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Symbolic Computation of Travelling Wave Solutions of Nonlinear Differential-Difference Equations

Prof. Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO-80401, U.S.A.
http://www.mines.edu/fs_home/whereman/
whereman@mines.edu

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Collaborators: Douglas Baldwin & Ünal Göktaş (Wolfram Research, Inc.)

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OUTLINE

Purpose & Motivation

Typical Examples of ODEs, PDEs, and DDEs

Part I: Tanh Method for PDEs

Quick Review of Tanh Algorithm

A Typical Example

Other Types of Functions

Demo of Mathematica Software: PDESpecialSolutions.m

Part II: Tanh Method for DDEs (Lattices)

Algorithm for Tanh Solutions

Typical Examples

Table with Results

Demo of Mathematica Software: DDESpecialSolutions.m

Conclusions & Future Research

Research Papers & Software

Purpose & Motivation

- **Develop** and implement various **methods** to find closed form solutions of nonlinear PDEs and DDEs: direct methods, Lie symmetry methods, similarity methods, etc.

- Fully **automate** the hyperbolic and elliptic function methods to compute travelling solutions of nonlinear PDEs.

- Fully **automate** the hyperbolic tanh method to compute travelling wave solutions of nonlinear differential-difference equations (DDEs or lattices).

- **Class** of nonlinear PDEs and DDEs solvable with such methods includes famous evolution and wave equations, and lattices.

  Examples PDEs: Korteweg-de Vries, Boussinesq, and Kuramoto-Sivashinsky equations.
  Fisher and FitzHugh-Nagumo equations.

  Examples ODEs: Duffing and nonlinear oscillator equations.

  Examples DDEs: Volterra, Toda, and Ablowitz-Ladik lattices.

- **PDEs:** Solutions of tanh (kink) or sech (pulse) type **model** solitary waves in fluids, plasmas, circuits, optical fibers, bio-genetics, etc.

  DDEs: discretizations of PDEs, lattice theory, queing and network problems, solid state and quantum physics.

- **Benchmark** solutions for numerical PDE and DDE solvers.

- **Research aspect**: Design high-quality application packages to compute solitary wave solutions of large classes of nonlinear evolution and wave equations and lattices.

- **Educational aspect**: Software as course ware for courses in nonlinear PDEs and DDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software.

  REU projects of NSF. Extreme Programming!

- **Users**: scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.

Typical Examples of ODEs and PDEs

- The Duffing equation:

$$u'' + u + \alpha u^3 = 0$$

Solutions in terms of elliptic functions:

$$u(x) = \pm \frac{\sqrt{c_1^2 - 1}}{\sqrt{\alpha}} \, \mathrm{cn}(c_1 x + \Delta; \frac{c_1^2 - 1}{2c_1^2}),$$

and

$$u(x) = \pm \frac{\sqrt{2(c_1^2 - 1)}}{\sqrt{\alpha}} \, \mathrm{sn}(c_1 x + \Delta; \frac{1 - c_1^2}{c_1^2}).$$

- The Korteweg-de Vries (KdV) equation:

$$u_t + 6\alpha u u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x,t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 \left[\mathrm{c_1 x + c_2 t} + \Delta \right],$$

or, equivalently,

$$u(x,t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \mathrm{sech}^2 \left[\mathrm{c_1 x + c_2 t} + \Delta \right].$$

Cnoidal wave solution:

$$u(x,t) = \frac{4c_1^3(1 - 2m) - c_2}{\alpha c_1} + \frac{12m\, c_1^2}{\alpha} \, \mathrm{cn}^2(c_1 x + c_2 t + \Delta; m),$$

modulus m.

- The modified Korteweg-de Vries (mKdV) equation:

$$u_t + \alpha u^2 u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x,t) = \pm\sqrt{\frac{6}{\alpha}}\,c_1 \,\mathrm{sech}\left[\mathrm{c_1x - c_1^3t + \Delta}\right].$$

- Three-dimensional modified Korteweg-de Vries equation:

$$u_t + 6u^2 u_x + u_{xyz} = 0.$$

Solitary wave solution:

$$u(x,y,z,t) = \pm\sqrt{c_2 c_3}\,\mathrm{sech}\left[\mathrm{c_1x + c_2y + c_3z - c_1c_2c_3t + \Delta}\right].$$

- The combined KdV-mKdV equation:

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + \gamma u_{3x} = 0.$$

Real solitary wave solution:

$$u(x,t) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\gamma}{\beta}}c_1\ \mathrm{sech}(\mathrm{c_1x} + \frac{\mathrm{c_1}}{2\beta}(3\alpha^2 - 2\beta\gamma \mathrm{c_1^2})\mathrm{t} + \Delta).$$

Complex solutions:

$$u(x,t) = -\frac{\alpha}{2\beta} \pm i\sqrt{\frac{\gamma}{\beta}}c_1\ \tanh(\mathrm{c_1x} + \frac{\mathrm{c_1}}{2\beta}(3\alpha^2 + 4\beta\gamma \mathrm{c_1^2})\mathrm{t} + \Delta),$$

$$u(x,t) = -\frac{\alpha}{2\beta} + \frac{1}{2}\sqrt{\frac{\gamma}{\beta}}c_1\,(\,\mathrm{sech}\xi \pm \mathrm{i}\tanh\xi),$$

and

$$u(x,t) = -\frac{\alpha}{2\beta} - \frac{1}{2}\sqrt{\frac{\gamma}{\beta}}c_1\,(\,\mathrm{sech}\xi \mp \mathrm{i}\tanh\xi)$$

with $\xi = c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + \beta\gamma c_1^2)t + \Delta$.

- The Fisher equation:

$$u_t - u_{xx} - u\,(1-u) = 0.$$

Solitary wave solution:

$$u(x,t) = \frac{1}{4} \pm \frac{1}{2}\,\text{tanh}\xi + \frac{1}{4}\,\text{tanh}^2\xi,$$

with

$$\xi = \pm\frac{1}{2\sqrt{6}}\,x \pm \frac{5}{12}\,t + \Delta.$$

- The generalized Kuramoto-Sivashinski equation:

$$u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0.$$

Solitary wave solutions

(ignoring symmetry $u \to -u, x \to -x, \sigma \to -\sigma$) :

For $\sigma = 4$:

$$u(x,t) = 9 - 2c_2 - 15\,\text{tanh}\xi\;(1 + \;\text{tanh}\xi - \;\text{tanh}^2\xi)$$

with $\xi = \frac{x}{2} + c_2 t + \Delta$.

For $\sigma = \frac{12}{\sqrt{47}}$:

$$u(x,t) = \frac{45 \mp 4418 c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}}\,\text{tanh}\xi - \frac{45}{47\sqrt{47}}\,\text{tanh}^2\xi \pm \frac{15}{47\sqrt{47}}\,\text{tanh}^3\xi$$

with $\xi = \pm\frac{1}{2\sqrt{47}}\,x + c_2 t + \Delta$.

For $\sigma = 16/\sqrt{73}$:

$$u(x,t) = \frac{2\,(30 \mp 5329 c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}}\tanh\xi - \frac{60}{73\sqrt{73}}\tanh^2\xi \pm \frac{15}{73\sqrt{73}}\tanh^3\xi$$

with $\xi = \pm\frac{1}{2\sqrt{73}}\,x + c_2 t + \Delta.$

For $\sigma = 0$:

$$u(x,t) = -2\sqrt{\frac{19}{11}}c_2 - \frac{135}{19}\sqrt{\frac{11}{19}}\tanh\xi + \frac{165}{19}\sqrt{\frac{11}{19}}\tanh^3\xi$$

with $\xi = \frac{1}{2}\sqrt{\frac{11}{19}}\,x + c_2 t + \Delta.$

- The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0,$$

or written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right], \\ v(x,t) &= b_0 + 4\alpha c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right]. \end{aligned}$$

- The Broer-Kaup system:

$$\begin{aligned} u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xxy} &= 0, \\ v_t + 2(uv)_x + v_{xx} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x,t) &= -\frac{c_3}{2c_1} + c_1 \tanh\left[\mathrm{c_1 x + c_2 y + c_3 t} + \Delta\right], \\ v(x,t) &= c_1 c_2 - c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 y + c_3 t} + \Delta\right]. \end{aligned}$$

- System of three nonlinear coupled equations (Gao & Tian, 2001):

$$
\begin{aligned}
u_t - u_x - 2v &= 0, \\
v_t + 2uw &= 0, \\
w_t + 2uv &= 0.
\end{aligned}
$$

Solutions:

$$
\begin{aligned}
u(x,t) &= \pm c_2 \tanh\xi, \\
v(x,t) &= \mp \frac{1}{2} c_2 (c_1 - c_2) \operatorname{sech}^2\xi, \\
w(x,t) &= -\frac{1}{2} c_2 (c_1 - c_2) \operatorname{sech}^2\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
u(x,t) &= \pm i c_2 \operatorname{sech}\xi, \\
v(x,t) &= \pm \frac{1}{2} i c_2 (c_1 - c_2) \tanh\xi \operatorname{sech}\xi, \\
w(x,t) &= \frac{1}{4} c_2 (c_1 - c_2) \left(1 - 2\operatorname{sech}^2\xi\right),
\end{aligned}
$$

and also

$$
\begin{aligned}
u(x,t) &= \pm \frac{1}{2} i c_2 \left(\operatorname{sech}\xi + \mathrm{i} \tanh\xi\right), \\
v(x,t) &= \pm \frac{1}{4} c_2 (c_1 - c_2) \operatorname{sech}\xi \left(\operatorname{sech}\xi + \mathrm{i} \tanh\xi\right), \\
w(x,t) &= -\frac{1}{4} c_2 (c_1 - c_2) \operatorname{sech}\xi \left(\operatorname{sech}\xi + \mathrm{i} \tanh\xi\right)
\end{aligned}
$$

with $\xi = c_1 x + c_2 t + \Delta$.

- Nonlinear sine-Gordon equation (light cone coordinates):

$$\Phi_{xt} = \sin\Phi.$$

Set $u = \Phi_x, \quad v = \cos(\Phi) - 1$,

$$\begin{aligned} u_{xt} - u - u\,v &= 0, \\ u_t^2 + 2v + v^2 &= 0. \end{aligned}$$

Solitary wave solution (kink):

$$\begin{aligned} u &= \pm\frac{1}{\sqrt{-c}}\,\mathrm{sech}[\frac{1}{\sqrt{-\mathrm{c}}}(\mathrm{x} - \mathrm{ct}) + \Delta], \\ v &= 1 - 2\,\mathrm{sech}^2[\frac{1}{\sqrt{-\mathrm{c}}}(\mathrm{x} - \mathrm{ct}) + \Delta]. \end{aligned}$$

Solution:

$$\Phi(x,t) = \int u(x,t)dx = \pm 4 \text{ arctan}\left[\exp\left(\frac{1}{\sqrt{-c}}(x - ct) + \Delta\right)\right].$$

- ODEs from quantum field theory:

$$\begin{aligned} u_{xx} &= -u + u^3 + auv^2, \\ v_{xx} &= bv + cv^3 + av(u^2 - 1). \end{aligned}$$

Solitary wave solutions:

$$\begin{aligned} u &= \pm\tanh[\sqrt{\frac{\mathrm{a}^2 - \mathrm{c}}{2(\mathrm{a} - \mathrm{c})}}\mathrm{x} + \Delta], \\ v &= \pm\sqrt{\frac{1 - a}{a - c}}\,\mathrm{sech}[\sqrt{\frac{\mathrm{a}^2 - \mathrm{c}}{2(\mathrm{a} - \mathrm{c})}}\mathrm{x} + \Delta], \end{aligned}$$

provided $b = \sqrt{\frac{a^2 - c}{2(a - c)}}$.

Typical Examples of DDEs (lattices)

- The Volterra lattice:

$$\begin{aligned} \dot{u}_n &= u_n(v_n - v_{n-1}), \\ \dot{v}_n &= v_n(u_{n+1} - u_n). \end{aligned}$$

Travelling wave solution:

$$\begin{aligned} u_n(t) &= -c_1 \ \coth(d_1) + c_1 \tanh\left[\mathrm{d_1 n + c_1 t} + \delta\right], \\ v_n(t) &= -c_1 \ \coth(d_1) - c_1 \tanh\left[\mathrm{d_1 n + c_1 t} + \delta\right]. \end{aligned}$$

- The Toda lattice:

$$\ddot{u}_n = (1 + \dot{u}_n)\,(u_{n-1} - 2u_n + u_{n+1})\,.$$

Travelling wave solution:

$$u_n(t) = a_{10} \pm \sinh(d_1) \tanh\left[\mathrm{d_1 n} \pm \mathrm{\sinh(d_1)\, t} + \delta\right].$$

- The Relativistic Toda lattice:

$$\begin{aligned} \dot{u}_n &= (1 + \alpha\, u_n)(v_n - v_{n-1}), \\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha\, v_{n+1} - \alpha\, v_{n-1}). \end{aligned}$$

Travelling wave solution:

$$\begin{aligned} u_n(t) &= -\frac{1}{\alpha} - c_1 \ \coth(d_1) + c_1 \tanh\left[\mathrm{d_1 n + c_1 t} + \delta\right], \\ v_n(t) &= \frac{c_1 \ \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh\left[\mathrm{d_1 n + c_1 t} + \delta\right]. \end{aligned}$$

- The Ablowitz-Ladik lattice:

$$\begin{aligned} \dot{u}_n(t) &= (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \\ \dot{v}_n(t) &= -(\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n. \end{aligned}$$

Travelling wave solution:

$$\begin{aligned} u_n(t) &= \frac{\alpha \sinh^2(d_1)}{a_{21}} \left(\pm 1 - \tanh\left[\mathrm{d_1 n} + 2\alpha \mathrm{t} \sinh^2(\mathrm{d_1}) + \delta\right]\right), \\ v_n(t) &= a_{21}(\pm 1 + \tanh\left[\mathrm{d_1 n} + 2\alpha \sinh^2(\mathrm{d_1})\mathrm{t} + \delta\right]). \end{aligned}$$

- 2D Toda lattice:

$$\frac{\partial^2 u_n}{\partial x \partial t}(x,t) = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}).$$

Travelling wave solution:

$$u_n(x,t) = a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh\left[\mathrm{d_1 n} + \frac{\sinh^2(\mathrm{d_1})}{\mathrm{c_2}}\mathrm{x} + \mathrm{c_2 t} + \delta\right].$$

- Hybrid lattice:

$$\dot{u}_n(t) = (1 + \alpha u_n + \beta u_n^2)(u_{n-1} - u_{n+1}),$$

Travelling wave solution:

$$u_n(t) = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}\, \tanh(\mathrm{d_1})}{2\beta} \tanh\left[\mathrm{d_1 n} + \frac{\alpha^2 - 4\beta}{2\beta} \tanh(\mathrm{d_1})\mathrm{t} + \delta\right].$$

Algorithm for Tanh Solutions for system of PDEs

System of nonlinear PDEs of order m

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(m)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u}$ has M components u_i (or $u, v, w, ...$).
Independent variable $\mathbf{x}$ has N components x_j (or $x, y, z, ..., t$).

Step T1:

- Seek solution $\mathbf{u}(\mathbf{x}) = \mathbf{U}(T)$, with

$$T = \tanh\xi = \tanh\left[\sum_{\mathrm{j}}^{\mathrm{N}} \mathrm{c_j x_j} + \delta\right].$$

- Observe $\tanh'\xi = 1 - \tanh^2\xi$ or $T' = 1 - T^2$. Hence, *all* derivative of T are polynomial in T. For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{\partial \xi}{\partial x_j} \frac{dT}{d\xi} \frac{d\bullet}{dT} = c_j(1 - T^2)\frac{d\bullet}{dT}$$

  Produces a nonlinear system of ODEs

$$\mathbf{\Delta}(T, \mathbf{U}(T), \mathbf{U}'(T), \mathbf{U}''(T), \ldots, \mathbf{U}^{(m)}(T)) = \mathbf{0}.$$

  NOTE: Compare with the ultra-spherical (linear) ODE:

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0$$

  with integer $n \geq 0$ and α real. Includes:

  * Legendre equation ($\alpha = \frac{1}{2}$),
  * ODE for Chebeyshev polynomials of type I ($\alpha = 0$),
  * ODE for Chebeyshev polynomials of type II ($\alpha = 1$).

- Example: For the Boussinesq system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

after cancelling common factors $1 - T^2$,

$$\begin{aligned} &c_2 U' + c_1 V' = 0, \\ &c_2 V' + c_1 U' - 3c_1 UU' \\ &\quad + \alpha c_1^3 \left[2(1-3T^2)U' + 6T(1-T^2)U'' - (1-T^2)^2 U''' \right] = 0. \end{aligned}$$

Step T2:

- Seek polynomial solutions

$$U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j.$$

Determine the highest exponents $M_i \geq 1$.

Substitute $U_i(T) = T^{M_i}$ into the LHS of ODE.

Gives polynomial $\mathbf{P}(T)$.

For every P_i consider all possible balances of the highest exponents in T.

Solve the resulting linear system(s) for the unknowns M_i.

- Example: Balance highest exponents for the Boussinesq system

$$M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1.$$

So, $M_1 = M_2 = 2$.

Hence,

$$\begin{aligned} U(T) &= a_{10} + a_{11} T + a_{12} T^2, \\ V(T) &= a_{20} + a_{21} T + a_{22} T^2. \end{aligned}$$

Step T3:

- Derive algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the power terms in T.
- Example: Algebraic system for Boussinesq case

$$
\begin{aligned}
&a_{11}\, c_1\, (3a_{12} + 2\alpha\, c_1^2) = 0, \\
&a_{12}\, c_1\, (a_{12} + 4\alpha\, c_1^2) = 0, \\
&a_{21}\, c_1 + a_{11}\, c_2 = 0, \\
&a_{22}\, c_1 + a_{12}\, c_2 = 0, \\
&a_{11}\, c_1 - 3a_{10}\, a_{11}\, c_1 + 2\alpha a_{11}\, c_1^3 + a_{21}\, c_2 = 0, \\
&-3a_{11}^2\, c_1 + 2\, a_{12}\, c_1 - 6a_{10}\, a_{12}\, c_1 + 16\alpha\, a_{12}\, c_1^3 + 2a_{22}\, c_2 = 0.
\end{aligned}
$$

Step T4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for Boussinesq system

$$
\begin{aligned}
a_{10} &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0, \\
a_{12} &= -4\alpha c_1^2, \quad a_{20} = \text{free}, \\
a_{21} &= 0, \quad a_{22} = 4\alpha c_1 c_2.
\end{aligned}
$$

Step T5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.
- Example: Solitary wave solution for Boussinesq system:

$$
\begin{aligned}
u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right], \\
v(x,t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right].
\end{aligned}
$$

Other Types of Solutions for PDEs

Function	ODE $\quad (y' = \frac{dy}{d\xi})$	Chain Rule
$\tanh(\xi)$	$y' = 1 - y^2$	$\frac{\partial \bullet}{\partial x_j} = c_j (1 - \text{T}^2) \frac{d\bullet}{d\text{T}}$
$\text{sech}(\xi)$	$y' = -y\sqrt{1 - y^2}$	$\frac{\partial \bullet}{\partial x_j} = -c_j \text{S} \sqrt{1 - \text{S}^2} \frac{d\bullet}{d\text{S}}$
$\tan(\xi)$	$y' = 1 + y^2$	$\frac{\partial \bullet}{\partial x_j} = c_j (1 + \text{TAN}^2) \frac{d\bullet}{d\text{TAN}}$
$\exp(\xi)$	$y' = y$	$\frac{\partial \bullet}{\partial x_j} = c_j \text{E} \frac{d\bullet}{d\text{E}}$
$\text{cn}(\xi; \text{m})$	$y' = -\sqrt{(1-y^2)(1-m+m\, y^2)}$	$\frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{(1-\text{CN}^2)(1-m+m\,\text{CN}^2)} \frac{d\bullet}{d\text{CN}}$
$\text{sn}(\xi; \text{m})$	$y' = \sqrt{(1-y^2)(1-m\, y^2)}$	$\frac{\partial \bullet}{\partial x_j} = c_j \sqrt{(1-\text{SN}^2)(1-m\text{SN}^2)} \frac{d\bullet}{d\text{SN}}$

Algorithm for Tanh Solutions for System of DDEs

Nonlinear differential-difference equations (DDEs) of order m

$$\mathbf{\Delta}(\mathbf{u_{n+p_1}}(\mathbf{x}), \mathbf{u_{n+p_2}}(\mathbf{x}), \cdots, \mathbf{u_{n+p_{\mathit{k}}}}(\mathbf{x}), \mathbf{u'_{n+p_1}}(\mathbf{x}), \mathbf{u'_{n+p_2}}(\mathbf{x}), \cdots, \mathbf{u'_{n+p_{\mathit{k}}}}(\mathbf{x}), \cdots,$$
$$\mathbf{u}^{(r)}_{\mathbf{n+p_1}}(\mathbf{x}), \mathbf{u}^{(r)}_{\mathbf{n+p_2}}(\mathbf{x}), \cdots, \mathbf{u}^{(r)}_{\mathbf{n+p_{\mathit{k}}}}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u_n}$ has M components $u_{i,\mathbf{n}}$ (or $u_\mathbf{n}, v_\mathbf{n}, w_\mathbf{n}, \cdots$)
Independent variable $\mathbf{x}$ has N components x_i (or $t, x, y, \cdots$).
Shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$.
$\mathbf{u}^{(r)}(\mathbf{x})$ is collection of mixed derivatives of order r.

Simplest case for independent variable (t), and one lattice point (n):

$$\Delta(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \dot{\mathbf{u}}_{n-1}, \dot{\mathbf{u}}_n, \dot{\mathbf{u}}_{n+1}, ..., \mathbf{u}^{(r)}_{n-1}, \mathbf{u}^{(r)}_{n}, \mathbf{u}^{(r)}_{n+1}, ...) = \mathbf{0}.$$

Step D1:

- Seek solution $\mathbf{u_n}(\mathbf{x}) = \mathbf{U_n}(T_\mathbf{n})$, with $T_\mathbf{n} = \tanh(\xi_\mathbf{n})$,

$$\xi_\mathbf{n} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \delta = \mathbf{d}\cdot\mathbf{n} + \mathbf{c}\cdot\mathbf{x} + \delta.$$

- Repeatedly apply the operator rule

$$\frac{d\bullet}{dx_j} = \frac{\partial \xi_\mathbf{n}}{\partial x_j}\frac{dT_\mathbf{n}}{d\xi_\mathbf{n}}\frac{d\bullet}{dT_\mathbf{n}} = c_j(1-T_\mathbf{n}^2)\frac{d\bullet}{dT_\mathbf{n}},$$

  transforms DDE into

$$\mathbf{\Delta}(\mathbf{U_{n+p_1}}(T_\mathbf{n}), \cdots, \mathbf{U_{n+p_{\mathit{k}}}}(T_\mathbf{n}), \mathbf{U'_{n+p_1}}(T_\mathbf{n}), \cdots, \mathbf{U'_{n+p_{\mathit{k}}}}(T_\mathbf{n}), \cdots$$
$$\mathbf{U}^{(r)}_{\mathbf{n+p_1}}(T_\mathbf{n}), \cdots, \mathbf{U}^{(r)}_{\mathbf{n+p_{\mathit{k}}}}(T_\mathbf{n})) = \mathbf{0}.$$

  Note: $\mathbf{U_{n+p_{\mathit{s}}}}$ is function of $T_\mathbf{n}$ not of $T_{\mathbf{n+p_{\mathit{s}}}}$.

- Example: Toda lattice

$$\ddot{u}_n = (1 + \dot{u}_n)\,(u_{n-1} - 2u_n + u_{n+1})$$

transforms into

$$c_1^2(1-T_n^2)\left[2T_nU_n' - (1-T_n^2)U_n''\right] + \left[1 + c_1(1-T_n^2)U_n'\right][U_{n-1} - 2U_n + U_{n+1}] = 0.$$

Step D2:

- Seek polynomial solutions

$$U_{i,\mathbf{n}}(T_{\mathbf{n}}) = \sum_{j=0}^{M_i} a_{ij} T_{\mathbf{n}}^j.$$

Use $\tanh(\mathrm{x}+\mathrm{y}) = \dfrac{\tanh\mathrm{x} + \tanh\mathrm{y}}{1 + \tanh\mathrm{x}\tanh\mathrm{y}}$ to deal with the shift:

$$T_{\mathbf{n}+\mathbf{p}_s} = \frac{T_{\mathbf{n}} + \tanh\phi_\mathrm{s}}{1 + T_{\mathbf{n}}\tanh\phi_\mathrm{s}},$$

where

$$\phi_s = \mathbf{p}_s \cdot \mathbf{d} = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q,$$

Substitute $U_{i,\mathbf{n}}(T_{\mathbf{n}}) = T_{\mathbf{n}}^{M_i}$, and

$$U_{i,\mathbf{n}+\mathbf{p}_s}(T_{\mathbf{n}}) = T_{\mathbf{n}+\mathbf{p}_s}^{M_i} = \left[\frac{T_{\mathbf{n}} + \tanh\phi_\mathrm{s}}{1 + T_{\mathbf{n}}\tanh\phi_\mathrm{s}}\right]^{M_i},$$

and balance the highest exponents in $T_{\mathbf{n}}$ to determine M_i.

Note: $U_{i,\mathbf{n}+\mathbf{0}}(T_{\mathbf{n}}) = T_{\mathbf{n}}^{M_i}$ is of degree M_i in $T_{\mathbf{n}}$.

$U_{i,\mathbf{n}+\mathbf{p}_s}(T_{\mathbf{n}}) = \left[\frac{T_{\mathbf{n}} + \tanh\phi_\mathrm{s}}{1 + T_{\mathbf{n}}\tanh\phi_\mathrm{s}}\right]^{M_i}$ is of degree zero in $T_{\mathbf{n}}$.

- Example: Balance of exponents for Toda lattice

$$2M_1 + 1 = M_1 + 2.$$

So, $M_1 = 1$. Hence,

$$U_n(T_n) = a_{10} + a_{11} T_n,$$

$$U_{n\pm1}(T(n\pm1)) = a_{10} + a_{11} T(n\pm1) = a_{10} + a_{11} \frac{T_n \pm \tanh(\mathrm{d}_1)}{1 \pm T_n \tanh(\mathrm{d}_1)}.$$

Step D3:

- Determine the algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the powers in T_n.
- Example: Algebraic system for Toda lattice

$$c_1^2 - \tanh^2(\mathrm{d}_1) - \mathrm{a}_{11}\mathrm{c}_1 \tanh^2(\mathrm{d}_1) = 0,$$
$$c_1 - a_{11} = 0.$$

Step D4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution of algebraic system for Toda lattice

$$a_{10} = \text{free}, \quad a_{11} = \pm \sinh(d_1), \quad c_1 = \pm \sinh(d_1).$$

Step D5:

- Return to the original variables. Test solution(s) of DDE. Reject trivial ones.
- Example: Solitary wave solution for Toda lattice:

$$u_n(t) = a_{10} \pm \sinh(d_1) \tanh\left[\mathrm{d}_1 \mathrm{n} \pm \sinh(\mathrm{d}_1)\, \mathrm{t} + \delta\right].$$

Example of System of DDEs: Relativistic Toda Lattice

$$
\begin{aligned}
\dot{u}_n &= (1+\alpha u_n)(v_n - v_{n-1}), \\
\dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).
\end{aligned}
$$

Change of variables

$$u_n(t) = U_n(T_n), \qquad v_n(t) = V_n(T_n),$$

with

$$T_n(t) = \tanh\left[\mathrm{d_1 n + c_1 t} + \delta\right].$$

gives

$$
\begin{aligned}
& c_1(1-T^2)U_n' - (1+\alpha U_n)(V_n - V_{n-1}) = 0, \\
& c_1(1-T^2)V_n' - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) = 0.
\end{aligned}
$$

Seek polynomial solutions

$$U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, \qquad V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j.$$

Balance the highest exponents in T_n to determine M_1, and M_2 :

$$M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2.$$

So, $M_1 = M_2 = 1$. Hence,

$$U_n = a_{10} + a_{11} T_n, \qquad V_n = a_{20} + a_{21} T_n.$$

Algebraic system for a_{ij} :

$$-a_{11}\, c_1 + a_{21}\ \tanh(\mathrm{d}_1) + \alpha\, \mathrm{a}_{10}\, \mathrm{a}_{21}\ \tanh(\mathrm{d}_1) = 0,$$
$$a_{11}\ \tanh(\mathrm{d}_1)\, (\alpha\, \mathrm{a}_{21} + \mathrm{c}_1) = 0,$$
$$-a_{21}\, c_1 + a_{11}\, a_{20}\ \tanh(\mathrm{d}_1) + 2\alpha\, \mathrm{a}_{20}\, \mathrm{a}_{21}\ \tanh(\mathrm{d}_1) = 0,$$
$$\tanh(\mathrm{d}_1)\, (\mathrm{a}_{11}\, \mathrm{a}_{21} + 2\alpha\, \mathrm{a}_{21}^2 - \mathrm{a}_{11}\, \mathrm{a}_{20}\ \tanh(\mathrm{d}_1)) = 0,$$
$$a_{21}\ \tanh^2(\mathrm{d}_1)\, (\mathrm{c}_1 - \mathrm{a}_{11}) = 0.$$

Solution of the algebraic system

$$\begin{aligned}
a_{10} &= -\frac{1}{\alpha} - c_1 \coth(d_1), \\
a_{11} &= c_1, \\
a_{20} &= \frac{c_1\ \coth(d_1)}{\alpha}, \\
a_{21} &= -\frac{c_1}{\alpha}.
\end{aligned}$$

Solitary wave solution in original variables:

$$\begin{aligned}
u_n(t) &= -\frac{1}{\alpha} - c_1\ \coth(d_1) + c_1 \tanh\left[\mathrm{d}_1\mathrm{n} + \mathrm{c}_1\mathrm{t} + \Delta\right], \\
v_n(t) &= \frac{c_1\ \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh\left[\mathrm{d}_1\mathrm{n} + \mathrm{c}_1\mathrm{t} + \Delta\right].
\end{aligned}$$

Multi-dimensional Example: 2D Toda Lattice

2D Toda lattice:

$$\frac{\partial^2 y_n}{\partial x \partial t} = \exp\left(y_{n-1} - y_n\right) - \exp\left(y_n - y_{n+1}\right),$$

$y_n(x,t)$ is displacement from equilibrium of the n-th unit mass under an exponential decaying interaction force between nearest neighbors.

Set

$$\frac{\partial u_n}{\partial t} = \exp(y_{n-1} - y_n) - 1. \qquad (*)$$

Then,

$$\exp(y_{n-1} - y_n) = \frac{\partial u_n}{\partial t} + 1,$$

and the 2D-Toda lattice becomes

$$\frac{\partial^2 y_n}{\partial x \partial t} = \frac{\partial u_n}{\partial t} + 1 - \left(\frac{\partial u_{n+1}}{\partial t} + 1\right) = \frac{\partial u_n}{\partial t} - \frac{\partial u_{n+1}}{\partial t}.$$

Integrate with respect to t to get

$$\frac{\partial y_n}{\partial x} = u_n - u_{n+1}.$$

Differentiate (*) with respect to x and

$$\begin{aligned}
\frac{\partial^2 u_n}{\partial x \partial t} &= \frac{\partial}{\partial x}\left(\exp(y_{n-1} - y_n) - 1\right) \\
&= \exp(y_{n-1} - y_n)\left(\frac{\partial y_{n-1}}{\partial x} - \frac{\partial y_n}{\partial x}\right), \\
&= \left(\frac{\partial u_n}{\partial t} + 1\right)[(u_{n-1} - u_n) - (u_n - u_{n+1})], \\
&= \left(\frac{\partial u_n}{\partial t} + 1\right)\left(u_{n-1} - 2u_n + u_{n+1}\right).
\end{aligned}$$

So, the 2D Toda lattice is written in polynomial form:

$$\frac{\partial^2 u_n}{\partial x \partial t}(x,t) = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}) .$$

Travelling wave solution:

$$u_n(x,t) = a_{10} + \frac{1}{c_2}\sinh^2(d_1)\tanh\left[\mathrm{d_1 n} + \frac{\sinh^2(\mathrm{d_1})}{\mathrm{c_2}}\mathrm{x} + \mathrm{c_2 t} + \delta\right].$$

Complicated Example: Ablowitz-Ladik Lattice

The Ablowitz-Ladik lattice:

$$\begin{aligned}\dot{u}_n(t) &= (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n,\\ \dot{v}_n(t) &= -(\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n.\end{aligned}$$

Travelling wave solution:

$$\begin{aligned}u_n(t) &= \frac{\alpha\sinh^2(d_1)}{a_{21}}\left(\pm 1 - \tanh\left[\mathrm{d_1 n} + 2\alpha \mathrm{t}\sinh^2(\mathrm{d_1}) + \delta\right]\right),\\ v_n(t) &= a_{21}(\pm 1 + \tanh\left[\mathrm{d_1 n} + 2\alpha\sinh^2(\mathrm{d_1})\mathrm{t} + \delta\right]).\end{aligned}$$

Analyzing and Solving Nonlinear Parameterized Systems

Assumptions:

- All $c_i \neq 0$ and $d_i \neq 0$ (and modulus $m \neq 0$).
- Parameters $(\alpha, \beta, \gamma, ...)$. Otherwise the maximal exponents M_i may change.
- All $M_i \geq 1$.
- All $a_{i\,M_i} \neq 0$. Highest power terms in U_i must be present, except in mixed sech-tanh-method.
- Solve for a_{ij}, then c_i, $\tanh(\mathrm{d_i})$, and m. Then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in a_{ij} first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in c_i, $\tanh(\mathrm{d_i}), \mathrm{m}$ if they are free of a_{ij}.
- Solve linear equations in parameters if they free of a_{ij}, c_i, $\tanh(\mathrm{d_i}), \mathrm{m}$.
- Solve quasi-linear equations for a_{ij}, c_i, $\tanh(\mathrm{d_i}), \mathrm{m}$.
- Solve quadratic equations for a_{ij}, c_i, $\tanh(\mathrm{d_i}), \mathrm{m}$.
- Eliminate cubic terms for a_{ij}, c_i, $\tanh(\mathrm{d_i}), \mathrm{m}$, without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner bases techniques.
- Use Ritt-Wu characteristic sets method.
- Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$.

Implementation Issues – Software Demo – Future Work

- Demonstration of Mathematica package for hyperbolic and elliptic function methods for PDEs and tanh function for DDEs.
- Long term goal: Develop PDESolve and DDESolve for analytical solutions of nonlinear PDEs and DDEs.
- Implement various methods: Lie symmetry methods, etc.
- Look at other types of explicit solutions involving
  - other hyperbolic and elliptic functions $\sinh, \cosh, \text{dn}, \ldots$.
  - complex exponentials combined with sech or $\tanh$.
- Other applications (of the nonlinear algebraic solver):

  Computation of conservation laws, symmetries, first integrals, etc. leading to **linear** parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).

- Preprints:
  D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller, *Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs*, Journal of Symbolic Computation (2003). 39 pages. In press.

  D. Baldwin, Ü. Göktaş, and W. Hereman, *Symbolic computation of exact tanh solutions of nonlinear differential-difference equations*, Computer Physics Communications (2003). Submitted.

  Available from http://www.mines.edu/fs_home/whereman/

- Software:
  D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller, **PDESpecialSolutions.m**: A Mathematica program for the symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for systems of nonlinear partial differential equations (2001-2003).
  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/pde-sols;
  or via Internet URL: http://www.mines.edu/fs_home/whereman/

  D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller, **DDESpecialSolutions.m**: A Mathematica program for the symbolic computation of tanh solutions for systems of nonlinear differential-difference equations (2001).
  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/dde-sols;
  or via Internet URL: http://www.mines.edu/fs_home/whereman/