Using Symmetries to Investigate the Complete Integrability of Nonlinear PDEs and Differential-Difference Equations

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Nonlinear PDEs

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  - Lax pair
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- Making equations scaling invariant
- Using the scaling symmetry (KdV equation)

Nonlinear DDEs

- Scaling symmetry of Kac-van Moerbeke lattice
- Analogy between PDEs and DDEs
- Scaling invariant quantities of Kac-van Moerbeke lattice
- Scaling invariant quantities of the Toda lattice
- Using the scaling symmetry (Toda lattice)

Conclusions

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Scaling symmetry of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

has scaling (dilation) symmetry

$$(x, t, u) \rightarrow (\frac{x}{\kappa}, \frac{t}{\kappa^3}, \kappa^2 u) \; = (\tilde{x}, \tilde{t}, \tilde{u})$$

where κ is an arbitrary parameter.

Replacing (x, t, u) in terms of $(\tilde{x}, \tilde{t}, \tilde{u})$ yields

$$\frac{1}{\kappa^5}\left(\tilde{u}_{\tilde{t}} + 6\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}\right) = 0$$

Weights: $W(x) = -1$, $W(t) = -3$, and $W(u) = 2$.
Equivalently, $W(\mathrm{D}_x) = 1$ and $W(\mathrm{D}_t) = 3$.

Rank is the weight of a monomial, e.g., $\mathrm{rank}(6uu_x) = 5$.

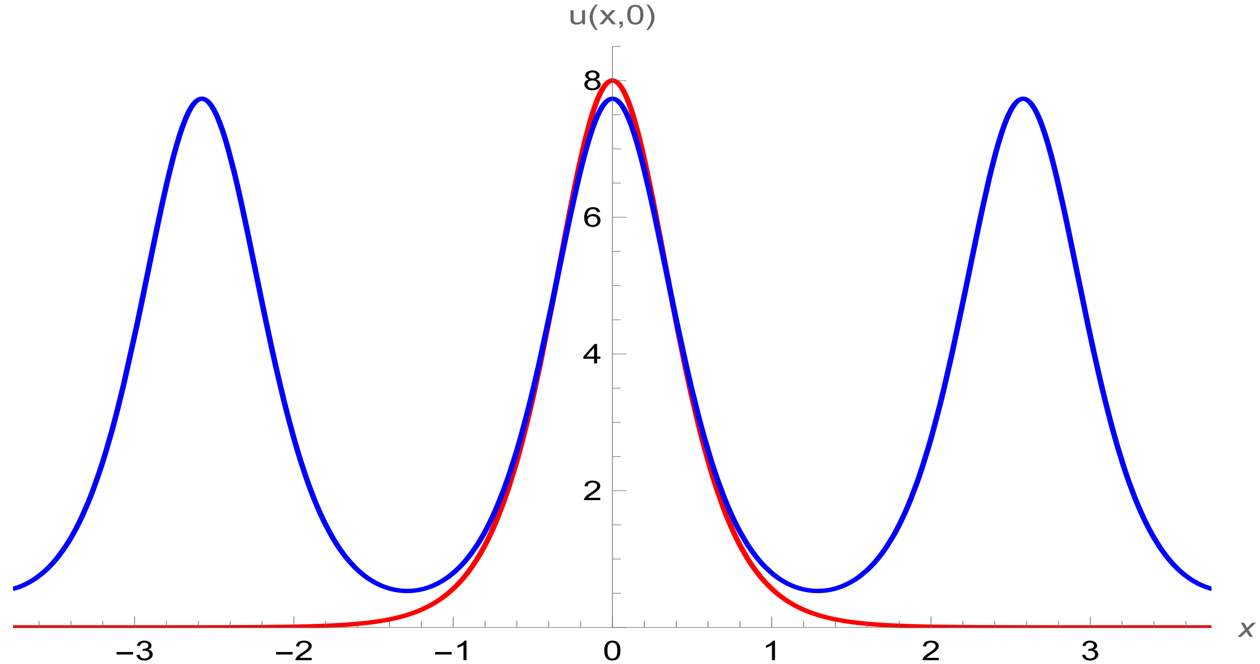
The KdV equation is uniform in rank!

Solitary wave and periodic solutions

$$u(x,t) = 2k^2 \operatorname{sech}^2(kx - 4k^3 t + \delta) \quad \text{and}$$
$$u(x,t) = \tfrac{4}{3}k^2(1-m) + 2k^2 m \operatorname{cn}^2(kx - 4k^3 t + \delta; m)$$

Since $W(x) = -1$, $W(t) = -3$, and $W(u) = 2$ we have $W(k) = 1$. Obviously, $W(\delta) = W(m) = 0$.

Solitary and cnoidal waves for $k = 2$, $m = \frac{9}{10}$, $\delta = 0$.



Solitons using Hirota's method

Substitution of $u(x,t) = 2(\ln f)_{xx}$, which is uniform in rank, into

$$\partial_t\left(\int^x u\,dx\right) + 3u^2 + u_{2x} = 0$$

yields $\quad f(f_{xt} + f_{4x}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} = 0$

which is homogenous of degree and uniform in rank!

Introducing Hirota's bilinear operators

$$D_x^m D_t^n (f\cdot g) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')\Big|_{x'=x,t'=t}$$

one gets $\quad \boxed{\left(D_x D_t + D_x^4\right)(f\cdot f) = 0}$

The bilinear equation is uniform in rank!

Explicitly, $D_x D_t(f\cdot g) = f_{xt}g - f_t g_x - f_x g_t + f g_{xt}$ and

$$D_x^4(f\cdot g) = f_{4x}g - 4f_{3x}g_x + 6f_{xx}g_{xx} - 4f_x g_{3x} + f g_{4x}.$$

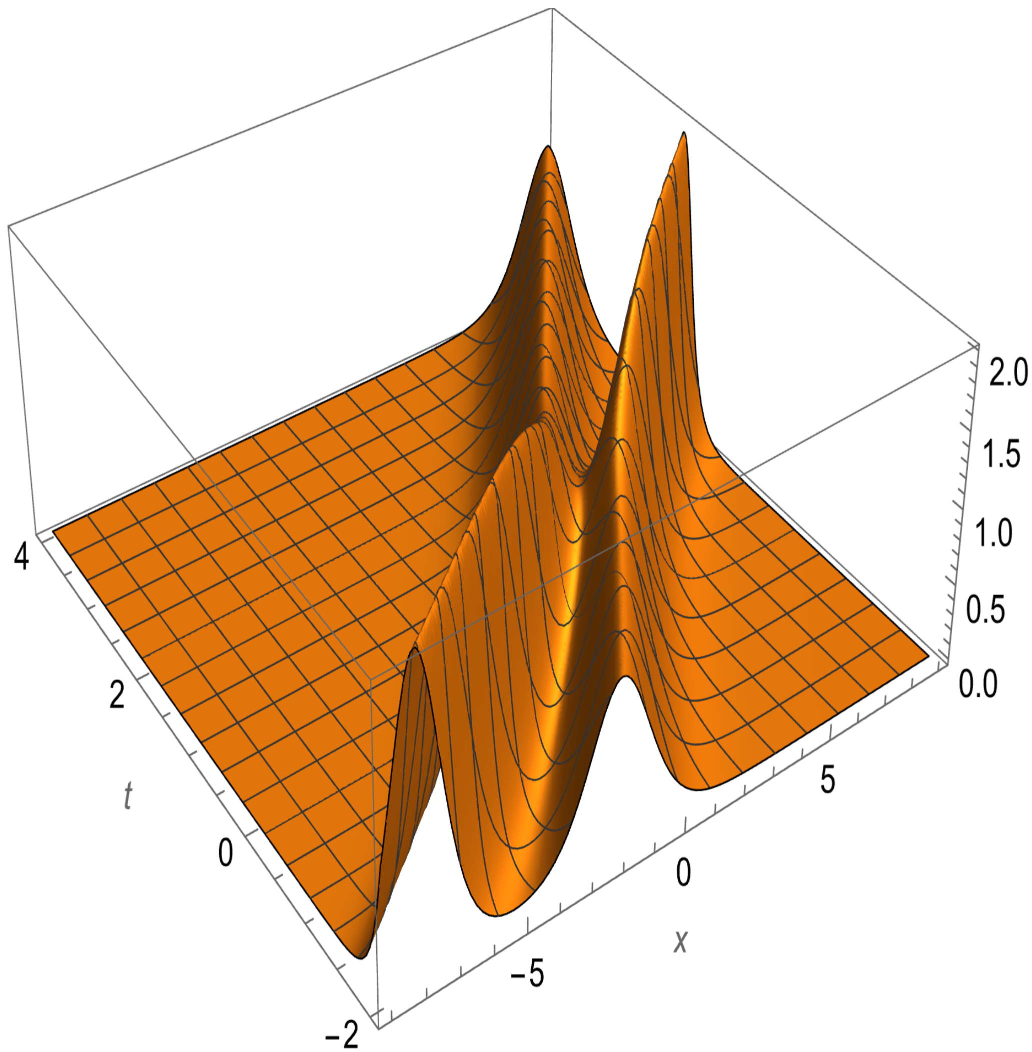
Leibniz rule for derivatives of products with every other sign flipped.

Two-soliton solution of the KdV equation

Using $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2}$, with $\theta_i = k_i x - k_i^3 t + \delta_i$ and $a_{12} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$,

$$u(x,t) = \frac{2\left[k_1^2\mathrm{e}^{\theta_1} + k_2^2\mathrm{e}^{\theta_2} + 2(k_1-k_2)^2\mathrm{e}^{\theta_1+\theta_2} + a_{12}(k_2^2\mathrm{e}^{\theta_1} + k_1^2\mathrm{e}^{\theta_2})\mathrm{e}^{\theta_1+\theta_2}\right]}{\left(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}\right)^2}$$

Bird's eye view of a 2-soliton collision for the KdV equation; $k_1 = 2$, $k_2 = \frac{3}{2}$, $\delta_1 = \delta_2 = 0$.



Lie point symmetries of the KdV equation

Making new solutions

If $u = f(x,t)$ is a solution of the KdV equation, so are

$$u = f(x - \varepsilon, t) \qquad space\ translation$$

$$u = f(x, t - \varepsilon) \qquad time\ translation$$

$$u = f(x - \varepsilon t, t) + \varepsilon \qquad Galilean\ boost$$

$$u = \frac{1}{\kappa^2} f(\frac{x}{\kappa}, \frac{t}{\kappa^3}) \qquad scaling\ (dilation)$$

Integrability of the KdV equation

Defining equation of conservation law: $\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J \doteq 0}$

$$\mathrm{D}_t \Big(u \Big) + \mathrm{D}_x \Big(3u^2 + u_{xx} \Big) \doteq 0$$

$$\mathrm{D}_t \Big(u^2 \Big) + \mathrm{D}_x \Big(4u^3 - {u_x}^2 + 2uu_{xx} \Big) \doteq 0$$

$$\mathrm{D}_t \Big(u^3 - \frac{1}{2} u_x^2 \Big) + \mathrm{D}_x \Big(\tfrac{9}{2} u^4 - 6uu_x^2 + \ldots - u_x u_{xxx} \Big) \doteq 0$$

$$\mathrm{D}_t \Big(u^4 - 2uu_x^2 + \tfrac{1}{5} u_{xx}^2 \Big) + \mathrm{D}_x \Big(\tfrac{24}{5} u^5 - 18uu_x^2 + 4u^3 u_{xx}$$

$$+ 2u_x^2 u_{xx} + \tfrac{16}{5} uu_{xx}^2 - 4uu_x u_{xxx} - \tfrac{1}{5} u_{xxx}^2 + \tfrac{2}{5} u_{xx} u_{4x} \Big) \doteq 0$$

Conservation laws are uniform in rank!

$$\mathrm{rank}(\rho) + \mathrm{rank}(\mathrm{D}_t) = \mathrm{rank}(J) + \mathrm{rank}(\mathrm{D}_x)$$

Single evolution equation $u_t = F$ or system $\mathbf{u}_t = \mathbf{F}$.

Defining equation of a generalized symmetry:

$$\boxed{\mathrm{D}_t G \;\doteq\; F'(u)[G]}$$

where $F(u)'[G]$ is the Fréchet derivative of $F(u)$ in the direction of G:

$$F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0} = \sum_k \left(\mathrm{D}_x^k G\right) \frac{\partial F}{\partial u_{kx}}$$

In practice, introduce $u_\tau = G$. Compatibility with $u_t = F$ yields $\mathrm{D}_\tau F \doteq \mathrm{D}_t G$.

Eliminate all $u_\tau, u_{\tau x}, \ldots$ and $u_t, u_{tx}, \ldots$.

Generalized symmetries of the KdV equation:

$$
\begin{aligned}
G^{(1)} &= u_x \\
G^{(2)} &= 6uu_x + u_{xxx} \\
G^{(3)} &= 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x} \\
G^{(4)} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{xxx} \\
&\quad +70u_{xx}u_{xxx} + 42u_xu_{4x} + 14uu_{5x} + u_{7x}
\end{aligned}
$$

Generalized symmetries are uniform in rank!

Recursion operator connects generalized symmetries:

$$\boxed{G^{(j+1)} = \mathcal{R} G^{(j)}} \quad (j = 1, 2, \ldots)$$

Recursion operator for the KdV equation:

$$\mathcal{R} = \mathrm{D}_x^2 + 2u\mathrm{I} + 2\mathrm{D}_x u \mathrm{D}_x^{-1} = \mathrm{D}_x^2 + 4u\mathrm{I} + 2u_x \mathrm{D}_x^{-1}$$

Recursion operator is uniform in rank!

Sequential symmetries of the KdV equation:

$$\mathcal{R} u_x = (\mathrm{D}_x^2 + 2u\mathrm{I} + 2\mathrm{D}_x u \mathrm{D}_x^{-1}) u_x = 6uu_x + u_{xxx}$$

$$\mathcal{R}(6uu_x + u_{xxx}) = (\mathrm{D}_x^2 + 2u\mathrm{I} + 2\mathrm{D}_x u \mathrm{D}_x^{-1})(6uu_x + u_{xxx})$$

$$= 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x}$$

Defining equation for the recursion operator:

$$\boxed{\mathrm{D}_t \mathcal{R} + [\mathcal{R}, F'(u)] \ \doteq \ 0}$$

Explicitly,

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[F] + \mathcal{R} \circ F'(u) - F'(u) \circ \mathcal{R} \ \doteq \ 0$$

where $[\ ,\ ]$ is the commutator, $\circ$ is composition, and $\mathcal{R}'[F]$ is the Fréchet derivative of $\mathcal{R}$ in the direction of F.

Explicitly,

$$F'(u) = \sum_k \frac{\partial F}{\partial u_{nk}} \mathrm{D}_x^k$$

$$\mathcal{R}'[F] = \sum_k (\mathrm{D}_x^k F) \ \frac{\partial \mathcal{R}}{\partial u_{nk}}$$

Lax pair: Replace the PDE with a compatible linear system: $\mathcal{L}\psi = \lambda\psi, \;\; \mathrm{D}_t\psi = \mathcal{M}\psi$

$\mathcal{L}$ and $\mathcal{M}$ are differential operators; ψ is eigenfunction; λ is constant eigenvalue $(\lambda_t = 0)$ (isospectral).

Defining equation of a Lax pair: $\boxed{\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}}$

with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

Lax pair for the KdV equation:

$$\mathcal{L} = \mathrm{D}_x^2 + u\,\mathbf{I}$$

$$\mathcal{M} = -\left(4\,\mathrm{D}_x^3 + 6u\mathrm{D}_x + 3u_x\,\mathbf{I}\right)$$

Lax pair is uniform in rank!

Scaling symmetry of the mKdV equation

$$u_t + \alpha u^2 u_x + u_{xxx} = 0$$

is invariant under the transformation

$$(x, t, u) \rightarrow (\frac{x}{\kappa}, \frac{t}{\kappa^3}, \kappa\, u)$$

Thus, $W(u) = W(\mathsf{D}_x) = 1$ and $W(\mathsf{D}_t) = 3$.

Lax pair of the mKdV equation:

$$\mathcal{L} = \mathsf{D}_x^2 + 2u\,\mathsf{D}_x + \Big(u^2 + u_x\Big)\,\mathrm{I}$$

$$\mathcal{M} = \Big(3\alpha u^3 + u_{xx}\Big)\,\mathrm{I}$$

The Lax pair is uniform in rank!

General Lax pair for the mKdV equation:

$$\mathcal{L} = \mathsf{D}_x^2 + 2\epsilon u \mathsf{D}_x + \frac{1}{6}\left(\left(6\epsilon^2 + \alpha\right) u^2 + \left(6\epsilon \pm \sqrt{-6\alpha}\right) u_x\right) \mathrm{I}$$

$$\begin{aligned}\mathcal{M} = &-4\mathsf{D}_x^3 - 12\epsilon u \mathsf{D}_x^2 \\ &- \left(\left(12\epsilon^2 + \alpha\right) u^2 + \left(12\epsilon \pm \sqrt{-6\alpha}\right) u_x\right) \mathsf{D}_x \\ &- \Big((4\epsilon^3 + \tfrac{2}{3}\epsilon\alpha)u^3 + (12\epsilon^2 \pm \epsilon\sqrt{-6\alpha} + \alpha)u u_x \\ &+ (3\epsilon \pm \tfrac{1}{2}\sqrt{-6\alpha})u_{xx}\Big) \mathrm{I}\end{aligned}$$

where ϵ is arbitrary parameter (M. Wadati, J. Phys. Soc. Jpn., 1972-1973).

The Lax pair is uniform in rank!

Making PDEs scaling invariant

The Boussinesq equation

$$u_{tt} - u_{xx} + 3u_x^2 + 3uu_{xx} + \alpha u_{4x} = 0$$

is not scaling invariant.

Introduce an auxiliary parameter β (with weight)

$$u_{tt} - \beta\, u_{xx} + 3u_x^2 + 3uu_{xx} + \alpha u_{4x} = 0.$$

Then $(x, t, u, \beta) \rightarrow (\frac{x}{\kappa}, \frac{t}{\kappa^2}, \kappa^2\, u, \kappa^2\, \beta)$. Thus,
$W(\mathsf{D}_x) = 1$, $W(\mathsf{D}_t) = 2$, $W(u) = 2$, and $W(\beta) = 2$.

After the computations are done set β equal to one.

The short pulse equation

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx}$$

is not scaling invariant. But

$$u_{xt} = \beta\, u + (u^3)_{xx} = \beta\, u + 6uu_x^2 + 3u^2u_{xx}$$

is uniform in rank with weights
$W(\mathsf{D}_x) = 1$, $W(\mathsf{D}_t) = 3$, $W(u) = 1$, and $W(\beta) = 4$.

The (non-polynomial) conservation law

$$\mathrm{D}_t \left(\sqrt{\beta + 6u_x^2}\right) - \mathrm{D}_x \left(3u^2\sqrt{\beta + 6u_x^2}\right) = 0$$

is uniform in rank!

After the computations are done set β equal to one.

Discrete symmetries

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is invariant under $(x, t, u) \to (-x, -t, u)$.

The mKdV equation

$$u_t + \alpha u^2 u_x + u_{xxx} = 0$$

is invariant under $(x, t, u) \to (x, t, -u)$ and $(x, t, u) \to (-x, -t, u)$.

Using the scaling symmetry

Compute recursion operator for KdV equation

$$\mathcal{R} = \mathrm{D}_x^2 + 4u\mathrm{I} + 2u_x\mathrm{D}_x^{-1}$$

Structure of the recursion operator: $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$.

- Rank: $R \equiv \mathrm{r}ank(\mathcal{R}) = \mathrm{r}ank(G^{(2)}) - \mathrm{r}ank(G^{(1)})$.
- Differential operator $\mathcal{R}_0$ is scaling invariant.
- Integral operator $\mathcal{R}_1$ is scaling invariant.
- $\mathcal{R}_1 = \sum_j \sum_k \mathbf{G}^{(j)} D_x^{-1} \otimes \mathcal{L}_{\mathbf{u}}(\rho^{(k)})$ where $\otimes$ is outer product and $\mathcal{L}_{\mathbf{u}}$ is Euler operator $\mathcal{L}_{\mathbf{u}} = \sum_i (-1)^i D_x^i \frac{\partial}{\partial \mathbf{u}_{ix}}$.
- Indices j and k are taken such that $\mathrm{r}ank(G^{(j)}) + \mathrm{r}ank(\rho^{(k)}) - 2 = R$.

- Compute the first few conserved densities:
  $\rho^{(1)} = u, \;\; \rho^{(2)} = u^2, \ldots .$

- Compute the first few generalized symmetries:
  $G^{(1)} = u_x, \;\; G^{(2)} = 6uu_x + u_{xxx}, \ldots .$

- Compute $R = \mathrm{r}ank(G^{(2)}) - \mathrm{r}ank(G^{(1)}) = 2$.

- Build $\mathcal{R}_0 = c_1 \mathrm{D}_x^2 + c_2 u \mathrm{I}$.

- Build $\mathcal{R}_1 = c_3 u_x \mathrm{D}_x^{-1}$.

- Substitute $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ into the determining equation and find the c_i.

- Use $G^{(j+1)} = \mathcal{R}\, G^{(j)}$ to verify $\mathcal{R}$.

Result: $\mathcal{R} = \mathrm{D}_x^2 + 4u\mathrm{I} + 2u_x \mathrm{D}_x^{-1}$

Nonlinear Differential-Difference Equations

Scaling symmetry of Kac-van Moerbeke lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1})$$

is invariant under the scaling symmetry

$$(t, u_n) \rightarrow (\frac{t}{\kappa}, \kappa\, u_n).$$

Thus, $W(\mathrm{D}_t) = 1$ and $W(u_n) = 1$.

Analogy between PDEs and DDEs

	PDEs	DDEs
System	$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \ldots)$	$\dot{\mathbf{u}}_n = \mathbf{F}(\ldots, \mathbf{u}_n, \ldots)$
Cons. Law	$\mathrm{D}_t \rho + \mathrm{D}_x J \doteq 0$	$\dot{\rho} + \Delta J \doteq 0$
Symmetry	$\mathrm{D}_t \mathbf{G} \doteq \mathbf{F}'(\mathbf{u})[\mathbf{G}]$	$\mathrm{D}_t \mathbf{G} \doteq \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$
Rec. oper.	$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(u)] \doteq 0$	$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] \doteq 0$

Typical Examples

	PDEs	DDEs
Equation	KdV equation	KvM lattice
	$u_t = 6uu_x + u_{xxx}$	$\dot{u}_n = u_n \left(u_{n+1} - u_{n-1} \right)$
Densities	$\rho = u$ $\rho = u^2$ $\rho = u^3 - \frac{1}{2} u_x^2$	$\rho = u_n$ $\rho = u_n (\frac{1}{2} u_n + u_{n+1})$ $\rho = \frac{1}{3} u_n^3 +$ $u_n u_{n+1} (u_n + u_{n+1} + u_{n+2})$

The KvM lattice has a conserved density $\rho = \ln(u_n)$ of rank zero with flux $J = -(u_n + u_{n-1})$ of rank 1.

Typical Examples – continued

Equation	KdV equation	KvM lattice
	$u_t = 6uu_x + u_{xxx}$	$\dot{u}_n = u_n(u_{n+1} - u_{n-1})$
Symm.	$G = u_x$ $G = 6uu_x + u_{xxx}$ $G = 30u^2u_x + 20u_xu_{xx}$ $+10uu_{xxx} + u_{5x}$	$G = u_n(u_{n+1} - u_{n-1})$ $G = u_n\Big(u_{n+1}(u_n +$ $u_{n+1} + u_{n+2}) - u_{n-1}$ $(u_{n-2} + u_{n-1} + u_n)\Big)$
Rec. oper.	$\mathcal{R} = \mathrm{D}_x^2 + 4u + 2u_x\mathrm{D}_x^{-1}$	$\mathcal{R} = u_n(\mathrm{D} + \mathrm{I})(u_n\mathrm{D}$ $-\mathrm{D}^{-1}u_n)(\mathrm{D} - \mathrm{I})^{-1}\frac{1}{u_n}$

Scaling symmetry of Toda lattice

$$\ddot{y}_n = \exp\left(y_{n-1} - y_n\right) - \exp\left(y_n - y_{n+1}\right)$$

y_n is displacement from equilibrium of nth particle with unit mass under an exponentially decaying interaction force between nearest neighbors.

Change of variables: $u_n = \dot{y}_n, \quad v_n = \exp\left(y_n - y_{n+1}\right)$

yields $\boxed{\dot{u}_n = v_{n-1} - v_n, \;\; \dot{v}_n = v_n(u_n - u_{n+1})}$

which is invariant under the scaling symmetry

$$(t, u_n, v_n) \rightarrow (\frac{t}{\kappa}, \kappa\, u_n, \kappa^2\, v_n).$$

Hence, $W(\mathrm{D}_t) = 1, \;\; W(u_n) = 1, \;\; w(v_n) = 2.$

Scaling invariant quantities for Toda lattice

- First three density-flux pairs:

$$\rho^{(0)} = \ln(v_n) \qquad J^{(0)} = u_n$$

$$\rho^{(1)} = u_n \qquad J^{(1)} = v_{n-1}$$

$$\rho^{(2)} = \frac{1}{2}u_n^2 + v_n \qquad J^{(2)} = u_n v_{n-1}$$

- First three generalized symmetries:

$$\mathbf{G}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{G}^{(2)} = \begin{pmatrix} v_n - v_{n-1} \\ v_n(u_n - u_{n+1}) \end{pmatrix}$$

$$\mathbf{G}^{(3)} = \begin{pmatrix} v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n) \\ v_n(u_{n+1}^2 - u_n^2 + v_{n+1} - v_{n-1}) \end{pmatrix}$$

• Recursion operator:

$$\mathcal{R} = \begin{pmatrix} -u_n \mathrm{I} & -\mathrm{D}^{-1} - \mathrm{I} + (v_{n-1} - v_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ -v_n \mathrm{I} - v_n \mathrm{D} & u_{n+1} \mathrm{I} + v_n (u_n - u_{n+1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \end{pmatrix}$$

For r-component systems, $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ is an $r \times r$ matrix.

Defining equation for $\mathcal{R}$:

$$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] = \frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[\mathbf{F}] + \mathcal{R} \circ \mathbf{F}'(\mathbf{u}_n) - \mathbf{F}'(\mathbf{u}_n) \circ \mathcal{R} \doteq 0$$

where $[\ ,\ ]$ is the commutator, $\circ$ is composition, and $\mathbf{F}'(\mathbf{u}_n)$ is the Fréchet derivative operator.

For a system with two components, u_n and v_n,

$$\mathbf{F}'(\mathbf{u}_n) = \begin{pmatrix} \sum_k \frac{\partial F_1}{\partial u_{n+k}} \mathrm{D}^k & \sum_k \frac{\partial F_1}{\partial v_{n+k}} \mathrm{D}^k \\ \\ \sum_k \frac{\partial F_2}{\partial u_{n+k}} \mathrm{D}^k & \sum_k \frac{\partial F_2}{\partial v_{n+k}} \mathrm{D}^k \end{pmatrix}.$$

Applied to $\mathbf{G} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$,

$$F_i{}'(\mathbf{u}_n)[\mathbf{G}] = \sum_k \frac{\partial F_i}{\partial u_{n+k}} \mathrm{D}^k G_1 + \sum_k \frac{\partial F_i}{\partial v_{n+k}} \mathrm{D}^k G_2$$

with $i = 1, 2.$

Furthermore,

$$\mathcal{R}'[\mathbf{F}] = \sum_k (\mathrm{D}^k \mathbf{F}) \, \frac{\partial \mathcal{R}}{\partial \mathbf{u}_{n+k}}$$

For lattice systems matrix $\mathcal{R}_1$ is of the form

$$\sum_j \sum_k \mathbf{G}^{(j)} (\mathrm{D} - \mathrm{I})^{-1} \otimes \mathcal{L}_{\mathbf{u}_n}(\rho^{(k)})$$

where $\otimes$ is outer product, and $\mathcal{L}$ is the discrete Euler operator.

Explicitly, for systems with u_n and v_n

$$\mathcal{L}_{u_n}(\rho^{(k)}) = \sum_i D^{-i} \frac{\partial \rho^{(k)}}{\partial u_{n+i}}$$

Similar formula for $\mathcal{L}_{v_n}(\rho^{(k)})$.

Recursion operator for Toda lattice

$$\mathcal{R} = \begin{pmatrix} -u_n \mathrm{I} & -\mathrm{D}^{-1} - \mathrm{I} + (v_{n-1} - v_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ -v_n \mathrm{I} - v_n \mathrm{D} & u_{n+1} \mathrm{I} + v_n (u_n - u_{n+1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \end{pmatrix}$$

The recursion operator can be factored as $\mathcal{R} = \mathcal{H}\mathcal{S}$ with Hamiltonian (symplectic) operator

$$\mathcal{H} = \begin{pmatrix} \mathrm{D}^{-1} v_n \mathrm{I} - v_n \mathrm{D} & -u_n v_n \mathrm{I} + u_n \mathrm{D}^{-1} v_n \mathrm{I} \\ -v_n \mathrm{D} u_n \mathrm{I} + u_n v_n \mathrm{I} & -v_n \mathrm{D} v_n \mathrm{I} + v_n \mathrm{D}^{-1} v_n \mathrm{I} \end{pmatrix}$$

and co-symplectic operator

$$\mathcal{S} = \begin{pmatrix} 0 & (\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ \frac{1}{v_n} \mathrm{D} (\mathrm{D} - \mathrm{I})^{-1} & 0 \end{pmatrix}$$

Conclusions

- Conservation laws, generalized symmetries, recursion operators, and Lax pairs inherit the scaling symmetry (and other Lie symmetries).
- Solutions of PDEs and DDEs have the scaling and discrete symmetries of the equations.
- Use the method of undetermined coefficients to construct scaling invariant quantities.
- Use tools of the calculus of variations and differential geometry (Fréchet derivatives, Euler and homotopy operators).
- Implementation in *Maple* or *Mathematica* lead to software the automate the computations.

Thank You