sin^2\theta}\left(\frac{1}{2}\lambda MY - L\frac{\partial C}{\partial\theta} + M\frac{\partial B}{\partial\theta}\right),
\end{aligned}$$

whence

$$\begin{aligned}
\nabla^2\sigma_{\theta\Phi} + \frac{2}{r^2}\Omega_7 = &-\frac{\lambda}{2}\left(\nabla^2 M - \frac{\lambda^2}{r^2}M\right)Y + (\nabla^2 L)\frac{\partial C}{\partial\theta} - (\nabla^2 M)\frac{\partial B}{\partial\theta} + \frac{2(1+\cos^2\theta)}{r^2\sin^2\theta}\left(\frac{1}{2}\lambda MY - L\frac{\partial C}{\partial\theta} + M\frac{\partial B}{\partial\theta}\right)\\
&+\frac{L}{r^2}\left\{\mathcal{D}\frac{\partial C}{\partial\theta} + \frac{4\cot\theta}{\sin\theta}\frac{\partial^2 B}{\partial\theta\,\partial\Phi} + 2\lambda^2 C\cot\theta\right\} - \frac{M}{r^2}\left\{\mathcal{D}\frac{\partial B}{\partial\theta} - \frac{4\cot\theta}{\sin\theta}\frac{\partial^2 C}{\partial\theta\,\partial\Phi}\right\}.
\end{aligned}$$

Using Equations (C8) and (C9), the last line of this equation becomes

$$\frac{L}{r^2}\,\frac{4-\lambda^2\sin^2\theta}{\sin^2\theta}\,\frac{\partial C}{\partial\theta} - \frac{M}{r^2}\left(\frac{4-\lambda^2\sin^2\theta}{\sin^2\theta}\,\frac{\partial B}{\partial\theta} + \frac{2\lambda Y}{\sin^2\theta}\right)$$

whence

$$\nabla^2\sigma_{\theta\Phi} + \frac{2}{r^2}\Omega_7 = -\frac{\lambda}{2}\left(\nabla^2 M - \frac{\mu}{r^2}M\right)Y + \left(\nabla^2 L - \frac{\mu}{r^2}L\right)\frac{\partial C}{\partial\theta} - \left(\nabla^2 M - \frac{\mu}{r^2}M\right)\frac{\partial B}{\partial\theta}.$$

For Ω_8, we have

$$\begin{aligned}
\Omega_8 &= \frac{1}{\sin\theta}\frac{\partial}{\partial\Phi}(T_B B + T_C C) + \frac{\partial}{\partial\theta}(T_B C - T_C B) - (T_B C - T_C B)\cot\theta\\
&= T_B\left(\frac{1}{\sin\theta}\frac{\partial B}{\partial\Phi} + \frac{\partial C}{\partial\theta} - C\cot\theta\right) + T_C\left(\frac{1}{\sin\theta}\frac{\partial C}{\partial\Phi} - \frac{\partial B}{\partial\theta} + B\cot\theta\right) = 2T_B\frac{\partial C}{\partial\theta} - T_C\left(\lambda Y + 2\,\frac{\partial B}{\partial\theta}\right).
\end{aligned}$$

Finally, we obtain the formula (75).

APPENDIX D: SPHERICAL SYMMETRY: AN ODE FOR Θ

For motions with spherical symmetry, the governing equations are Equations (90)–(92). If we drop the subscript P and eliminate $\mathcal{P}$ using Equation $(90)_{1,3}$, we obtain

$$\begin{aligned}
&3\mathrm{i}\omega\Theta = 2\overset{\bullet}{Q} + 2\overset{\bullet}{V},\ 3\mathrm{i}\omega G = -2\,\mathcal{L}Q - 3\omega^2 V - 5\,\mathcal{L}V,\\
&Q - \frac{3}{2}\mathrm{i}\omega\mathbb{K}Q - \frac{27}{5}\mathbb{K}^2\,\mathcal{L}Q + \frac{3}{2}\mathbb{K}\,G = -\frac{15}{4}\mathbb{K}\Theta',\\
&(1-\mathrm{i}\omega\mathbb{K})G - \frac{6}{5}\mathbb{K}^2\,\mathcal{L}G + \frac{4}{3}\mathbb{K}\,\mathcal{L}V + \frac{8}{15}\mathbb{K}\,\mathcal{L}Q = 0.
\end{aligned}$$

Apply the overdot operator to the last three equations using $\overset{\bullet}{\overline{\Theta'}} = \nabla^2\Theta$ and $\overset{\bullet}{\overline{\mathcal{L}Q}} = \nabla^2\overset{\bullet}{Q}$:

$$3\mathrm{i}\omega\Theta = 2\overset{\bullet}{Q} + 2\overset{\bullet}{V},\ 3\mathrm{i}\omega\overset{\bullet}{G} = -2\ \nabla^2\overset{\bullet}{Q} - 3\omega^2\overset{\bullet}{V} - 5\ \nabla^2\overset{\bullet}{V},$$
$$\overset{\bullet}{Q} - \frac{3}{2}\mathrm{i}\omega\mathbb{K}\overset{\bullet}{Q} - \frac{27}{5}\mathbb{K}^2\ \nabla^2\overset{\bullet}{Q} + \frac{3}{2}\mathbb{K}\overset{\bullet}{G} = -\frac{15}{4}\mathbb{K}\nabla^2\Theta,$$
$$(1-\mathrm{i}\omega\mathbb{K})\overset{\bullet}{G} - \frac{6}{5}\mathbb{K}^2\ \nabla^2\overset{\bullet}{G} + \frac{4}{3}\mathbb{K}\ \nabla^2\overset{\bullet}{V} + \frac{8}{15}\mathbb{K}\ \nabla^2\overset{\bullet}{Q} = 0.$$

Eliminate $\overset{\bullet}{V}$ using the first equation, $2\overset{\bullet}{V} = 3\mathrm{i}\omega\Theta - 2\overset{\bullet}{Q}$:

$$6\mathrm{i}\omega\overset{\bullet}{G} = 6\ \nabla^2\overset{\bullet}{Q} + 6\omega^2\overset{\bullet}{Q} - 9\mathrm{i}\omega^3\Theta - 15\mathrm{i}\omega\ \nabla^2\Theta,$$
$$\overset{\bullet}{Q} - \frac{3}{2}\mathrm{i}\omega\mathbb{K}\overset{\bullet}{Q} - \frac{27}{5}\mathbb{K}^2\ \nabla^2\overset{\bullet}{Q} + \frac{3}{2}\mathbb{K}\overset{\bullet}{G} = -\frac{15}{4}\mathbb{K}\nabla^2\Theta,$$
$$(1-\mathrm{i}\omega\mathbb{K})\overset{\bullet}{G} - \frac{6}{5}\mathbb{K}^2\ \nabla^2\overset{\bullet}{G} + 2\mathrm{i}\omega\mathbb{K}\ \nabla^2\Theta - \frac{4}{5}\mathbb{K}\ \nabla^2\overset{\bullet}{Q} = 0.$$

Eliminate $\overset{\bullet}{G}$ using the first equation:

$$\mathrm{i}\omega\overset{\bullet}{Q} + 3\omega^2\mathbb{K}\overset{\bullet}{Q} + \left(\frac{3}{2}\mathbb{K} - \frac{27}{5}\mathrm{i}\omega\mathbb{K}^2\right)\nabla^2\overset{\bullet}{Q} = \frac{9}{4}\mathrm{i}\omega^3\mathbb{K}\Theta,$$
$$\omega^2(1-\mathrm{i}\omega\mathbb{K})\overset{\bullet}{Q} + \left(1 - \frac{9}{5}\mathrm{i}\omega\mathbb{K} - \frac{6}{5}\omega^2\mathbb{K}^2\right)\nabla^2\overset{\bullet}{Q} - \frac{6}{5}\mathbb{K}^2\nabla^4\overset{\bullet}{Q} = \mathrm{i}\omega\left[\frac{3}{2}\omega^2(1-\mathrm{i}\omega\mathbb{K})\Theta + \left(\frac{5}{2} - \frac{9}{2}\mathrm{i}\omega\mathbb{K} - \frac{9}{5}\omega^2\mathbb{K}^2\right)\nabla^2\Theta - 3\mathbb{K}^2\nabla^4\Theta\right].$$

Finally, eliminate $\overset{\bullet}{Q}$ between these two equations:

$$\left[\omega^2(1-\mathrm{i}\omega\mathbb{K}) + \left(1 - \frac{9}{5}\mathrm{i}\omega\mathbb{K} - \frac{6}{5}\omega^2\mathbb{K}^2\right)\nabla^2 - \frac{6}{5}\mathbb{K}^2\nabla^4\right]\frac{9}{4}\omega^2\mathbb{K}\Theta$$
$$-\left[\mathrm{i}\omega + 3\omega^2\mathbb{K} + \left(\frac{3}{2}\mathbb{K} - \frac{27}{5}\mathrm{i}\omega\mathbb{K}^2\right)\nabla^2\right]\left[\frac{3}{2}\omega^2(1-\mathrm{i}\omega\mathbb{K})\Theta + \left(\frac{5}{2} - \frac{9}{2}\mathrm{i}\omega\mathbb{K} - \frac{9}{5}\omega^2\mathbb{K}^2\right)\nabla^2\Theta - 3\mathbb{K}^2\nabla^4\Theta\right] = 0.$$

This has the form

$$A_6\nabla^6\Theta + A_4\nabla^4\Theta + A_2\nabla^2\Theta + A_0\Theta = 0, \tag{D1}$$

where

$$A_6 = \frac{9}{2}\mathbb{K}^3\left(1 - \frac{18}{5}\mathrm{i}\omega\mathbb{K}\right),\ A_4 = -\frac{15}{4}\mathbb{K}\left(1 - \frac{31}{5}\mathrm{i}\omega\mathbb{K} - \frac{222}{25}\omega^2\mathbb{K}^2 + \frac{324}{125}\mathrm{i}\omega^3\mathbb{K}^3\right), \tag{D2}$$

$$A_2 = -\frac{5}{2}\mathrm{i}\omega\left(1 - \frac{24}{5}\mathrm{i}\omega\mathbb{K} - \frac{216}{25}\omega^2\mathbb{K}^2 + \frac{108}{25}\mathrm{i}\omega^3\mathbb{K}^3\right),\ A_0 = -\frac{3}{2}\mathrm{i}\omega^3(1-\mathrm{i}\omega\mathbb{K})\left(1 - \frac{3}{2}\mathrm{i}\omega\mathbb{K}\right). \tag{D3}$$

An equation equivalent to Equation (D1) has been given by Ben-Ami and Manela.[9, eq. 3.12] To see this equivalence, denote their frequency and Knudsen number by $\tilde{\omega}$ and Kn, respectively. They are related to our ω and $\mathbb{K}$ by $\omega = \sqrt{2}\,\tilde{\omega}$ and $\mathbb{K} = \sqrt{2}\ Kn$. (The extra $\sqrt{2}$ comes from their scalings; see the definition of U^*_{th} above.[9, eq. 2.4]) Using these relations (which give $\omega\mathbb{K} = 2\tilde{\omega}\ Kn$ and $2(\tilde{\omega}/\omega)^2 = 1$) in Equations (D2)–(D3), and noting that we used $e^{-\mathrm{i}\omega t}$ rather than $e^{+\mathrm{i}\omega t}$,[9, eq. 3.1] we find complete agreement with the corresponding terms in Ben-Ami and Manela.[9, eq. 3.12]