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Scattering by a sphere in a tube, and related problems

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ABSTRACT:
Time-harmonic waves propagate along a cylindrical waveguide in which there is an obstacle. The problem is to calculate the reflection and transmission coefficients. Simple explicit approximations are found assuming that the waves are long compared to the diameter of the cross-section d. Simpler but useful approximations are found when the lateral dimensions of the obstacle are small compared to d. Results for spheres, discs, and spheroids are given.
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I. INTRODUCTION

A basic problem in acoustics concerns propagation along a cylindrical waveguide containing an obstacle. Practical applications arise in aeroacoustics (interaction of sound with rotor and stator in turbomachinery[1]), noise control (muffler, silencer, and duct systems design[2]), and ultrasound diagnostics (flow meters, particle and bubble counters, and localisation of blockage in pipelines[3]). One prototypical problem has a cylinder of circular cross-section (a tube) containing a spherical obstacle. Another important application is to the problem of scattering by a bi-periodic array (lattice) of identical objects: a cylindrical waveguide of rectangular cross-section can represent a periodic element of such a lattice. These problems are attracting an increasing interest due to their relevance in the modeling and tailored design of certain metamaterials and metasurfaces.

Some of these problems can be solved by numerical methods or by semi-analytical methods. References will be given later for acoustic problems; for electromagnetic scattering of a waveguide mode by a sphere inside a cylinder, see Refs. 4–6; for analogous elastodynamic problems, see Refs. 7–9. Perhaps inevitably, such methods are complicated.

In this paper, relatively simple approximations are obtained under certain simplifying assumptions. To begin, it is assumed that the cylindrical boundary of the waveguide is sound hard (Neumann boundary condition), implying that a time-harmonic plane wave can propagate along the waveguide. Next, it is assumed that the wavelength is long compared to the diameter of the waveguide cross-section, long enough so that no other propagating waves can exist. Then, when a plane wave is incident upon an obstacle in the waveguide, the basic problem is to compute the reflection and transmission coefficients. The long-wave/low-frequency assumption suggests using matched asymptotic expansions, in the spirit of Lamb (Sec. 307 of Ref. 10). This approach was developed in a previous paper[11] for two-dimensional problems with sound-hard obstacles. Here, extensions to three-dimensional problems and sound-soft obstacles are made.

The approximations derived require information obtained by solving certain related boundary value problems for Laplace's equation. One of these is uniform potential flow along the cylinder past the obstacle. The difference between the values of the potential at the two ends of the (infinite) cylinder is a constant, $2L$ (in a dimensionless form), known as the *blockage coefficient*. For sound-hard obstacles, the quantity L arises in the context of the theory of *Coulter counters*, electrical devices used to measure the size and concentration of small objects in a liquid using the change in resistance due to their presence; for reviews and collections of approximations to L, see Refs. 12 and 13. For sound-soft obstacles, however, no published computations of L have been found.

In addition, for sound-soft scatterers only, it is found that a second potential problem must be analysed and two dimensionless constants (denoted by P and Q) extracted. The boundary condition on the harmonic potential is that it be constant on the obstacle.

Exact integral formulas are derived for L, P, and Q in Sec. IV. These are used to obtain various approximations, which are compared with earlier work when available. Simple explicit approximations are obtained for discs (Sec. V), spheres (Sec. VI), and spheroids (Sec. VII). For small sound-soft obstacles, it turns out that P provides the dominant contribution to the reflection and transmission coefficients.

In Sec. VIII, the problem of scattering by a sphere in a tube is considered in detail. The approximations obtained for the reflection and transmission coefficients are shown to agree with Linton's asymptotic approximations.[14] Concluding remarks are made in Sec. IX.

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II. FORMULATION

Consider a long cylindrical waveguide containing a bounded obstacle B. Take the z-axis along the cylinder (parallel to the cylinder's generators). Assume B is symmetric with respect to the cross-sectional plane $z = 0$.

Assume the cylinder wall W is sound hard. Then, a time-harmonic plane wave can propagate along the cylinder; it is partly reflected by the obstacle and partly transmitted. The problem is to calculate the reflection and transmission coefficients. The motion is governed by the three-dimensional Helmholtz equation,

$$(\nabla^2 + k^2)u = 0, \tag{1}$$

where $k = \omega/c$, c is the speed of sound, and there is a suppressed time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$. The normal derivative of u is zero on the sound-hard cylinder wall, $\partial u/\partial n = 0$ on W. There is also a boundary condition on the surface of the scatterer, S, written concisely as $\mathcal{B}u = 0$ on S. Two cases are considered here

$$\begin{aligned} &\mathcal{B}u = u = 0 \text{ on } S, && \text{sound-soft scatterer,}\\ &\mathcal{B}u = \partial u/\partial n = 0 \text{ on } S, && \text{sound-hard scatterer.} \end{aligned}$$

As a special case, situations where the scatterer shrinks to a screen occupying a piece of the cross-sectional plane at $z = 0$ will be considered. For example, B could be a thin circular disk. There are also complementary problems; for example, the cross-sectional plane could be rigid apart from a circular hole or aperture. These will be referred to as *iris problems*.

Define a length a by equating the cylinder's cross-sectional area to πa^2. For a tube (a cylinder with a circular cross section), a is the cross-sectional radius.

The incident wave is $u^{\mathrm{in}} = \mathrm{e}^{\mathrm{i}kz}$. Assume that ka is sufficiently small so that the only propagating modes are $\mathrm{e}^{\pm \mathrm{i}kz}$. With u as the total field, define the scattered field by $u^{\mathrm{sc}} = u - u^{\mathrm{in}}$; this field must be outgoing, whence

$$u \sim \begin{cases} T\,\mathrm{e}^{\mathrm{i}kz}, & z \to \infty,\\ \mathrm{e}^{\mathrm{i}kz} + R\,\mathrm{e}^{-\mathrm{i}kz}, & z \to -\infty, \end{cases}$$

or, alternatively,

$$u^{\mathrm{sc}} \sim \begin{cases} (T-1)\,\mathrm{e}^{\mathrm{i}kz}, & z \to \infty,\\ R\,\mathrm{e}^{-\mathrm{i}kz}, & z \to -\infty. \end{cases} \tag{2}$$

Here, R and T are the complex reflection and transmission coefficients, respectively. They satisfy

$$|R|^2 + |T|^2 = 1 \quad \text{and} \quad RT^* + R^*T = 0, \tag{3}$$

where the asterisk denotes complex conjugation.

As the obstacle is symmetric about the plane $z = 0$, the whole problem can be separated into two, an antisymmetric problem (odd function of z, subscript a) and a symmetric problem (even function of z, subscript s), using

$$u = u_{\mathrm{a}} + u_{\mathrm{s}}, \quad u^{\mathrm{in}} = u_{\mathrm{a}}^{\mathrm{in}} + u_{\mathrm{s}}^{\mathrm{in}}, \quad u^{\mathrm{sc}} = u_{\mathrm{a}}^{\mathrm{sc}} + u_{\mathrm{s}}^{\mathrm{sc}},$$

with $u_{\mathrm{a}}^{\mathrm{in}} = \mathrm{i}\sin kz$ and $u_{\mathrm{s}}^{\mathrm{in}} = \cos kz$. Equation (2) defines u^{sc} in the outer region, where any evanescent terms have been discarded. In that region,

$$u_{\mathrm{a}}^{\mathrm{sc}} = D_{\mathrm{a}}\,\mathrm{e}^{\mathrm{i}k|z|}\,\mathrm{sgn}\,z, \quad u_{\mathrm{s}}^{\mathrm{sc}} = D_{\mathrm{s}}\,\mathrm{e}^{\mathrm{i}k|z|}, \tag{4}$$

where D_{a} and D_{s} are (dimensionless) constants. Comparison with Eq. (2) gives

$$R = D_{\mathrm{s}} - D_{\mathrm{a}}, \quad T - 1 = D_{\mathrm{s}} + D_{\mathrm{a}}. \tag{5}$$

Direct numerical computations of R and T for specific geometries are feasible but complicated. If the cylinder W has a square or rectangular cross-section, the problem is equivalent to a plane wave at normal incidence to a two-dimensional bi-periodic array of identical scatterers. Such *lattice problems* have extensive literature and efficient numerical methods have been developed.[15,16] For lattice iris problems (an infinite thin rigid screen with a bi-periodic array of holes), see Ref. 17.

For the problem of scattering by a sound-hard sphere in a sound-hard tube, see Refs. 14, 18, and 19. Boström[18] also gives results for spheroids and for penetrable spheres. Linton[14] also gives results for soft spheres and for soft tubes. Kubenko and Dzyuba[19] do not consider the reflected or transmitted waves. For the iris problem of a tube with a rigid screen at $z = 0$ containing a circular hole, see Table V in Ref. 20.

Approximations to R and T are sought, assuming that $ka \ll 1$. For lattice problems, all the details were worked out by Twersky.[21] Subsequent papers include Refs. 17, 22, and 23.

The focus here will be on axisymmetric tube problems. For such problems, introduce cylindrical polar coordinates so that the wall W is at $r = a$. Also, introduce the smallest finite circular cylinder (with cross-sectional radius b and length $2h$) that contains B; for a sphere of radius b, $h = b$; for a disk of radius b, $h = 0$; for a prolate spheroid, h is the length of the semi-major axis. Finally, introduce two dimensionless parameters,

$$\kappa = ka \quad \text{and} \quad \mu = b/a.$$

III. LONG-WAVE APPROXIMATIONS

The inner expansions of the outer solutions, Eq. (4), are

$$u_{\mathrm{a}} = u_{\mathrm{a}}^{\mathrm{sc}} + u_{\mathrm{a}}^{\mathrm{in}} \sim D_{\mathrm{a}}\,\mathrm{sgn}\,z + (1 + D_{\mathrm{a}})\mathrm{i}\kappa\bar{z}, \tag{6}$$

$$\begin{aligned} u_{\mathrm{s}} &= u_{\mathrm{s}}^{\mathrm{sc}} + u_{\mathrm{s}}^{\mathrm{in}}\\ &\sim (1 + D_{\mathrm{s}}) + \mathrm{i}\kappa|\bar{z}|D_{\mathrm{s}} - \frac{1}{2}(1 + D_{\mathrm{s}})\kappa^2\bar{z}^2 \end{aligned} \tag{7}$$

as $\kappa|\bar{z}| \to 0$, where $\bar{z} = z/a$. These will be matched to the solutions of certain inner problems.

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A. Determination of D_a

Introduce scaled inner variables in a cross-sectional plane, $\bar{x} = x/a$ and $\bar{y} = y/a$, so that a scaled Laplacian $\bar{\nabla}^2 = a^2\nabla^2$ can be defined. Then, in the inner region, suppose that

$$u_a = \mathrm{i}\kappa u_a^{(1)}(\bar{x}, \bar{y}, \bar{z}) + O(\kappa^2) \quad \text{as } \kappa = ka \to 0.$$

Substitution in Eq. (1) gives $\bar{\nabla}^2 u_a^{(1)} = 0$ together with suitable scaled boundary conditions on the wall W and the obstacle S.

Introduce a potential Φ_a with $\nabla^2\Phi_a = 0$ in the fluid, $\partial\Phi_a/\partial n = 0$ on W, $\mathcal{B}\Phi_a = 0$ on S and

$$\Phi_a = z/a + L\,\mathrm{sgn}\,z + o(1) \quad \text{as } z \to \pm\infty. \tag{8}$$

This represents potential flow past the obstacle. The dimensionless constant L is sometimes known as the *blockage coefficient*. Its computation is discussed in Sec. IV A.

Writing $u_a^{(1)} = A_a\Phi_a$, comparison with Eq. (6) gives $A_a = 1 + D_a$ and $D_a/(\mathrm{i}\kappa) = A_a L$, whence

$$D_a = \frac{\mathrm{i}\kappa L}{1 - \mathrm{i}\kappa L} \quad \text{and} \quad A_a = \frac{1}{1 - \mathrm{i}\kappa L}. \tag{9}$$

B. Determination of D_s

Inspection of Eq. (7) suggests the following expansion in the inner region:

$$u_s = 1 + \mathrm{i}\kappa u_s^{(1)}(\bar{x}, \bar{y}, \bar{z}) - \kappa^2 u_s^{(2)}(\bar{x}, \bar{y}, \bar{z}) - \cdots. \tag{10}$$

Substitution in Eq. (1) gives

$$\bar{\nabla}^2 u_s^{(1)} = 0, \quad \bar{\nabla}^2 u_s^{(2)} = 1, \tag{11}$$

together with suitable scaled boundary conditions on W and S.

It turns out that the boundary condition on S matters. For a hard scatterer (Neumann condition), we have to determine an additive constant, whereas this complication does not arise when the scatterer is soft (Dirichlet condition).

1. Soft scatterer

Introduce a potential Φ_s with $\nabla^2\Phi_s = 0$ in the fluid, $\partial\Phi_s/\partial n = 0$ on W, $\Phi_s = 1$ on S and

$$\Phi_s = P|z|/a + Q + o(1) \quad \text{as } z \to \pm\infty, \tag{12}$$

where P and Q are dimensionless constants; they are both to be found by solving the boundary value problem for Φ_s (see Sec. IV B). Note that, unlike Φ_a, Φ_s satisfies an inhomogeneous boundary condition on S.

As the condition $u_s = 0$ on S is required, try the approximation

$$u_s = A_s(1 - \Phi_s),$$

and then match with the first two terms on the right-hand side of Eq. (7); this gives $-A_s P = \mathrm{i}\kappa D_s$ and $A_s(1 - Q) = 1 + D_s$, whence $A_s = \mathrm{i}\kappa/\{P + \mathrm{i}\kappa(1 - Q)\}$ and

$$D_s = \frac{-P}{P + \mathrm{i}\kappa(1 - Q)}. \tag{13}$$

2. Hard scatterer

Motivated by the last term on the right-hand side of Eq. (7) and the second of Eq. (11), introduce a function Ψ_s with $a^2\nabla^2\Psi_s = 1$ in the fluid with $\partial\Psi_s/\partial n = 0$ on both W and S, together with $\Psi_s \sim (1/2)(z/a)^2$ as $z \to \pm\infty$. It is convenient to write

$$\Psi_s = \frac{1}{2}(z/a)^2 + \Upsilon_s.$$

As $a^2\nabla^2\Psi_s = 1$, Υ_s is harmonic, $\nabla^2\Upsilon_s = 0$ in the fluid, with $\partial\Upsilon_s/\partial n = -\partial/\partial n\left\{(1/2)(z/a)^2\right\}$ on S, $\partial\Upsilon_s/\partial n = 0$ on W, and

$$\Upsilon_s = M|z|/a + o(1) \quad \text{as } z \to \pm\infty, \tag{14}$$

where M is a constant. The condition Eq. (14) eliminates arbitrary additive constants from the problem for Υ_s. In fact, as in Sec. 2.2.2 of Ref. 11, M can be calculated exactly, using a simple application of Green's theorem,

$$M = -\frac{|B|}{2\pi a^3}, \tag{15}$$

where $|B|$ is the volume of the scatterer B.

Next, also as in Ref. 11, try the approximation

$$u_s = B_s\Psi_s + C_s,$$

where B_s and C_s are constants. Matching with the right-hand side of Eq. (7) yields $B_s = -(1 + D_s)\kappa^2$, $B_s M = \mathrm{i}\kappa D_s$ and $C_s = 1 + D_s$, whence $B_s = -\kappa^2/(1 - \mathrm{i}\kappa M)$, $C_s = (1 - \mathrm{i}\kappa M)^{-1}$ and

$$D_s = \frac{\mathrm{i}\kappa M}{1 - \mathrm{i}\kappa M}. \tag{16}$$

C. Reflection and transmission coefficients

The reflection and transmission coefficients are given by Eq. (5) in terms of D_a and D_s; these are given by Eqs. (9) and (13) when the scatterer is soft, and by Eqs. (9) and (16) when the scatterer is hard. The resulting approximations are found to satisfy the constraints Eq. (3) exactly.

An application to scattering by a sphere in a tube is given in Sec. VIII.

IV. POTENTIAL PROBLEMS: GENERAL RESULTS

The low-frequency approximations derived in Sec. III involve the four dimensionless constants L, M, P, and Q.

The constant M is given explicitly by the formula from Eq. (15) in terms of the volume of the scatterer. The other three (when they are needed) have to be determined by solving an appropriate boundary value problem for Laplace's equation. In general, this is not a trivial task, although good approximations can be obtained for small scatterers ($\mu \ll 1$).

In this section, exact integral representations for L, P, and Q are derived. In principle, formulas of this kind could be derived using Green's function for the empty tube, but the results would be complicated. Instead, simple applications of Green's theorem are used, leading to effective and useful formulas for L, P, and Q.

A. The blockage coefficient *L*

For L, $\nabla^2\Phi_a = 0$ has to be solved, with $\partial\Phi_a/\partial n = 0$ on W, the far-field condition Eq. (8), and a homogeneous boundary condition on S.

If S is hard ($\partial\Phi_a/\partial n = 0$ on S), an application of Green's theorem to Φ_a and z gives

$$L = \frac{1}{2\pi a^2}\int_S \Phi_a \frac{\partial z}{\partial n}\,\mathrm{d}S, \tag{17}$$

where the normal vector on S points outwards. This exact formula is due to Hurley.[24] Similarly, if S is soft ($\Phi_a = 0$ on S),

$$L = -\frac{1}{2\pi a^2}\int_S \frac{\partial \Phi_a}{\partial n}\, z\,\mathrm{d}S. \tag{18}$$

These formulas will be used later to obtain approximations to L. They are quite general: they do not assume that the waveguide is a tube and they do not assume that S is symmetric.

For axisymmetric tube problems, with S symmetric about $z = 0$, the method of separation of variables can be used. As Φ_a is an odd function of z, consider a semi-infinite tube $0 \le r < a$, $z \ge h$. (Recall that S is inside the finite cylinder defined by $0 \le r < b < a$, $|z| < h$.) Then, separation of variables gives

$$\Phi_a(r,z) = \frac{z}{a} + L + \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\,\mathrm{e}^{-\lambda_n z}, \quad z > h, \tag{19}$$

where J_0 is a Bessel function, the quantities λ_n are given in terms of the positive zeros of $J_0'(x) = -J_1(x)$, $J_1(\lambda_n a) = 0$, $n = 1, 2, \ldots$, and the coefficients L and c_n are to be determined.

The right-hand side of Eq. (19) is known as a Dini–Bessel series; see Sec. 18.3 of Ref. 25 with $H = \nu = 0$ therein, Eq. (2.2.20) in Ref. 26 or Sec. 3.3 in Ref. 27. Orthogonality leads to exact formulas for L and c_n, although we are mainly interested in determining L; thus

$$\frac{z_0}{a} + L = \frac{2}{a^2}\int_0^a \Phi_a(r,z_0)\, r\,\mathrm{d}r \quad \text{for any } z_0 > h. \tag{20}$$

Writing $\Phi_a(r,z) = z/a + \Phi_0(r,z)$, Eq. (20) becomes

$$L = \frac{2}{a^2}\int_0^a \Phi_0(r,z_0)\, r\,\mathrm{d}r \quad \text{for any } z_0 > h. \tag{21}$$

B. The coefficients *P* and *Q*

For soft scatterers, P and Q are required. They are determined by solving $\nabla^2\Phi_s = 0$, with $\partial\Phi_s/\partial n = 0$ on W, the far-field condition Eq. (12) and the boundary condition $\Phi_s = 1$ on S.

An application of Green's theorem gives

$$P = \frac{1}{2\pi a}\int_S \frac{\partial \Phi_s}{\partial n}\,\mathrm{d}S, \tag{22}$$

where the normal vector on S points outwards. This exact formula shows that P is related to the *capacity* of the object S in the waveguide. As with Eqs. (17) and (18), Eq. (22) does not assume that the waveguide is a tube and it does not assume that S is symmetric.

If Φ_s is replaced in Eq. (22) by the corresponding potential for an unbounded fluid (thus ignoring the presence of W), the result would be

$$P \simeq -2\mathcal{C}/a, \tag{23}$$

where $\mathcal{C}$ is the capacity of S; see Eq. (8.10) in Ref. 28. Approximations of this kind will be used extensively later; they are often surprisingly accurate.

For Q, let $\mathcal{G}$ denote the semi-infinite waveguide $z > 0$, and let S_+ denote the half of S in $\mathcal{G}$. Apply Green's theorem to Φ_s and z in the subregion of $\mathcal{G}$ bounded by W, S_+, the plane $z = 0$ and the plane $z = z_1$. To fix ideas, consider a single scatterer, so that S is bisected by the plane $z = 0$. The piece of this plane outside S but inside W is denoted by F_+ and the piece inside S is denoted by F_-. The piece of the plane $z = z_1$ inside W is denoted by F_1; its area is πa^2 (which defines a). The result is

$$\int_{F_+\cup S_+\cup F_1}\left(z\frac{\partial\Phi_s}{\partial n} - \Phi_s\frac{\partial z}{\partial n}\right)\mathrm{d}S = 0,$$

where the normal vector points inwards (out of S_+). For F_1, use Eq. (12) and let $z_1 \to \infty$; the integral over F_1 evaluates to $Q\pi a^2$. On F_+, $z = 0$, whereas on S_+, $\Phi_s = 1$. The integral of $\partial z/\partial n$ over S_+ is equal to $|F_-|$, the area of F_-. Hence

$$Q\pi a^2 = \int_{F_+}\Phi_s\,\mathrm{d}S - \int_{S_+} z\frac{\partial\Phi_s}{\partial n}\,\mathrm{d}S + |F_-|. \tag{24}$$

This formula is exact and it does not assume that the waveguide is a tube, but it does assume that $z = 0$ is a symmetry plane.

For axisymmetric problems in a tube, F_- will be a disk of radius b_0, say, and then Eq. (24) becomes

$$Q = \frac{2}{a^2}\int_{b_0}^a \Phi_s(r,0)\, r\,\mathrm{d}r + \frac{b_0^2}{a^2} - \frac{1}{\pi a^2}\int_{S_+} z\frac{\partial\Phi_s}{\partial n}\,\mathrm{d}S. \tag{25}$$

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For such problems, Φ_s can be written in the separated form [see Eq. (19)],

$$\Phi_s(r,z) = P\frac{z}{a} + Q + \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\,\mathrm{e}^{-\lambda_n z}, \quad z > h. \tag{26}$$

Let $v(r,z) = \partial\Phi_s/\partial z$. Then orthogonality gives

$$P = \frac{2}{a}\int_0^a v(r,z_0)\,r\,\mathrm{d}r, \tag{27}$$

$$P\frac{z_0}{a} + Q = \frac{2}{a^2}\int_0^a \Phi_s(r,z_0)\,r\,\mathrm{d}r, \tag{28}$$

for any $z_0 > h$. Equation (27) is consistent with Eq. (22) because Green's theorem implies that the integral over S can be replaced by twice the integral over the disk at $z = z_0$.

V. DISCS, IRISES, AND OTHER THIN SCREENS

In this section, special cases are considered where the scatterer occupies a piece of the cross-sectional plane at $z = 0$. This includes disk problems and iris problems.

If the screen is soft, then the scattered field is an even function of z, where $D_a = 0$ and D_s is given by Eq. (13). From Eq. (5),

$$R = D_s \text{ and } T = 1 + D_s, \text{ whence } T - R = 1. \tag{29}$$

If the screen is hard, then the scattered field is an odd function of z, whence $D_s = 0$ and D_a is given by Eq. (9). From Eq. (5),

$$R = -D_a \text{ and } T = 1 + D_a, \text{ whence } T + R = 1. \tag{30}$$

For the complementary iris problem, there is Fock's 1941 paper.[29] Fock gave results for a tube with a small iris of radius b ($\mu = b/a \ll 1$) with an emphasis on computing the blockage coefficient. Leppington and Levine[30] obtained an explicit approximation for a large iris ($\mu \simeq 1$); for reviews, see Sec. 4.1 of Ref. 31 as well as Ref. 32. For numerical results, see Ref. 33.

In the remainder of this section, consideration is given to relevant axisymmetric potential problems for a circular disk in a tube, with the disk's center at the origin of cylindrical polar coordinates.

A. Hard circular disk in a hard tube

For a hard disk of radius b, D_a is required [see Eq. (30)], and this is given by Eq. (9) in terms of the dimensionless blockage coefficient L.

To estimate L, start from Eq. (21), in which $z_0 = 0$. The potential $\Phi_0(r,z)$ is a continuous odd function of z for $b < r < a$, and so Eq. (21) reduces to

$$L = \frac{2}{a^2}\int_0^b \Phi_0(r,0)\,r\,\mathrm{d}r. \tag{31}$$

This formula is exact.

Suppose the disk is small, $\mu \ll 1$. Then an approximation to L is obtained by inserting the known result for potential flow about a hard disk in an unbounded fluid into Eq. (31). Thus,

$$\Phi_0(r,0) \simeq \frac{2}{\pi a}\sqrt{b^2 - r^2},$$

which gives the approximation

$$L \simeq \frac{4}{\pi a^3}\int_0^b r\sqrt{b^2 - r^2}\,\mathrm{d}r = \frac{4\mu^3}{3\pi}. \tag{32}$$

The blockage coefficient for axisymmetric potential flow past a rigid disk in a tube was computed numerically by Smythe.[34] In fact, he discussed and computed a quantity ΔL called the "effective increase in length of tube" due to insertion of the disk into the tube; it is $2aL$. His numerical results are given in Table I, where they are compared with the simple approximation Eq. (32); the agreement is seen to be excellent when μ is small.

There is later work on this and related problems; see Ref. 35 and Sec. 3.3 in Ref. 27.

B. Soft circular disk in a hard tube

For a soft disk of radius b, D_s is required [see Eq. (29)], and this is given by Eq. (13) in terms of the coefficients P and Q. To estimate P and Q, use Eqs. (27) and (28) in which $z_0 = 0$. As $\Phi_s(r,0) = 1$ for $0 \le r < b$ and $v(r,0) = 0$ for $a < r < b$,

$$P = \frac{2}{a}\int_0^a v(r,0)\,r\,\mathrm{d}r = \frac{2}{a}\int_0^b v(r,0)\,r\,\mathrm{d}r, \tag{33}$$

$$Q = \frac{2}{a^2}\int_0^a \Phi_s(r,0)\,r\,\mathrm{d}r = \frac{b^2}{a^2} + \frac{2}{a^2}\int_b^a \Phi_s(r,0)\,r\,\mathrm{d}r. \tag{34}$$

Suppose the disk is small, $\mu = b/a \ll 1$. Then approximations to P and Q are obtained by inserting known expressions for $v(r,0)$ and $\Phi_s(r,0)$ coming from the solution to the problem of an electrified disk in the absence of the tube. This classic problem in potential theory can be solved in

TABLE I. Smythe's (Ref. 34) numerical results from 1964 for the blockage coefficient L for a hard disc of radius b in a tube of cross-sectional radius a (from Table VI of Ref. 34). His results (second column) compare well with the small-μ approximation Eq. (32) (third column).

$\mu = b/a$	L	$4\mu^3/(3\pi)$
0.1	0.000425	0.000424
0.2	0.003405	0.003395
0.3	0.011568	0.011459
0.4	0.02780	0.02716
0.5	0.05562	0.05305
0.6	0.1001	0.0917
0.7	0.17	0.15
0.8	0.3	0.22

several ways; for details and history, see Chap. III in Ref. 26. The results needed here are

$$v(r,0) = \frac{-(2/\pi)}{\sqrt{b^2 - r^2}} \quad \text{for } 0 \le r < b \quad \text{and}$$
$$\Phi_s(r,0) = \frac{2}{\pi}\arcsin(b/r) \quad \text{for } r > b.$$

Substitution for $v(r,0)$ in Eq. (33) leads to

$$P \simeq -\frac{4}{\pi a}\int_0^b \frac{r\,\mathrm{d}r}{\sqrt{b^2 - r^2}} = -\frac{4\mu}{\pi},$$

whereas substitution for $\Phi_s(r,0)$ in Eq. (34) leads to

$$Q \simeq \mu^2 + \frac{4}{\pi a^2}\int_b^a \arcsin(b/r)\, r\,\mathrm{d}r$$
$$= \mu^2 + \frac{4\mu^2}{\pi}\int_{\theta_0}^{\pi/2} \frac{\theta\cos\theta}{\sin^3\theta}\,\mathrm{d}\theta,$$

where $\mu = \sin\theta_0$. The remaining integral can be evaluated using integration by parts, giving

$$Q \simeq \mu^2 + \frac{4\mu^2}{\pi}\left(\frac{\theta_0}{2\mu^2} - \frac{\pi}{4} + \frac{\cos\theta_0}{2\mu}\right)$$
$$= \frac{2}{\pi}(\theta_0 + \mu\cos\theta_0) \sim \frac{4\mu}{\pi} \quad \text{as } \mu \to 0.$$

The estimates $P \simeq -4\mu/\pi$ and $Q \simeq 4\mu/\pi$ can be used in Eq. (13) to estimate D_s. The estimate for P can also be checked by using Eq. (22) (keeping in mind that the disk has two sides).

The problem for Φ_s has been solved by Hunter and Williams[36] using methods described on p. 148 of Sneddon's book[26] (see also example 3.3.2 in Ref. 27). However, they made a sign error in their C_0 (our P/a), as can be seen by comparing their Eq. (3) with Sneddon's Eq. (5.3.28). Taking this into account, their Eq. (10) can be written as

$$P(\mu) = -\frac{4\mu}{\pi f(\mu)},$$

where computations of $f(\mu)$ are presented in Fig. 2 from Hunter and Williams;[36] as $f(0) = 1$, the estimate $P \simeq -4\mu/\pi$ is verified. Hunter and Williams[36] did not include the constant Q; see Eq. (1) in Ref. 36 or Eq. (3.3.16) in Ref. 27.

VI. POTENTIAL PROBLEMS FOR A SPHERE IN A TUBE

Consider a sphere of radius b in a tube of cross-sectional radius a. The z-axis is along the axis of the tube and the sphere is centered at the origin. We define $\mu = b/a$. The problem is to solve Laplace's equation in the region bounded by the tube wall W and the sphere S together with boundary conditions on W and S and far-field conditions as $|z| \to \infty$.

In 1936, Knight[37] considered Dirichlet boundary conditions on all boundaries, with $\Phi_s = 0$ on W and Φ_s specified on the sphere S. He solved the symmetric problem (even functions of z) using a multipole method. In 1960, Smythe[38] solved the same problem, and he gave some numerical results. Implicit in both papers is the far-field condition $\Phi_s \to 0$ as $|z| \to \infty$. Approximate solutions of this sphere-tube problem when $\mu \ll 1$ (and for other small obstacles) have been derived.[39] For a soft disk in a soft tube, see Sec. 8.3 in Ref. 26.

A. Hard sphere in a hard tube

One year later, in 1961, Smythe[40] considered axisymmetric potential flow past a rigid sphere in a rigid tube. He computed $2L$, where L is the blockage coefficient. In a subsequent paper,[34] he corrected his computations; his results are in Table II. Subsequent studies include Refs. 41–43. The last of these contains approximations to L, obtained as follows. First, using a multipole method, an infinite algebraic system for certain coefficients $A_n(\mu)$ is obtained. This system is then solved recursively by writing $A_n(\mu) = \sum_{m=0}^{\infty} K_{nm}\mu^m$. It turns out that

$$L = \mu^3 A_1(\mu) \simeq \mu^3 A_1(0) = \mu^3, \tag{35}$$

where $K_{10} = 1$ and Eq. (2.10) from Ref. 43 have been used. This simple approximation is compared with Smythe's numerical results in Table II; the agreement is excellent for small μ. Jeffrey *et al.*[43] also gave an approximation for L when $\mu \simeq 1$.

Another derivation of Eq. (35) can be given using Hurley's formula, Eq. (17). Introduce spherical polar coordinates, R and Θ, so that $R^2 = r^2 + z^2$ and $z = R\cos\Theta$, where r and z are cylindrical polar coordinates. The sphere S is $R = b$. Replace Φ_a in Eq. (17) by the corresponding potential for uniform flow past a sphere in an unbounded fluid. Thus, insert the approximation

$$\Phi_a \simeq \left(\frac{R}{a} + \frac{b^3}{2aR^2}\right)\cos\Theta,$$

together with $\partial z/\partial n = \cos\Theta$, giving

TABLE II. Smythe's (Ref. 34) numerical results from 1964 for the blockage coefficient L for a hard sphere of radius b in a tube of cross-sectional radius a (see Table VI of Ref. 34). His results (second column) compare well with the small-μ approximation Eq. (35) (third column).

$\mu = b/a$	L	μ^3
0.1	0.001001	0.001000
0.2	0.008054	0.008000
0.3	0.027594	0.027000
0.4	0.067440	0.064000
0.5	0.13884	0.12500
0.6	0.26103	0.21600
0.7	0.47317	0.34300

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$$L \simeq \frac{1}{2\pi a^2}\int_S \frac{3b}{2a}\cos^2\Theta\,\mathrm{d}S = \frac{3b^3}{2a^3}\int_0^\pi \cos^2\Theta\sin\Theta\,\mathrm{d}\Theta = \mu^3.$$

For another derivation, use Eq. (21) with $z_0 = b$.

B. Soft sphere in a hard tube

There are two problems to be solved. First, L is needed so that D_a can be calculated using Eq. (9). Second, P and Q are needed so that D_s can be calculated using Eq. (13).

For L, use Eq. (18) into which the approximation

$$\Phi_a \simeq \left(\frac{R}{a} - \frac{b^3}{aR^2}\right)\cos\Theta$$

is inserted giving

$$\begin{aligned} L &\simeq -\frac{1}{2\pi a^2}\int_S \frac{3b}{a}\cos^2\Theta\,\mathrm{d}S \\ &= -\frac{3b^3}{a^3}\int_0^\pi \cos^2\Theta\sin\Theta\,\mathrm{d}\Theta = -2\mu^3 \quad \text{as } \mu \to 0. \end{aligned} \tag{36}$$

For another derivation of Eq. (36), use Eq. (21) with $z_0 = b$. Note that the blockage coefficient is $O(\mu^3)$ as $\mu \to 0$ for both hard and soft spheres.

For P and Q, use Eqs. (22) and (25). For an unbounded fluid with $\Phi_s = 1$ on the sphere, $\Phi_s = b/R$, so that Eq. (22) yields the approximation

$$P \simeq \frac{1}{2\pi a}\int_S \frac{\mathrm{d}S}{(-b)} = -2\mu. \tag{37}$$

Similarly, Eq. (25) with $b_0 = b$ gives

$$\begin{aligned} Q &\simeq \frac{2}{a^2}\int_b^a \frac{b}{r}\,r\,\mathrm{d}r + \frac{b^2}{a^2} - \frac{1}{\pi a^2}\int_{S_+} z\,\frac{\mathrm{d}S}{(-b)} \\ &= \frac{2b}{a} - \frac{b^2}{a^2} + \frac{2b^2}{a^2}\int_0^{\pi/2} \cos\Theta\sin\Theta\,\mathrm{d}\Theta = 2\mu. \end{aligned} \tag{38}$$

The estimates in Eqs. (37) and (38) can also be derived by using Eqs. (27) and (28) in which $z_0 = b$.

VII. POTENTIAL PROBLEMS FOR A PROLATE SPHEROID IN A TUBE

Suppose that S is a prolate spheroid

$$(r/b)^2 + (z/h)^2 = 1$$

with $h \geq b$. For potential flow about such a spheroid, use prolate spheroidal coordinates ξ, η, ϕ [as defined in Sec. 30.13(i) of Ref. 44] so that

$$r^2 = c^2(\xi^2 - 1)(1 - \eta^2) \quad \text{and} \quad z = c\xi\eta. \tag{39}$$

Then S is defined by $\xi = \xi_0$ where $\xi_0 = h/c$ and $c = \sqrt{h^2 - b^2}$; also $b^2 = c^2(\xi_0^2 - 1)$.

For Φ_a and an unbounded fluid, write (see Sec. 105 of Ref. 10)

$$\Phi_a = (c/a)\{\xi + EQ_1(\xi)\}\eta, \quad \xi > \xi_0,$$

where the constant E is to be determined from the boundary condition on S and Q_1 is a Legendre function,

$$\begin{aligned} Q_1(\xi) &= \frac{\xi}{2}\log\frac{\xi+1}{\xi-1} - 1, \\ Q_1'(\xi) &= \frac{1}{2}\log\frac{\xi+1}{\xi-1} - \frac{\xi}{\xi^2-1}. \end{aligned} \tag{40}$$

Also, given two axisymmetric functions, $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$,

$$\begin{aligned} \int_S \psi_1\frac{\partial\psi_2}{\partial n}\,\mathrm{d}S &= \int_{-1}^1\int_{-\pi}^{\pi} \psi_1\frac{\partial\psi_2}{\partial\xi}\frac{h_\eta h_\phi}{h_\xi}\,\mathrm{d}\phi\,\mathrm{d}\eta \\ &= 2\pi c(\xi_0^2 - 1)\int_{-1}^1 \psi_1\frac{\partial\psi_2}{\partial\xi}\bigg|_{\xi=\xi_0}\mathrm{d}\eta, \end{aligned} \tag{41}$$

using formulas for the metric coefficients h_ξ, h_η, and h_ϕ given in Sec. 30.13(ii) of Ref. 44.

A. Hard spheroid

For a hard spheroid, impose $\partial\Phi_a/\partial\xi = 0$ at $\xi = \xi_0$, where $1 + EQ_1'(\xi_0) = 0$. Then, using Hurley's formula, Eq. (17) and Eq. (41),

$$\begin{aligned} L &\simeq \frac{c}{a^2}(\xi_0^2 - 1)\int_{-1}^1 \Phi_a\frac{\partial z}{\partial\xi}\,\mathrm{d}\eta \\ &= \frac{c^3}{a^3}(\xi_0^2 - 1)\{\xi_0 + EQ_1(\xi_0)\}\int_{-1}^1 \eta^2\,\mathrm{d}\eta \\ &= -\frac{2c^3}{3a^3Q_1'(\xi_0)}, \end{aligned} \tag{42}$$

which is Hurley's estimate [Eq. (5) in Ref. 24].

The estimate from Eq. (42) can be compared with Smythe's computations[34] for a prolate spheroid with $h/b = 2$, giving $c/b = \sqrt{3}$, $\xi_0 = 2/\sqrt{3}$ and the estimate $L \simeq 1.6134\,\mu^3$; for small μ, the agreement is good (Table III). Cooke[45] also gave results for flow past a spheroid.

TABLE III. Smythe's (Ref. 34) numerical results for the blockage coefficient L for a hard prolate spheroid with semi-minor axis b and semi-major axis $2b$ in a tube of cross-sectional radius a (see Table VI of Ref. 34). His results (second column) compare well with the small-μ approximation Eq. (42) (third column).

$\mu = b/a$	L	$1.6134\,\mu^3$
0.1	0.001616	0.001613
0.2	0.013037	0.012907
0.3	0.04501	0.04356
0.4	0.11139	0.10326
0.5	0.2332	0.2017
0.6	0.448	0.349

The limit as the spheroid becomes a sphere can be examined. This implies that $\xi_0 \to \infty$ and $c \to 0$ in such a way that $c\xi_0 = h \to b$. From Eq. (40), it is found that $Q_1'(\xi) \sim -(2/3)\xi^{-3}$, so that Eq. (42) reduces to Eq. (35), as expected.

B. Soft spheroid

For a soft spheroid, impose $\Phi_a = 0$ at $\xi = \xi_0$, where $\xi_0 + EQ_1(\xi_0) = 0$. Then, using Eqs. (18) and (41),

$$\begin{aligned} L &\simeq -\frac{c}{a^2}(\xi_0^2 - 1)\int_{-1}^{1} z\,\frac{\partial \Phi_a}{\partial \xi}\,\mathrm{d}\eta \\ &= -\frac{c^3}{a^3}\xi_0(\xi_0^2 - 1)\{1 + EQ_1'(\xi_0)\}\int_{-1}^{1}\eta^2\,\mathrm{d}\eta \\ &= -\frac{2c^3\xi_0}{3a^3 Q_1(\xi_0)}. \end{aligned} \tag{43}$$

In the limit as the spheroid becomes a sphere, Eq. (36) is recovered from Eq. (43) using $Q_1(\xi) \sim (1/3)\xi^{-2}$ as $\xi \to \infty$.

For a soft scatterer, the constants P and Q also have to be determined. They can be estimated using Eqs. (22) and (25). For an unbounded fluid with $\Phi_s = 1$ on the spheroid,

$$\Phi_s = \frac{Q_0(\xi)}{Q_0(\xi_0)} \quad \text{with} \quad Q_0(\xi) = \frac{1}{2}\log\frac{\xi+1}{\xi-1}. \tag{44}$$

Then Eqs. (22) and (41) give the estimate

$$\begin{aligned} P &\simeq \frac{c}{a}(\xi_0^2 - 1)\int_{-1}^{1}\frac{\partial \Phi_s}{\partial \xi}\,\mathrm{d}\eta \\ &= \frac{2c}{a}(\xi_0^2 - 1)\frac{Q_0'(\xi_0)}{Q_0(\xi_0)} = -\frac{2c}{aQ_0(\xi_0)}, \end{aligned} \tag{45}$$

using $Q_0'(\xi) = (1 - \xi^2)^{-1}$.

For Q, Eqs. (25) (with $b_0 = b$) and (41) give

$$\begin{aligned} Q \simeq{}& \frac{2}{a^2 Q_0(\xi_0)}\int_b^a Q_0(\xi)|_{z=0}\, r\,\mathrm{d}r + \frac{b^2}{a^2} \\ &- \frac{2c^2\xi_0 Q_0'(\xi_0)}{a^2 Q_0(\xi_0)}(\xi_0^2 - 1)\int_0^1 \eta\,\mathrm{d}\eta. \end{aligned}$$

The last term simplifies to $c^2\xi_0/\{a^2 Q_0(\xi_0)\}$. For the first term, $r^2 = c^2(\xi^2 - 1)$ when $z = 0$, so $r\,\mathrm{d}r = c^2\xi\,\mathrm{d}\xi$. Hence

$$\begin{aligned} \frac{2}{c^2}\int_b^a Q_0(\xi)|_{z=0}\, r\,\mathrm{d}r &= 2\int_{\xi_0}^{\xi_a}\xi Q_0(\xi)\,\mathrm{d}\xi \\ &= \left[\xi^2 Q_0(\xi)\right]_{\xi_0}^{\xi_a} + \int_{\xi_0}^{\xi_a}\frac{\xi^2\,\mathrm{d}\xi}{\xi^2 - 1} \\ &= \left[\xi + (\xi^2 - 1)Q_0(\xi)\right]_{\xi_0}^{\xi_a}, \end{aligned}$$

where $a^2 = c^2(\xi_a^2 - 1)$ and $\xi^2(\xi^2 - 1)^{-1} = 1 - Q_0'(\xi)$ has been used. Thus,

$$\begin{aligned} Q &\simeq \frac{c^2}{a^2 Q_0(\xi_0)}\left[\xi + (\xi^2 - 1)Q_0(\xi)\right]_{\xi_0}^{\xi_a} + \frac{b^2}{a^2} + \frac{c^2\xi_0}{a^2 Q_0(\xi_0)} \\ &= \frac{c^2\xi_a}{a^2 Q_0(\xi_0)} + \frac{a^2 Q_0(\xi_a) - b^2 Q_0(\xi_0)}{a^2 Q_0(\xi_0)} + \frac{b^2}{a^2} \\ &= \frac{c^2\xi_a + a^2 Q_0(\xi_a)}{a^2 Q_0(\xi_0)}. \end{aligned} \tag{46}$$

In the limit when the spheroid becomes a sphere, Eq. (45) becomes

$$P \sim -\frac{2c\xi_0}{a} \sim -\frac{2b}{a},$$

in agreement with Eq. (37), where $Q_0(\xi) \sim \xi^{-1}$ as $\xi \to \infty$ and $(c\xi_0)^2 = b^2 + c^2$ have been used. Similarly, from Eq. (46),

$$Q \sim \frac{c^2\xi_a + a^2/\xi_a}{a^2/\xi_0} = \frac{2a^2 + c^2}{a^2\xi_a/\xi_0} \sim \frac{2\xi_0}{\xi_a} \sim \frac{2b}{a},$$

in agreement with Eq. (38), using $(c\xi_a)^2 = a^2 + c^2$.

VIII. SCATTERING BY A SPHERE IN A HARD TUBE

Formulas for R and T are given by the prescription described in Sec. III C. They are valid for long waves, which means $\kappa = ka \ll 1$, where k is the wavenumber and the length a is defined so that the cross-sectional area of the cylindrical waveguide is πa^2. Further approximations can be invoked if the obstacle is geometrically small, which means $\mu = b/a \ll 1$, where $2b$ is the lateral diameter of the obstacle.

A comparison with Linton's work[14] for the problem of a sphere (radius b) in a sound-hard tube (cross-sectional radius a) is made next. He developed an exact semi-analytical method, and he extracted rigorous asymptotic approximations. Precise agreement is found between these approximations and those obtained by the methods developed above.

A. Hard sphere

Consider a sound-hard sphere of radius b. Its volume is $|B| = (4/3)\pi b^3$. From Eq. (9),

$$D_a = \frac{\mathrm{i}\kappa L}{1 - \mathrm{i}\kappa L} \simeq \mathrm{i}\kappa L \simeq \mathrm{i}\kappa\mu^3,$$

where the approximation $L \simeq \mu^3$, Eq. (35), has been used. From Eqs. (15) and (16),

$$D_s = \frac{\mathrm{i}\kappa M}{1 - \mathrm{i}\kappa M} \simeq \mathrm{i}\kappa M = -\mathrm{i}\kappa\frac{|B|}{2\pi a^3} = -\frac{2}{3}\mathrm{i}\kappa\mu^3.$$

Then Eq. (5) gives

$$R = D_s - D_a \simeq -\frac{5}{3}\mathrm{i}\kappa\mu^3, \quad T - 1 = D_s + D_a \simeq \frac{1}{3}\mathrm{i}\kappa\mu^3.$$

These results agree with Linton's rigorous asymptotic analysis; see Eqs. (4.8) and (4.9) from Ref. 14.

B. Soft sphere

For a soft sphere, $D_a \simeq i\kappa L \simeq -2i\kappa\mu^3$, using Eq. (36). For D_s, Eqs. (13), (37), and (38) give

$$D_s = \frac{-P}{P + i\kappa(1-Q)} \simeq \frac{2\mu}{i\kappa(1-2\mu) - 2\mu}.$$

The approximation in Eq. (38) shows that Q is small compared to 1, and so it can be neglected, giving

$$D_s \simeq \frac{2\mu}{i\kappa - 2\mu} = \mathcal{D}(\kappa, \mu),$$

say. Notice that $\mathcal{D}(\kappa, 0) = 0$ and $\mathcal{D}(0, \mu) = -1$: $\mathcal{D}(\kappa, \mu)$ is not continuous at $(\kappa, \mu) = (0, 0)$. Note also that when κ and μ are both small, D_a is negligible, giving $R \simeq T - 1 \simeq D_s \simeq \mathcal{D}$. Also, for fixed κ, $R \simeq -2i\mu/\kappa$ as $\mu \to 0$, in agreement with Eq. (4.21) from Ref. 14.

IX. DISCUSSION AND CONCLUSIONS

Acoustic propagation along a cylinder containing an obstacle is considered. Simple explicit estimates for reflection and transmission coefficients have been presented, making use of certain coefficients that are extracted from the solutions to related boundary value problems for Laplace's equation. Good approximations to these coefficients are obtained by combining integral representations and solutions of similar boundary value problems without the waveguide boundary: the obstacle is surrounded by an unbounded fluid. This may seem crude but it has been shown to give excellent agreement with published numerical results, and it leads to approximations for R and T that agree with rigorous asymptotics.

One outcome of the study is the provision of a simple, but self-consistent, approach to modeling the effects of various thin "metascreens" on wave propagation, where a metascreen is a configuration of perforations and small obstacles nominally in the plane $z = 0$. The effect of such a screen is often modeled using certain transmission conditions across $z = 0$, leading to a one-dimensional problem. Write these "homogenized" conditions as

$$[u] = \beta_{11}\langle u\rangle + \beta_{12}\langle u'\rangle \text{ and } [u'] = \beta_{21}\langle u\rangle + \beta_{22}\langle u'\rangle, \tag{47}$$

where $[\cdot]$ denotes jump and $\langle\cdot\rangle$ denotes average, $[u] = u(0+) - u(0-)$ and $\langle u\rangle = (1/2)\{u(0+) + u(0-)\}$. The parameters β_{ij} are to be specified in terms of the geometry and the composition of the metascreen. For a soft screen, Eq. (47) may be simplified to

$$[u] = 0 \text{ and } [u'] = \beta_{21}u, \tag{48}$$

whereas for a hard screen,

$$[u] = \beta_{12}u' \text{ and } [u'] = 0. \tag{49}$$

The literature on such homogenized interface conditions is extensive.[46–51] Related homogenized boundary conditions arise in various diffusion problems.[52]

For a screen that is symmetric about $z = 0$, write $u(z) = e^{ikz} + R\,e^{-ikz}$ for $z < 0$ and $u(z) = T\,e^{ikz}$ for $z > 0$. Substituting in Eq. (48) and eliminating T gives

$$\beta_{21} = \frac{2ikR}{1+R} = \frac{2P}{a(Q-1)} \tag{50}$$

for a soft screen using Eqs. (13) and (29). Similarly, for a hard screen, Eq. (49) gives

$$\beta_{12} = \frac{2R}{ik(R-1)} = 2aL \tag{51}$$

using Eqs. (9) and (30). These formulas provide a simple characterization of the parameters β_{21} and β_{12} appearing in the homogenized boundary conditions, Eqs. (48) and (49), in terms of the constants L, P, and Q appearing in the analysis of the paper.

Another outcome of the study is that the dependence on the shape of the waveguide cross-section is weak, suggesting that the formulas obtained have broad applicability. In addition, the discussion of prolate spheroids in Sec. VII shows that the dependence on the longitudinal length $(2h)$ is also weak, suggesting that an analysis based on slender-body theory[53] should be pursued.

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