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Nonresonant Mode Coupling for Classes of Korteweg-de Vries Equations

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The method of mode coupling in third order is applied to generalized Korteweg-de Vries equations. These equations contain terms which are each a product of a function of the dependent variable u times one space derivative of u, up to a derivative of the fifth order. The resulting amplitude and phase equations are then specialized to three different classes of equations. For equations of the first class the amplitudes do not change, although all modes are coupled for the determination of the phases. For equations of the second class, all amplitude equations are truly coupled and the phases are constant. Finally, for equations of the third class, which includes the KdV equation, each mode behaves as if it were the only one present, so that only self-modulation is possible.

§1. Introduction

When in a previous paper[1)] the method of interacting waves was applied to the Korteweg-de Vries equation, it was discovered that the KdV equation has the remarkable property that the waves decouple also in third order. Such a decoupling is characteristic, of course, for linear media, but rather unusual for nonlinear media. The slow time variation of each wave is determined only by parameters of that wave, as if it were the only wave present, regardless of whatever other waves were present or not. The purpose of this paper is to generalize the classes of PDEs which can be handled in a similar way and to find out whether other equations besides the KdV equation exhibit the phenomenon of wave decoupling in third order. The PDEs considered contain one time derivative and are linear in each space derivative. The coefficients are functions of the dependent variable, so that the equation can be highly nonlinear indeed. In order to find results which would at least encompass the usual KdV equation, up to the fifth space derivative is included. Use is made of the method of mode coupling, which can be summarized as follows. An ordinary expansion in some ordering parameter is carried out for the dependent variable, together with the use of two time scales, one fast, one slow.[2–4)] The original PDE one started with is thereby replaced by a set of equations, of which the first is linear, and which can be solved order by order to determine the solution. For the firstorder part one then takes a superposition of plane waves, which amounts to a kind of Fourier treatment of the linear first-order equation and is hence a less particular form than would seem at first sight. The aim is to deduce the third-order equations governing the slow time behavior of the amplitudes and phases of the modes put into the linear part. In general these equations will be coupled, and some special cases are discussed. Some other equations besides the KdV equation exhibit indeed the property of wave decoupling in third order.

§2. Derivation of Amplitude and Phase Equations

We consider as classes of KdV equations the following nonlinear PDEs in one dependent variable u:

$$\frac{\partial u}{\partial t}=A_0(u)u+A_1(u)\frac{\partial u}{\partial x}+A_2(u)\frac{\partial^2 u}{\partial x^2}+A_3(u)\frac{\partial^3 u}{\partial x^3}+A_4(u)\frac{\partial^4 u}{\partial x^4}+A_5(u)\frac{\partial^5 u}{\partial x^5}=\sum_{s=0}^{5}A_s(u)\frac{\partial^s u}{\partial x^s}. \tag{1}$$

The KdV equation itself is included in (1) for the choice

$$A_0(u)=A_2(u)=A_4(u)=A_5(u)=0, \quad A_1(u)=\alpha+\beta u, \quad A_3(u)=\gamma, \tag{2}$$

where α, β and γ are constants. In order to apply the method of nonresonant wave interactions, as exposed before,[1,5] we begin by looking for possible homogeneous equilibria, characterized by a constant u_0, such that

$$A_0(u_0)u_0=0. \tag{3}$$

If $A_0(u)u$ is not identically zero, (3) is an algebraic equation in u_0. Whenever $A_0(u)$ is not included in (1), u_0 is undetermined and any constant will serve as an additional parameter. This is the case for the usual KdV equation. The next step is the introduction in (1) of a series expansion for u,

$$u=u_0+\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+\cdots, \tag{4}$$

together with a two-timescale method for the time derivative:

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t_0}+\varepsilon^2\frac{\partial}{\partial t_2}, \quad t_0=t, \quad t_2=\varepsilon^2 t. \tag{5}$$

This form is specifically chosen for the case of nonresonant interaction in third order. The special case of resonant interaction in second order is dealt with in Appendix A. Collecting terms of the same order in the smallness parameter ε yields a set of equations replacing (1):

$$\begin{aligned}
&Lu_1=0,\\
&Lu_2=\sum_{s=0}^{5} A_s'(u_0)u_1\frac{\partial^s u_1}{\partial x^s}+\frac{1}{2}A_0''(u_0)u_0u_1^2,\\
&Lu_3=-\frac{\partial u_1}{\partial t_2}+\sum_{s=0}^{5} A_s'(u_0)\left(u_2\frac{\partial^s u_1}{\partial x^s}+u_1\frac{\partial^s u_2}{\partial x^s}\right)+A_0''(u_0)u_0u_1u_2\\
&\qquad+\frac{1}{2}\sum_{s=0}^{5} A_s''(u_0)u_1^2\frac{\partial^s u_1}{\partial x^s}+\frac{1}{6}A_0'''(u_0)u_0u_1^3.
\end{aligned} \tag{6}$$

The dashes refer to the derivatives of the functions A_s with respect to their arguments u, and the linear operator L is defined as

$$L=\frac{\partial}{\partial t_0}-\sum_{s=0}^{5} A_s(u_0)\frac{\partial^s}{\partial x^s}-u_0A_0'(u_0). \tag{7}$$

Introducing now for u_1 a superposition of N plane waves,

$$\begin{aligned}
u_1&=\sum_{j=1}^{N} a_j(t_2)\cos(k_jx-\omega_jt_0+\alpha_j(t_2)),\\
&\equiv\sum_{j=1}^{N} a_j(t_2)\cos\phi_j(x,t_0,t_2),
\end{aligned} \tag{8}$$

into the first equation of (6) gives

$$\begin{aligned}
&\sum_{j=1}^{N}(\omega_j+A_1(u_0)k_j-A_3(u_0)k_j^3+A_5(u_0)k_j^5)a_j\sin\phi_j\\
&\qquad+\sum_{j=1}^{N}(-A_0(u_0)-u_0A_0'(u_0)+A_2(u_0)k_j^2-A_4(u_0)k_j^4)a_j\cos\phi_j=0.
\end{aligned} \tag{9}$$

In view of the linear independence of $\cos\phi_j$ and $\sin\phi_j$ for different arguments ϕ_j, one gets the dispersion law

$$\omega_j+A_1(u_0)k_j-A_3(u_0)k_j^3+A_5(u_0)k_j^5=0, \tag{10}$$

together with

$$A_0(u_0)+u_0A_0'(u_0)-A_2(u_0)k_j^2+A_4(u_0)k_j^4=0. \tag{11}$$

Equation (11) allows only two possibilities, either it is viewed as a biquadratic equation in k_j or it vanishes identically. In the former case only a two-wave interaction is possible (the sign of k_j is irrelevant, only its absolute value matters) and this is discussed in some detail in Appendix B.

In the latter case one requires

$$A_0(u_0)+u_0A_0'(u_0)=A_2(u_0)=A_4(u_0)=0, \tag{12}$$

and the linear operator L reduces to

$$L=\frac{\partial}{\partial t_0}-\sum_{s=1,3,5}A_s(u_0)\frac{\partial^s}{\partial x^s}, \tag{13}$$

containing only odd derivatives. This form of L will be used in the further discussion of (6). Turning now to the second equation in (6), the substitution of (8) gives

$$\begin{aligned}Lu_2=&\frac{1}{2}\sum_{j,l=1}^{N}\sum\left(A_0'(u_0)+\frac{1}{2}A_0''(u_0)u_0-A_2'(u_0)k_l^2+A_4'(u_0)k_l^4\right)a_ja_l(\cos(\phi_j+\phi_l)+\cos(\phi_j-\phi_l))\\&+\frac{1}{2}\sum_{j,l=1}^{N}\sum(-A_1'(u_0)+A_3'(u_0)k_l^2-A_5'(u_0)k_l^4)k_la_ja_l(\sin(\phi_j+\phi_l)-\sin(\phi_j-\phi_l)).\end{aligned} \tag{14}$$

In the summation with the cos-terms, the diagonal part, if allowed to remain present, would give a secular contribution to u_2. In order to avoid this, its coefficient must vanish, and hence

$$A_0'(u_0)+\frac{1}{2}A_0''(u_0)u_0=A_2'(u_0)=A_4'(u_0)=0, \tag{15}$$

placing further restrictions upon the allowable functions for A_0, A_2 and A_4. (14) is recast in the form

$$\begin{aligned}Lu_2=&\frac{1}{2}\sum_{j=1}^{N}(-A_1'(u_0)+A_3'(u_0)k_j^2-A_5'(u_0)k_j^4)k_ja_j^2\sin 2\phi_j\\&+\frac{1}{2}\sum_{\substack{j,l=1\\j<l}}^{N}\sum(\beta_{jl}^+(k_j+k_l)\sin(\phi_j+\phi_l)+\beta_{jl}^-(k_j-k_l)\sin(\phi_j-\phi_l))a_ja_l,\end{aligned} \tag{16}$$

where

$$\beta_{jl}^{\pm}=-A_1'(u_0)+(k_j^2\mp k_jk_l+k_l^2)A_3'(u_0)-(k_j^4\mp k_j^3k_l+k_j^2k_l^2\mp k_jk_l^3+k_l^4)A_5'(u_0). \tag{17}$$

From (16) the solution for u_2 follows as

$$u_2=\sum_{j=1}^{N}\frac{a_j^2}{4k_j^2}\mu_{jj}^+\cos 2\phi_j+\sum_{\substack{j,l=1\\j<l}}^{N}\sum\frac{a_ja_l}{2k_jk_l}(\mu_{jl}^+\cos(\phi_j+\phi_l)-\mu_{jl}^-\cos(\phi_j-\phi_l)), \tag{18}$$

with

$$\mu_{jl}^{\pm}=\beta_{jl}^{\pm}(-3A_3(u_0)+5A_5(u_0)(k_j^2\pm k_jk_l+k_l^2))^{-1}. \tag{19}$$

Finally, when u_1 as given in (8) and u_2 as in (18) are substituted into the third equation of (6), one gets

$$\begin{aligned}Lu_3=&\sum_{j=1}^{N}\left(-\frac{\partial a_j}{\partial t_2}\cos\phi_j+a_j\frac{\partial\alpha_j}{\partial t_2}\sin\phi_j\right)\\&+\frac{1}{8}\sum\sum_{l,m,n=1}^{N}\sum\left(A_0''(u_0)+\frac{1}{3}A_0'''(u_0)u_0-A_2''(u_0)k_n^2+A_4''(u_0)k_n^4\right)\\&\qquad\times(\cos(\phi_l+\phi_m+\phi_n)+\cos(\phi_l+\phi_m-\phi_n)+2\cos(\phi_l-\phi_m+\phi_n))a_la_ma_n\\&+\frac{1}{8}\sum\sum_{l,m,n=1}^{N}\sum(-A_1''(u_0)k_n+A_3''(u_0)k_n^3-A_5''(u_0)k_n^5)\\&\qquad\times(\sin(\phi_l+\phi_m+\phi_n)-\sin(\phi_l+\phi_m-\phi_n)+2\sin(\phi_l-\phi_m+\phi_n))a_la_ma_n\\&+\sum_{l,m=1}^{N}\sum\frac{a_la_m^2}{8k_m^2}\mu_{mm}^+(\rho_{lmm}^+\sin(\phi_l+2\phi_m)+\nu_{lmm}^+\sin(\phi_l-2\phi_m))\\&+\sum\sum_{\substack{l,m,n=1\\m<n}}^{N}\sum\frac{a_la_ma_n}{4k_mk_n}(\mu_{mn}^+\rho_{lmn}^+\sin(\phi_l+\phi_m+\phi_n)-\mu_{mn}^+\nu_{lmn}^+\sin(-\phi_l+\phi_m+\phi_n)\\&\qquad-\mu_{mn}^-\nu_{lmn}^-\sin(\phi_l-\phi_m+\phi_n)-\mu_{mn}^-\rho_{lmn}^-\sin(\phi_l+\phi_m-\phi_n)),\end{aligned} \tag{20}$$

if

$$\nu^{\pm}_{lmn} = -A_1'(k_l - k_m \mp k_n) + A_3'(k_l^3 - (k_m \pm k_n)^3) - A_5'(k_l^5 - (k_m \pm k_n)^5),$$
$$\rho^{\pm}_{lmn} = -A_1'(k_l + k_m \pm k_n) + A_3'(k_l^3 + (k_m \pm k_n)^3) - A_5'(k_l^5 + (k_m \pm k_n)^5). \tag{21}$$

Some terms on the right-hand side of (20) belong to the kernel of L and hence would lead to secular terms in u_3 if left in. In order to avoid such secularities some balancing is to occur in (20). The cos-terms yield a set of equations governing the slow time behavior of the amplitudes a_j:

$$\frac{\partial a_j}{\partial t_2} = \frac{1}{2} a_j \sum_{\substack{m=1 \\ m \neq j}}^{N} \left(A_0''(u_0) + \frac{1}{3} A_0'''(u_0) u_0 - A_2''(u_0) k_m^2 + A_4''(u_0) k_m^4 \right) a_m^2$$
$$+ \frac{1}{4} a_j (A_0''(u_0) + \frac{1}{3} A_0'''(u_0) u_0 - A_2''(u_0) k_j^2 + A_4''(u_0) k_j^4) \left(\sum_{m=1}^{N} a_m^2 + \frac{1}{2} a_j^2 \right), \tag{22}$$

On the other hand, the sine terms yield the variations in the phases α_j:

$$\frac{\partial \alpha_j}{\partial t_2} = \sum_{\substack{m=1 \\ m \neq j}}^{N} \frac{a_m^2}{4 k_j k_m} (\mu^+_{jm} \nu^+_{mmj} + \mu^-_{jm} \nu^-_{mmj} + A_1''(u_0) k_j^2 k_m - A_3''(u_0) k_j^4 k_m + A_5''(u_0) k_j^6 k_m)$$
$$+ \frac{a_j^2}{8 k_j^2} (\mu^+_{jj} \nu^+_{jjj} + A_1''(u_0) k_j^3 - A_3''(u_0) k_j^5 + A_5''(u_0) k_j^7). \tag{23}$$

§3. Discussion

In this section some special cases of (22) and (23) will be discussed, and hence also of (1). In the whole of the discussion, quartic and higher-order terms are omitted, as such nonlinearities do not influence a third-order wave coupling process. At the present level of expansion nothing can be learned anyway about such terms, one would have to go to higher orders. Also, in the examples given below, u_0 is always taken zero for simplicity. This is not really a restriction, as it amounts to a shift in the dependent variable. Furthermore, a rescaling of t, x and u has been carried out to achieve the simplest possible form for the types of equations considered. *The first class is nonlinear equations which leave the amplitudes of the interacting waves constant.* Together with the restrictions (3), (12) and (15) found already, requiring all a_j to remain constant leads to a further restriction

$$A_0''(u_0) + \frac{1}{3} A_0'''(u_0) = A_2''(u_0) = A_4''(u_0) = 0, \tag{24}$$

making the right-hand side of (22) zero. The corresponding types of equations are

$$u_t = A_1(u) u_x + A_3(u) u_{xxx} + A_5(u) u_{xxxxx}. \tag{25}$$

Derivatives will be denoted from now on by subscripts. Any nonlinear PDE containing only one odd-order derivative in each term gives constant wave amplitudes. As soon as the amplitudes are constant, the right-hand side of (23) is constant as well, and all phases α_j are a slow linear function of time:

$$\alpha_j(t_2) = \gamma_j t_2 + \alpha_j(0). \tag{26}$$

In general all amplitudes are coupled together in γ_j to determine each α_j, except for equations of the third class, discussed below. In the converse case, *we find the second class equations leading to constant phases for the interacting waves.* The vanishing of the right-hand side of (23) leads to

$$A_s'(u_0) = A_s''(u_0) = 0 \quad (s = 1, 3, 5). \tag{27}$$

Keeping also (3), (12) and (15) in mind gives equations of the type:

$$u_t = \lambda_0 u^3 + \lambda_1 u_x + \lambda_2 u^2 u_{xx} + \lambda_3 u_{xxx}$$
$$+ \lambda_4 u^2 u_{xxxx} + \lambda_5 u_{xxxxx}, \tag{28}$$

with all λ_i constant. The structure of the right-hand side of (28) is u^2 multiplied by a linear combination, with constant coefficients, of even-order derivatives (including u itself as the zeroth derivative), plus a similar linear combination of odd-order derivatives. To (28) belongs the special case of the Hirota equation[6] discussed in earlier papers.[1,5] It is clear from (22) that all amplitude equations are coupled. Finally, *in the third class one could look for equations such that the slow changes in amplitude and phase would for each wave* *depend only on the parameters of the wave itself.* For those equations the waves would decouple also in third order. Inspecting (22) leads again to (24), making the third class a subclass of the first class, with constant wave amplitudes. Furthermore, there follows from (23) that

$$\begin{aligned} &3A_3(u_0)A_1''(u_0)=4A_1'(u_0)A_3'(u_0),\\ &3A_3(u_0)A_3''(u_0)=4A_3'^2(u_0),\\ &A_5(u_0)=A_5'(u_0)=A_5''(u_0)=0. \end{aligned} \tag{29}$$

Together with (24) and earlier restrictions, one is led to nonlinear equations of the following types:

$$\begin{aligned} &\text{(i)} \quad u_t=(3+3u+2u^2)(u_x\pm u_{xxx}),\\ &\text{(ii)} \quad u_t=(3+3u+2u^2)u_{xxx},\\ &\text{(iii)} \quad u_t=uu_x+u_{xxx}. \end{aligned} \tag{30}$$

The last type is nothing but the KdV equation, for which the property of wave decoupling in third order was discovered earlier.[1] It was in fact this unexpected property of the KdV equation which started the research for a generalization. Writing (23) specifically for (30) respectively gives

$$\begin{aligned} &\text{(i)} \quad \frac{\partial \alpha_j}{\partial t_2}=\frac{a_j^2}{8k_j}(1-4k_j^2+3k_j^4),\\ &\text{(ii)} \quad \frac{\partial \alpha_j}{\partial t_2}=\frac{3}{8}a_j^2k_j^3,\\ &\text{(iii)} \quad \frac{\partial \alpha_j}{\partial t_2}=\frac{a_j^2}{24k_j}, \end{aligned} \tag{31}$$

leading again to phases which increase or decrease slowly with time.

As a final remark, similar investigations have shown that the Benjamin-Bona-Mahony equation[7,8)]

$$u_t=(1+u)u_x+u_{txx}, \tag{32}$$

and the Joseph-Egri or time regularized long wave equation,[8-10)]

$$u_t=(1+u)u_x+u_{ttx}, \tag{33}$$

also belong to the third class, but only in the long-wavelength limit, where the dispersion law tends to the KdV dispersion law.

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Appendix A: Resonant Second-Order Interaction

For resonant interaction in second order the wavevectors and frequencies have to obey the selection rules[11)]

$$k_1=k_2+k_3, \quad \omega_1=\omega_2+\omega_3, \tag{A·1}$$

which really express the conservation of momentum and energy when the waves are viewed as quanta. Furthermore, for each wave ω_j and k_j $(j=1, 2, 3)$ are linked through the dispersion law (10). Writing it as

$$\omega_j=-A_1(u_0)k_j+A_3(u_0)k_j^3-A_5(u_0)k_j^5, \quad (j=1, 2, 3) \tag{A·2}$$

and using (A·1) leads to

$$3A_3(u_0)=5A_5(u_0)(k_2^2+k_2k_3+k_3^2). \tag{A·3}$$

Nontrivial solutions for k_2 and k_3 are only possible whenever $A_3(u_0)$ and $A_5(u_0)$ both differ from zero. This is not the case for the interesting equations (30) of the third class. Also, when viewing (A·3) as a quadratic equation in k_3, given k_2, a real solution for k_3 requires that

$$k_2^2<\frac{4A_3(u_0)}{5A_5(u_0)}, \tag{A·4}$$

that is, one of the waves must be of a sufficiently long wavelength. If all the above is satisfied, the case of resonant three-wave interaction can be given in the usual fashion, requiring only a modification of the two-timescale method (5) to

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}, \quad t_0=t, \quad t_1=\varepsilon t. \tag{A·5}$$

These calculations, however, fall outside the scope of the present paper.

Appendix B: Two-Wave Interaction

Returning to the condition (11),

$$A_4(u_0)k_j^4-A_2(u_0)k_j^2+A_0(u_0)+u_0A_0'(u_0)=0, \quad (j=1, 2) \tag{B·1}$$

when this is now viewed as a biquadratic equation in k_j^2, several cases are possible. The most interesting one is where two different waves can exist, requiring two real, different, positive solutions for k_j^2, and this is possible if

$$A_2^2(u_0) > 4A_4(u_0)(A_0(u_0) + u_0 A_0'(u_0)),$$
$$A_2(u_0)/A_4(u_0) > 0,$$
$$(A_0(u_0) + u_0 A_0'(u_0))/A_4(u_0) > 0. \qquad (\mathrm{B}\cdot 2)$$

Otherwise, only one or no wave results. The corresponding equations are then of the form

$$u_t = (\lambda_0 + \kappa_0 u^2)u + A_1(u)u_x + (\lambda_2 + \kappa_2 u^2)u_{xx} + A_3(u)u_{xxx} + (\lambda_4 + \kappa_4 u^2)u_{xxxx} + A_5(u)u_{xxxxx}, \qquad (\mathrm{B}\cdot 3)$$

with

$$\lambda_2^2 > 4\lambda_0\lambda_4, \quad \lambda_2/\lambda_4 > 0, \quad \lambda_0/\lambda_4 > 0. \qquad (\mathrm{B}\cdot 4)$$

Substituting then the two values for k_j into (10) gives ω_1 and ω_2. From then on the procedure in the paper can be followed step by step, except that the linear operator L is not reduced as in (13), but keeps the form given in (7). This has some repercussions upon the determination of u_2. In all summations over wave quantities N is equal to 2 here, but no essential difficulties arise.

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