Symmetry – A Ubiquitous Concept in Nature and Science

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Martin Gardner: The New Ambidextrous Universe – Symmetry and Asymmetry

Keshlan Govinder: A Symmetric View of Life

Stephen Strogatz: Sync: How Order Emerges from Chaos in the Universe, Nature, and Daily Life

Goals and Outline

- What is symmetry?
- Nature and architecture
- Design and art
- Music and literature
- Types of symmetries
- Symmetry in mathematical terms
- Group theory
- Symmetry in physics
- Symmetry in mathematics
- Symmetry in my research:
    - Lie point symmetries
    - Nonlinear waves and solitons

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- Symmetry preserves distances, angles, sizes, shapes, etc.
- Symmetry creates stability, harmony, and order.
- Symmetry is associated with beauty and visually pleasing or pleasant to listen to.

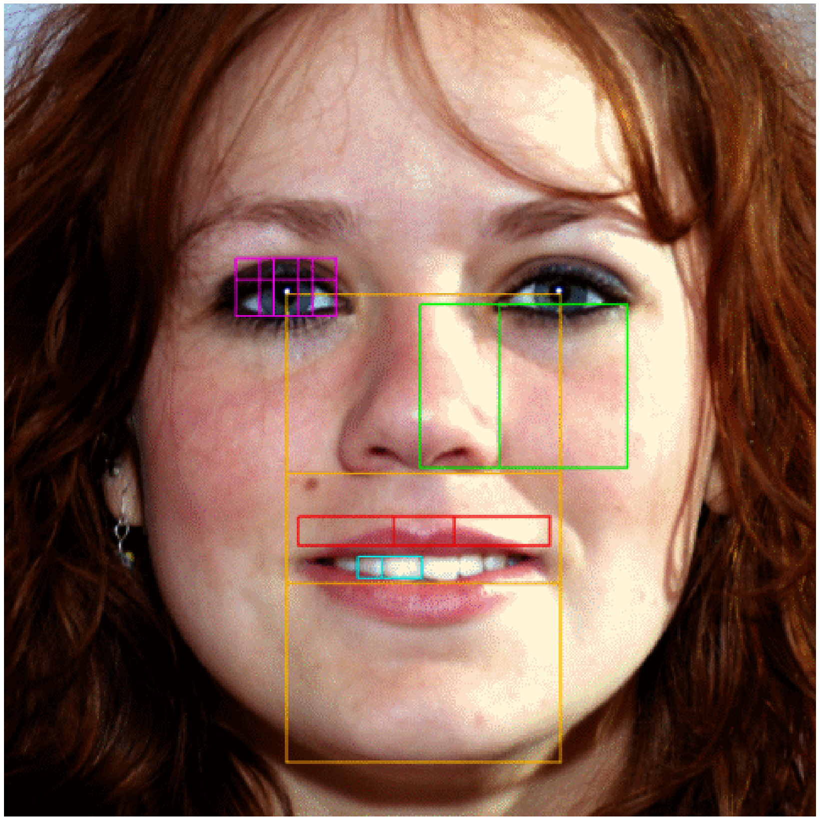
Nature

Nature

Human Face

Symmetry and divine proportions

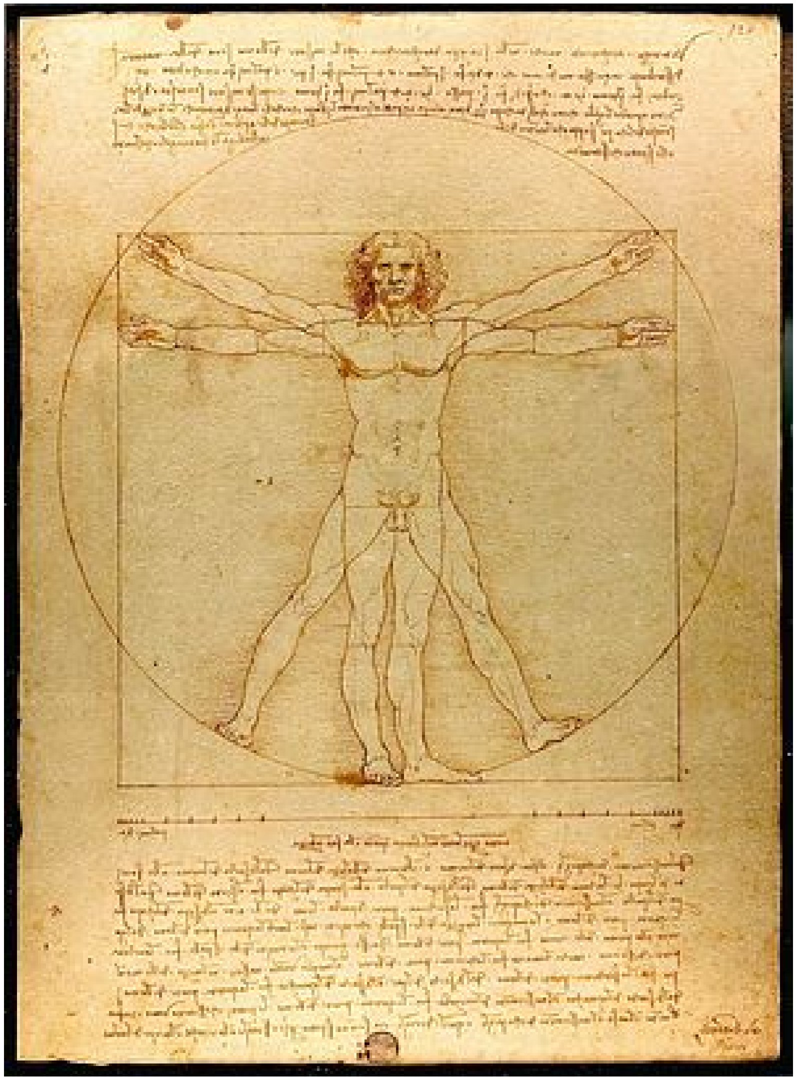
Golden Ratio: $\frac{1+\sqrt{5}}{2} \approx 1.618$



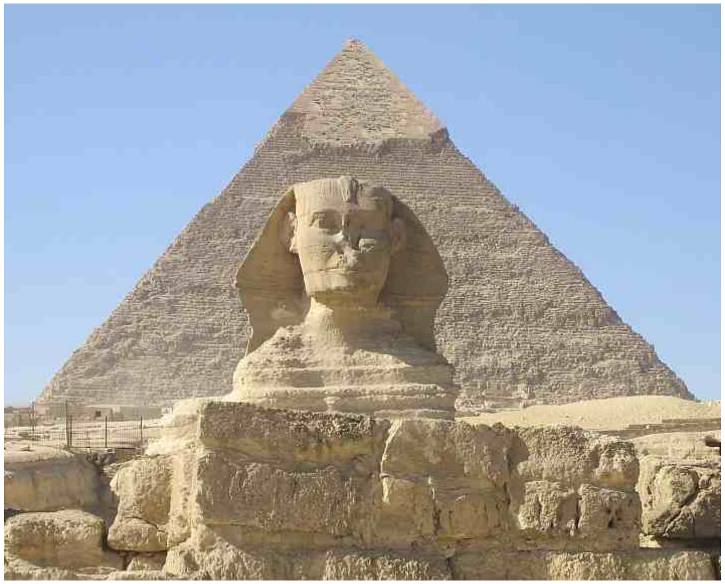
- A "beautiful" person's face is about 1.6 times longer than it is wide.
- The distance from the top of the nose to the center of the lips should be around 1.6 times the distance from the center of the lips to the chin.

Human Body

Vitruvian Man – Leonardo da Vinci (c. 1490)



Architecture

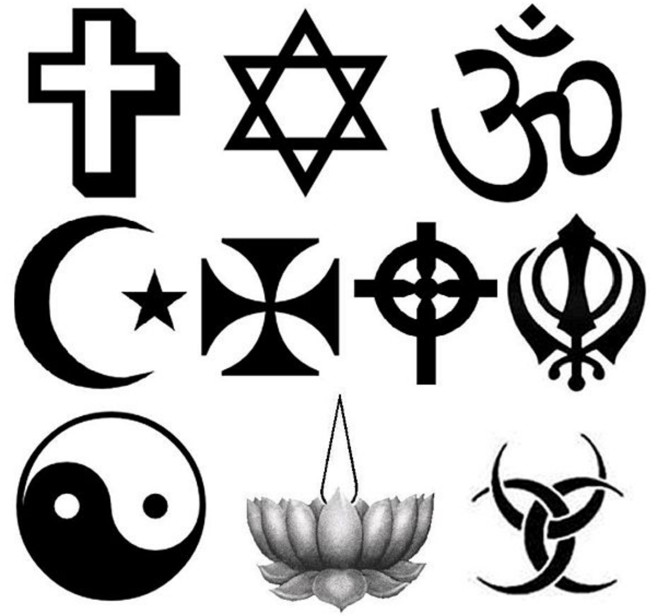


Architecture

Architecture: From Symmetry to Loss of Symmetry

Symmetric or Not?

Symmetric or Not?



Design and Art

Artwork of Maurits Cornelis Escher

Symmetries in Tiles in Alhambra

Music

Fugues – Bach: Play fugue.mp3

FIGURE 2. *Intervals in the well-tempered system.*

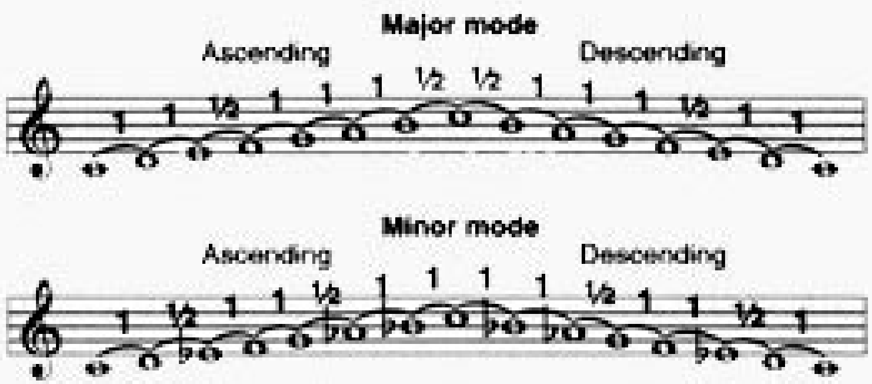
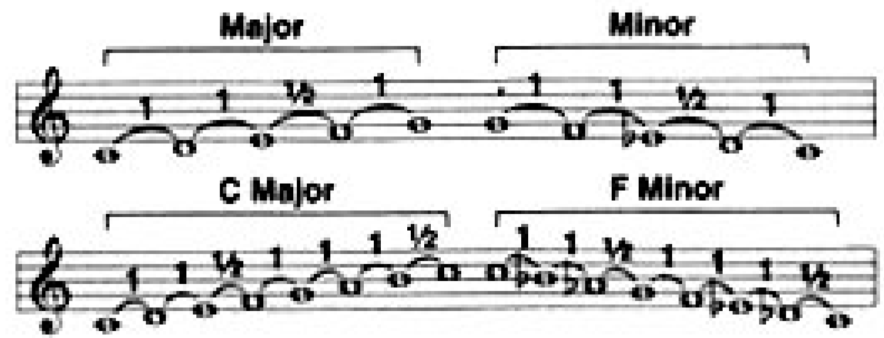


FIGURE 4. *Inversion of the major mode generates a minor mode of a different key.*



Arch form – ABCBA – Béla Bartók.

Scales/chords – Mozart

Literature

Palindromes: Words and Sentences

dud, civic, madam

A man, a plan, a canal: Panama

Girl, bathing on Bikini, eyeing boy, finds boy eying bikini on bathing girl.

Palindrome Story – 2002 words long; published on Web on February 20th, 2002 or 20-02-2002.

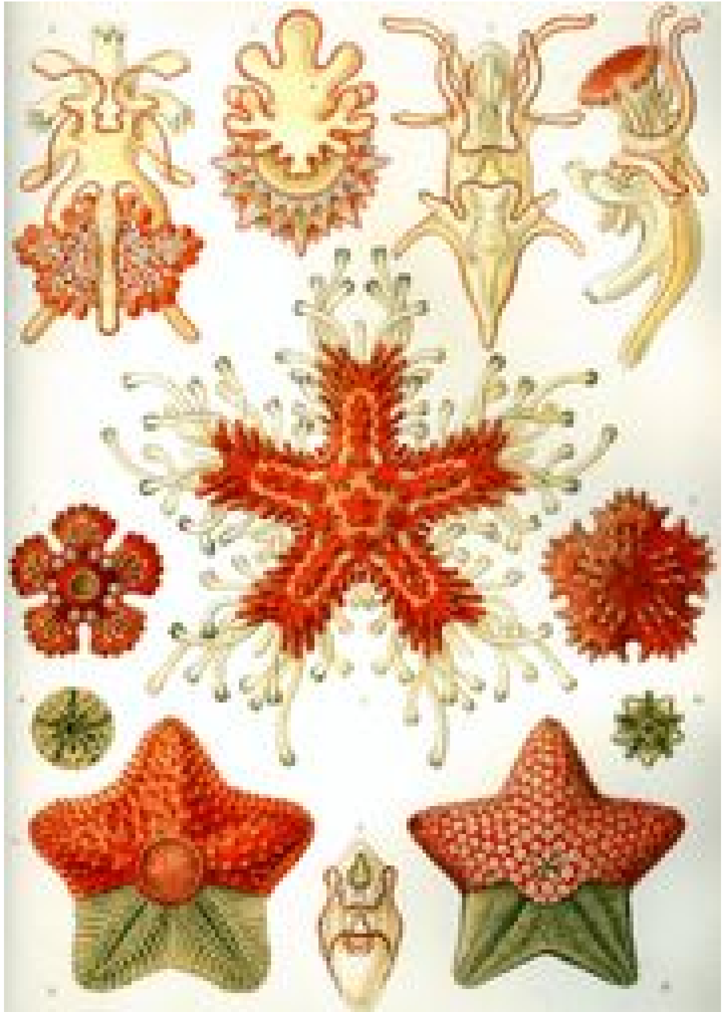
Longest sentence: 17,826 words!

Y chromosome: 6 million of its 50 million DNA letters form palindromic sequences (with A, T, C and G).

Types of Symmetries

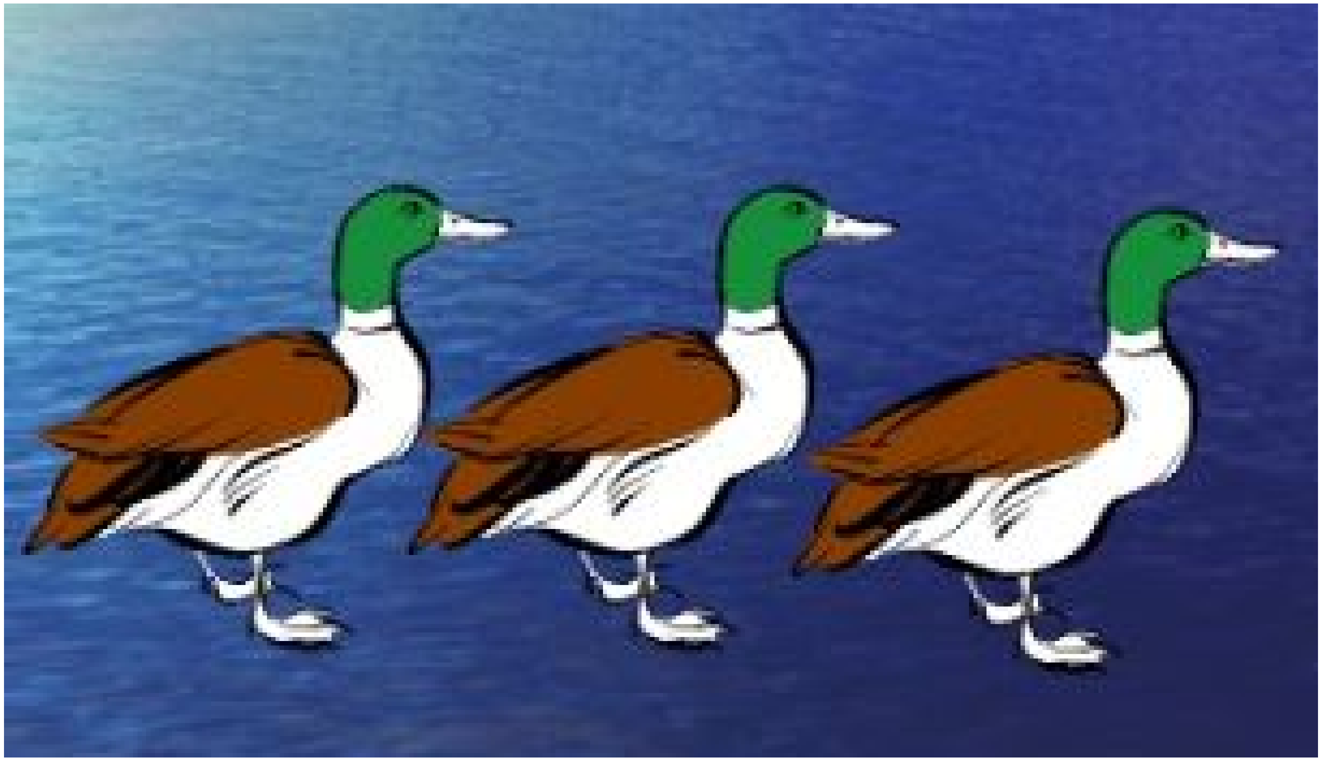
Reflection/Bilateral/Mirror Symmetry

Radial/Rotational Symmetry



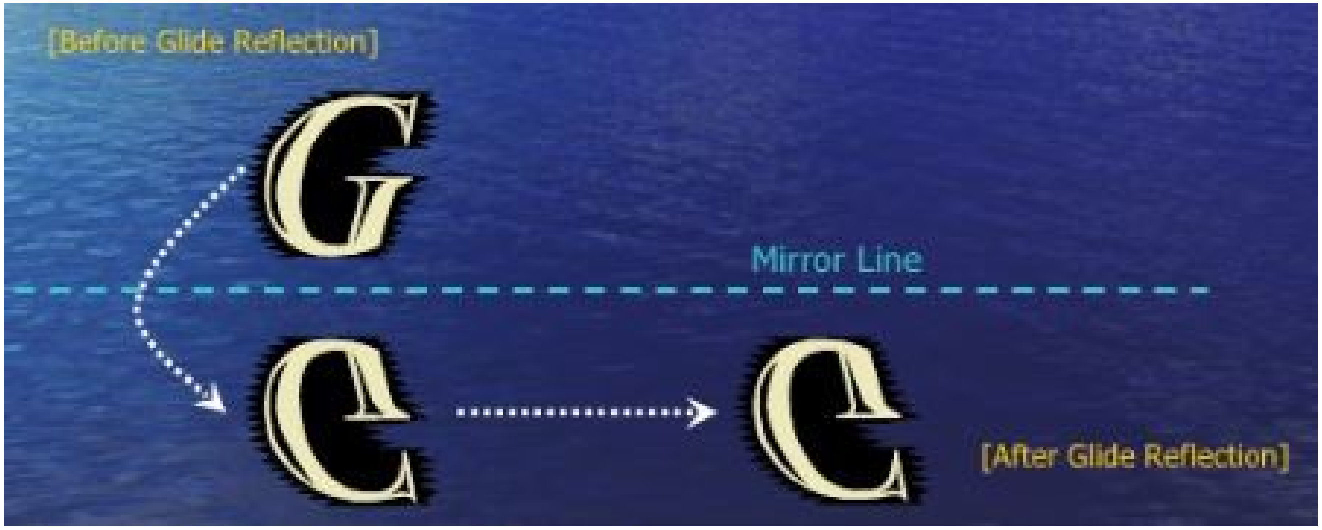
Radial/Rotational Symmetry

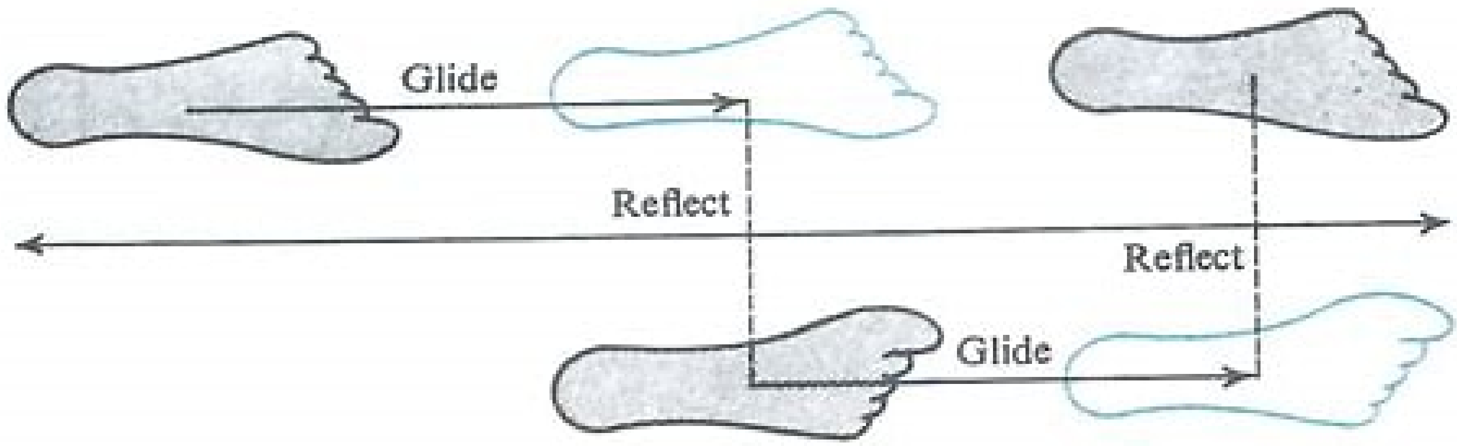
Translational Symmetry



Glide Reflection Symmetry

Combination of a reflection followed by a translation.

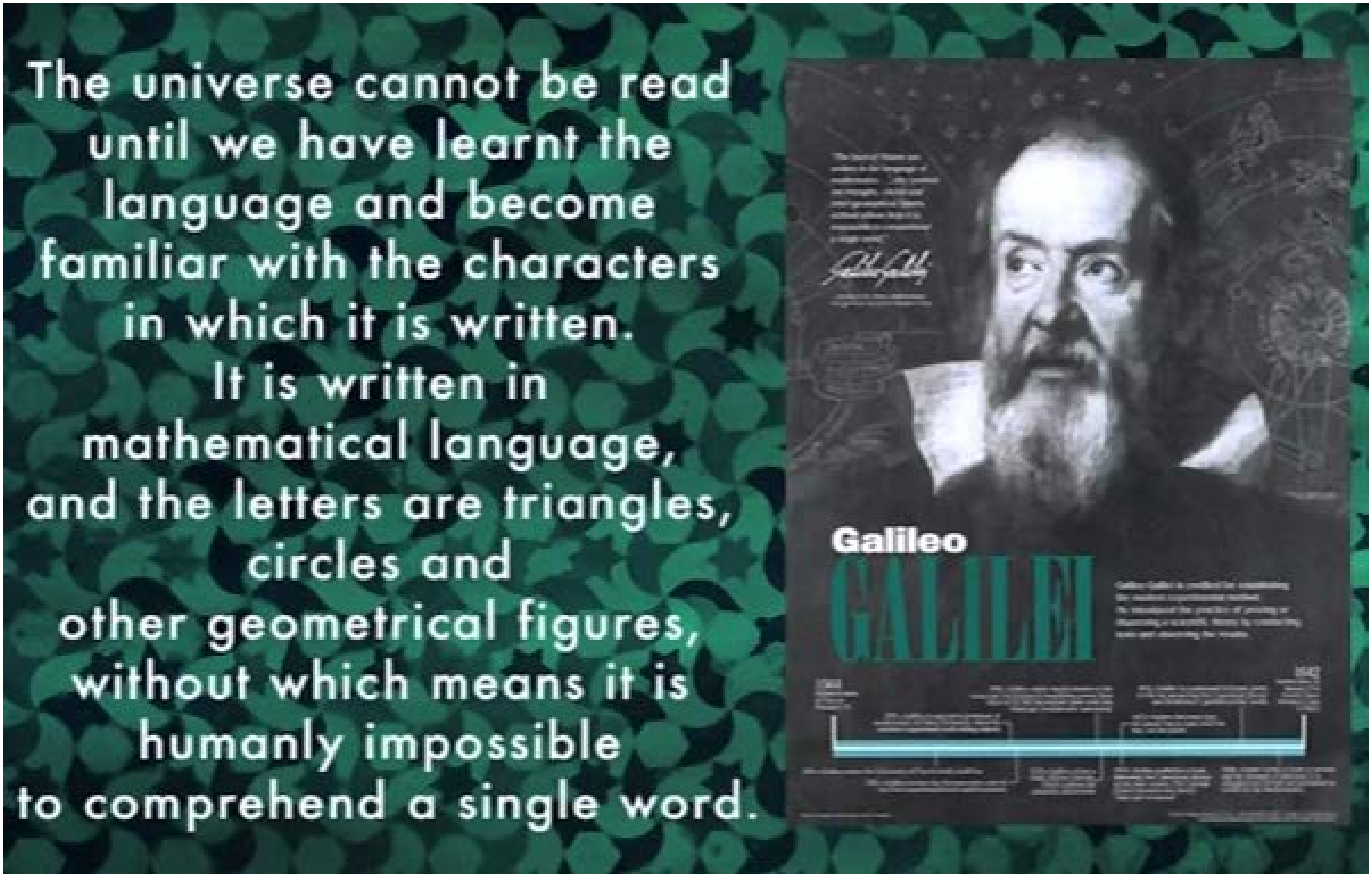




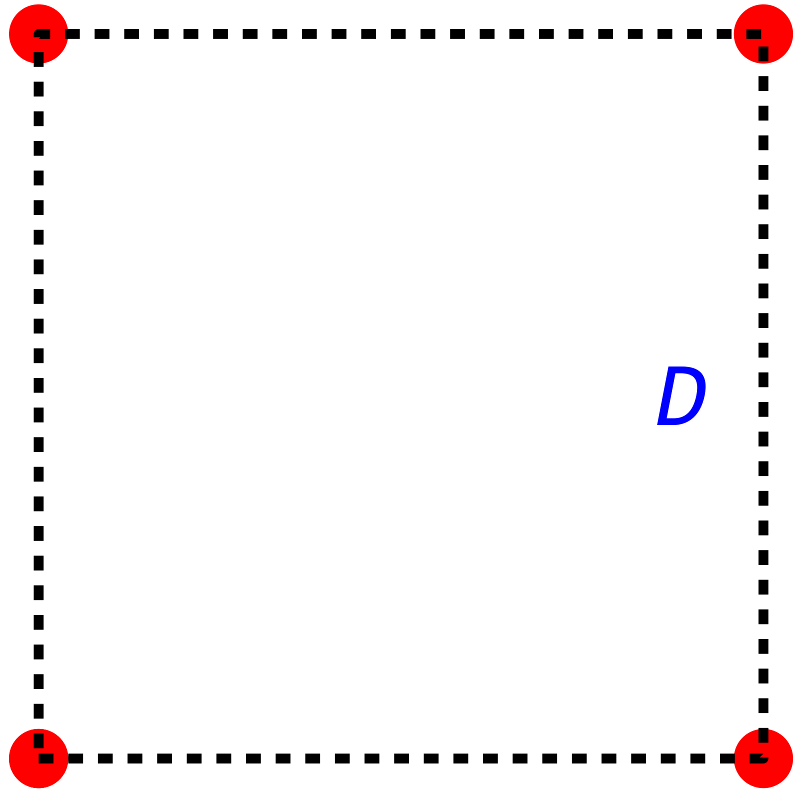
Wallpaper by William Morris (1834-1896)

What is Symmetry?

Galileo Galilei (1564–1642)



Symmetries of the Square



8 symmetries: Dihedral group $\mathbf{D}_4$

3 rotations $+$ identity: Cyclic group $\mathbf{Z}_4$

Group Theory

The official language of all symmetries

- Composition of symmetries gives a symmetry (closed under composition denoted by $\star$).

  For transformations, $\star$ means "followed by".

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- Inverse symmetry (e.g., inverse rotation, inverse translation).

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- Composition of symmetries gives a symmetry (closed under composition denoted by $\star$).

  For transformations, $\star$ means "followed by".

- Identity symmetry I (e.g., keep the figure fixed).
- Inverse symmetry (e.g., inverse rotation, inverse translation).
- Composition of symmetries is associative: $(a \star b) \star c = a \star (b \star c) = a \star b \star c$.

Cayley Table for the Square

Element (i, j) corresponds to $\mathrm{sym}_i \star \mathrm{sym}_j$.

All turns are counterclockwise.

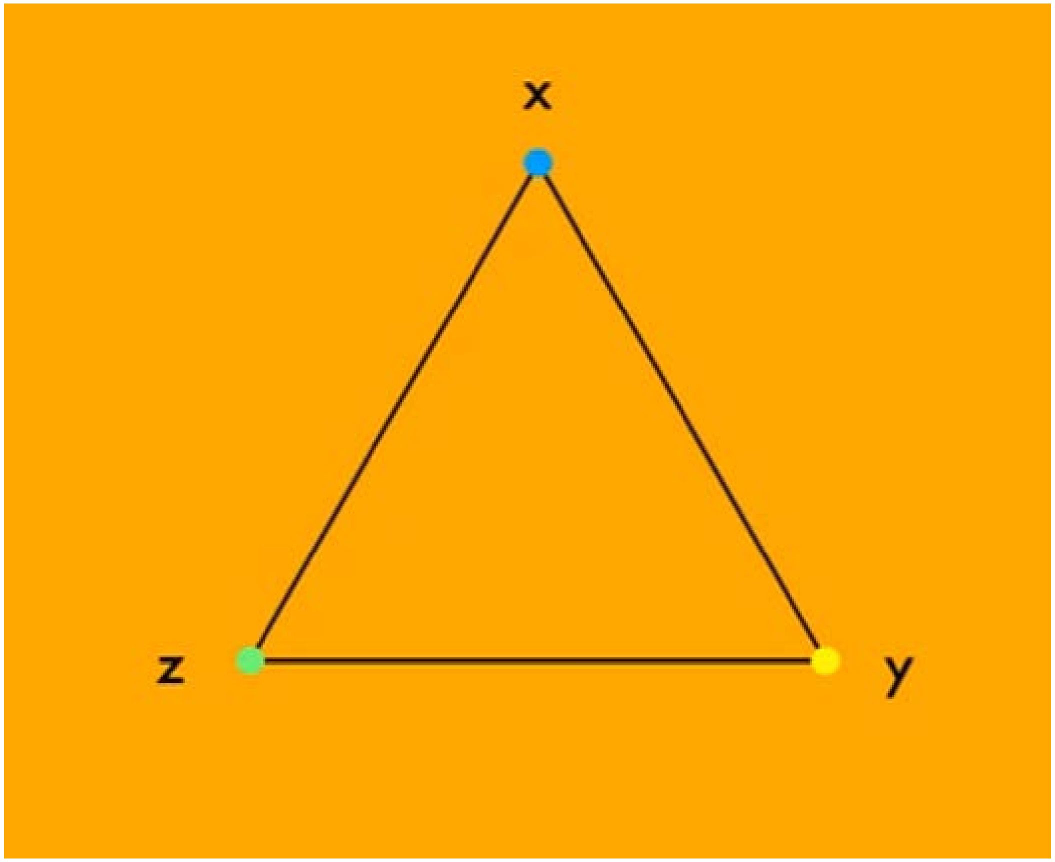
$\star$	I	R_{90}	R_{180}	R_{270}	X	Y	U	V
I	I	R_{90}	R_{180}	R_{270}	X	Y	U	V
R_{90}	R_{90}	R_{180}	R_{270}	I	V	U	X	Y
R_{180}	R_{180}	R_{270}	I	R_{90}	Y	X	V	U
R_{270}	R_{270}	I	R_{90}	R_{180}	U	V	Y	X
X	X	U	Y	V	I	R_{180}	R_{90}	R_{270}
Y	Y	V	X	U	R_{180}	I	R_{270}	R_{90}
U	U	Y	V	X	R_{270}	R_{90}	I	R_{180}
V	V	X	U	Y	R_{90}	R_{270}	R_{180}	I

I	leave in place
R_{90}	turn over $90°$
R_{180}	turn over $180°$
R_{270}	turn over $270°$
X	reflect over horizontal axis
Y	reflect over vertical axis
U	reflect over (downward) diagonal
V	reflect over upward diagonal

Non-abelian group, e.g., $R_{90} \star X = V \neq U = X \star R_{90}$

Isomorphic to permutation group S_4 (all permutations of four letters labeling the corners of the square)

Symmetries of an Equilateral Triangle



6 symmetries.

Permutation group S_3 (all permutations of 3 letters)

Cayley Table for an Equilateral Triangle

Element (i, j) corresponds to $\text{sym}_i \star \text{sym}_j$.

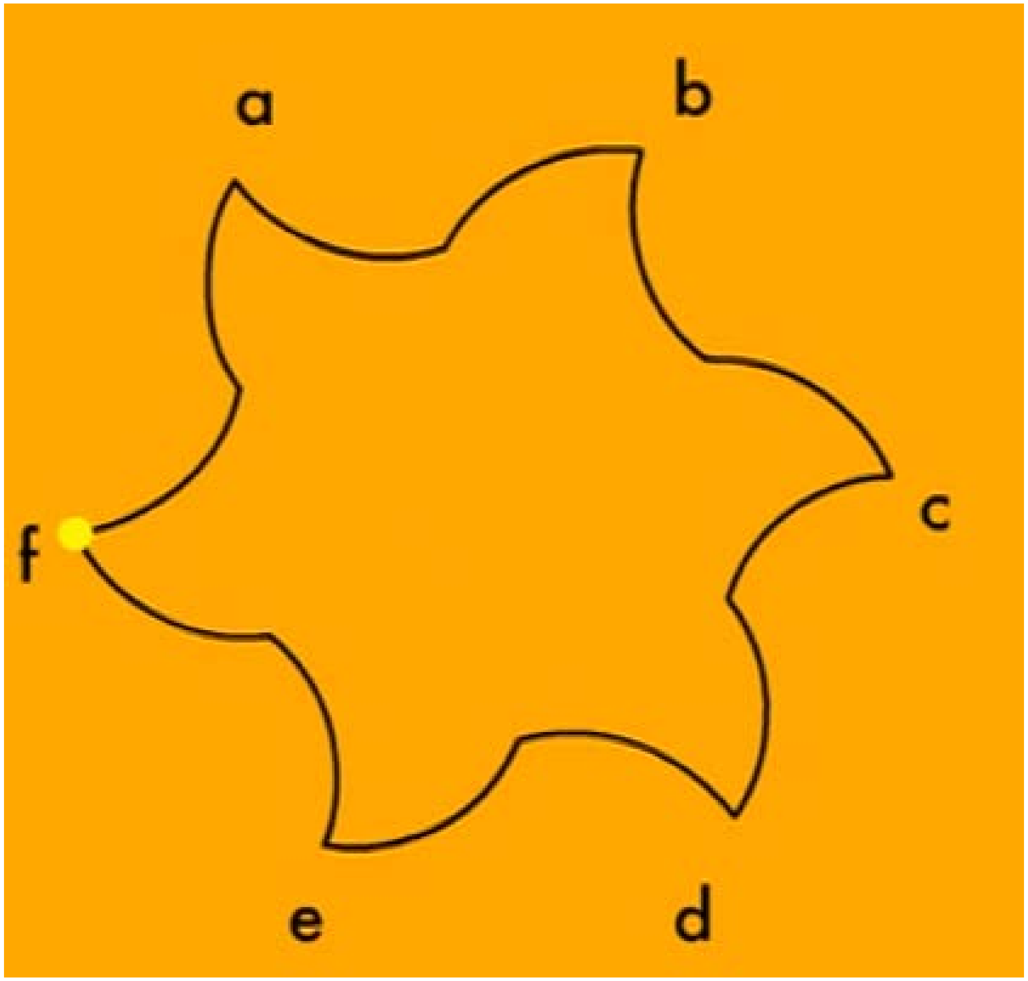
All turns are counterclockwise.

$\star$	I	R_{120}	R_{240}	X	Y	Z
I	I	R_{120}	R_{240}	X	Y	Z
R_{120}	R_{120}	R_{240}	I	Z	X	Y
R_{240}	R_{240}	I	R_{120}	Y	Z	X
X	X	Y	Z	I	R_{120}	R_{240}
Y	Y	Z	X	R_{240}	I	R_{120}
Z	Z	X	Y	R_{120}	R_{240}	I

I	leave in place
R_{120}	turn over $120°$
R_{240}	turn over $240°$
X	reflect over bisector of yz
Y	reflect over bisector of xy
Z	reflect over bisector of xz

Non-abelian group, e.g., $R_{120} \star X = Z \neq Y = X \star R_{120}$

Symmetries of the Starfish



6 rotations: Cyclic group $\mathbf{C}_6$

Cayley Table for the Starfish

Element (i, j) corresponds to $\text{sym}_i \star \text{sym}_j$.

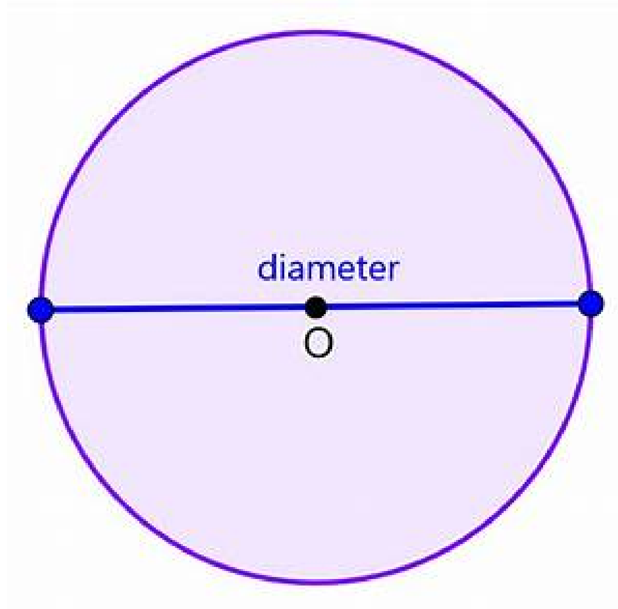
All turns are counterclockwise.

$\star$	I	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}
I	I	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}
R_{60}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	I
R_{120}	R_{120}	R_{180}	R_{240}	R_{300}	I	R_{60}
R_{180}	R_{180}	R_{240}	R_{300}	I	R_{60}	R_{120}
R_{240}	R_{240}	R_{300}	I	R_{60}	R_{120}	R_{180}
R_{300}	R_{300}	I	R_{60°	R_{120}	R_{180}	R_{240}

I	leave in place
R_{60}	turn over 60°
R_{120}	turn over 120°
R_{180}	turn over 180°
R_{240}	turn over 240°
R_{300}	turn over 300°

Abelian group, e.g., $R_{120} \star R_{60} = R_{180} = R_{60} \star R_{120}$

Symmetries of the Circle



The circle has an infinite group of symmetries.

Symmetry group of a circle is the orthogonal group $O(2)$ consisting of all rotations about a fixed point and reflections across any axis through that fixed point.

Examples of groups

- $\mathbb{Z}$: All integers with addition as group operation.
- S_n: Permutation group of n letters.

  Example (with letters A, B, and C)

A	B	C
A	C	B
B	A	C
B	C	A
C	A	B
C	B	A

  Group operation is composition of permutations.

There are 6 permutations possible.

In general $n! = n \times (n-1) \times \ldots \times 2 \times 1$.

- $\mathrm{SO}(2)$: Rotation group consisting of orthogonal matrices A with determinant equal to one.

$$\mathrm{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Group operation is matrix multiplication.

Note that $\mathrm{A}\mathrm{A}^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}}\mathrm{A} = \mathrm{I}$

- $\mathrm{U}(n)$: Group of unitary matrices

$$\mathrm{U}\bar{\mathrm{U}}^{\mathrm{T}} = \bar{\mathrm{U}}^{\mathrm{T}}\mathrm{U} = \mathrm{I}$$

Group operation is matrix multiplication.

General form:

$$\mathrm{U} = \begin{bmatrix} a & b \\ \mathrm{e}^{-i\theta}\,\bar{b} & \mathrm{e}^{i\theta}\,\bar{a} \end{bmatrix}$$

Example

$$\mathrm{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

- U(1): Unitary group of dimension one

  consisting of all complex number with magnitude equal to one

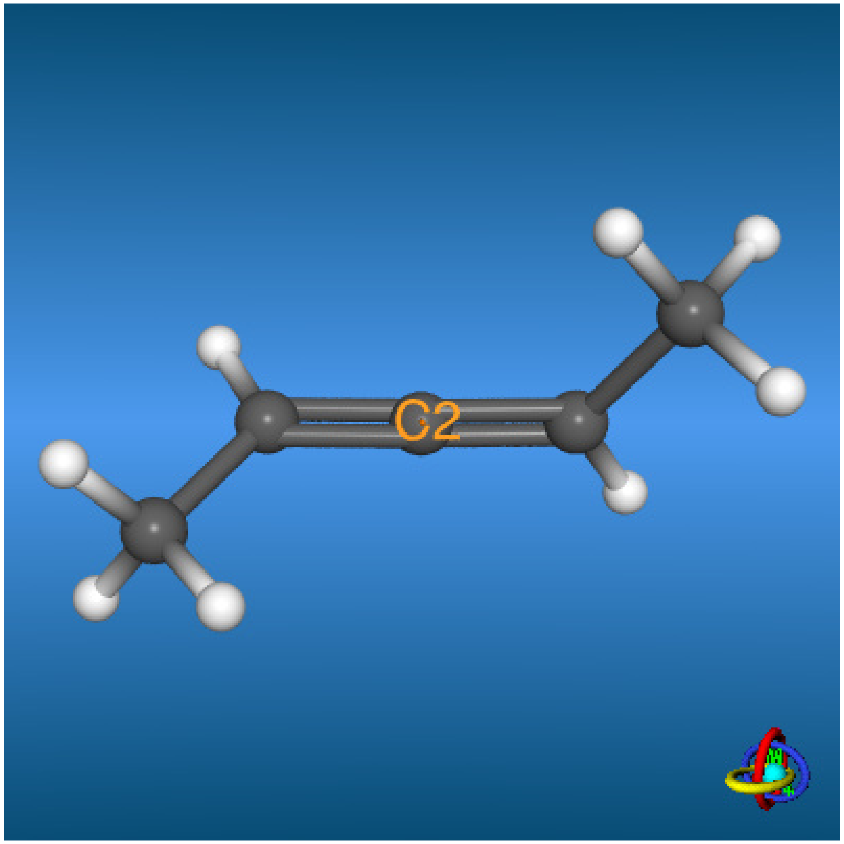
$$z = a + ib, \quad |z| = a^2 + b^2 = 1, \;\; \text{or} \;\; z = \mathrm{e}^{i\theta}$$

  Group operation is product.

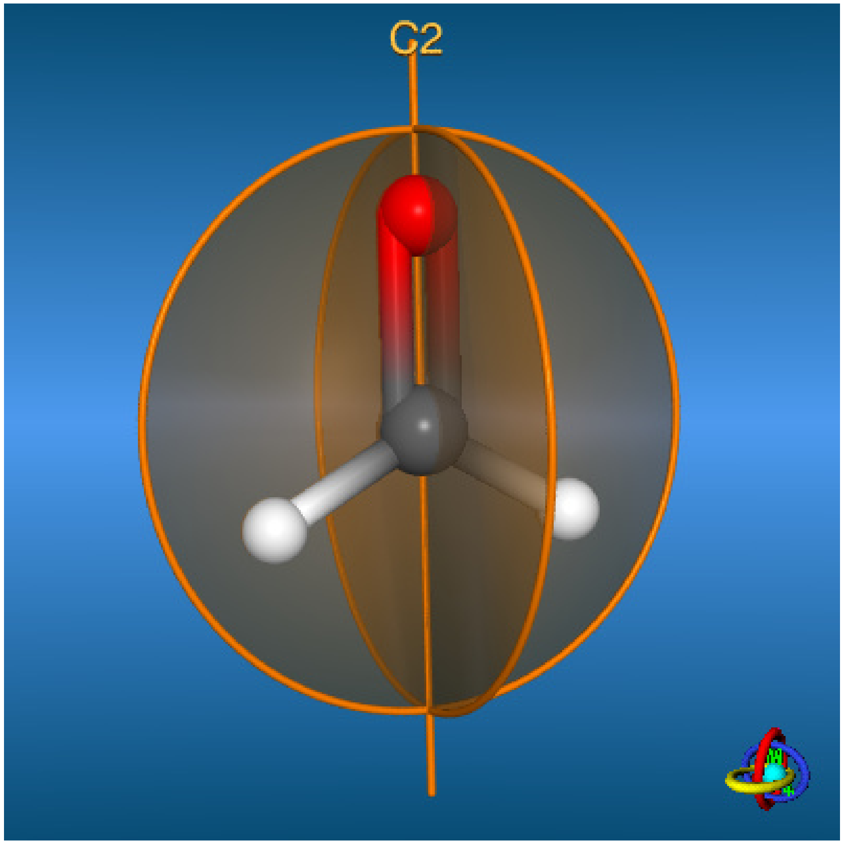
- Galois group: Solvability of polynomial equations in algebra.

  Group operation is permutation of the roots of the equation.

Symmetry in Chemistry



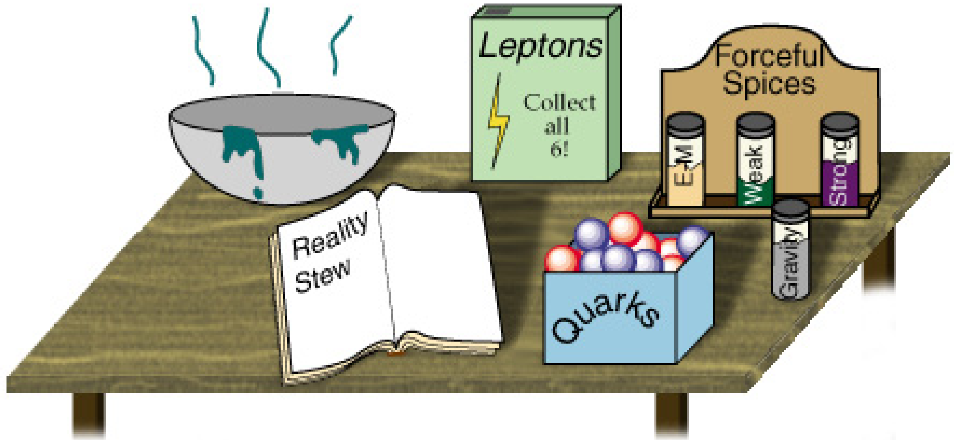
Allene



Methanal

Symmetry in Physics

Universality of the Laws of Physics



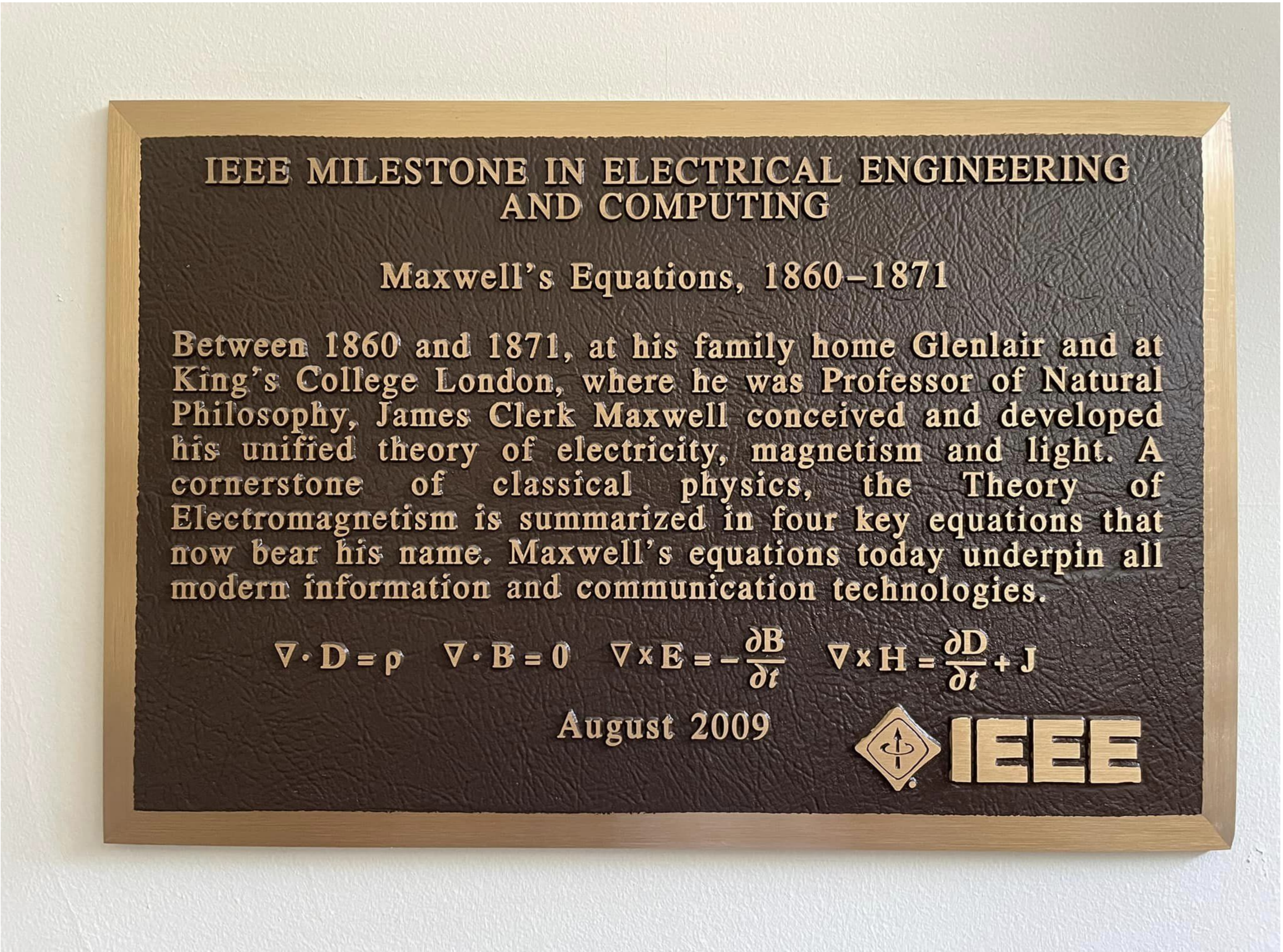
Laws of motion are independent of location (space invariant).

The Lorentz group (of Lorentz transformations) expresses the fundamental symmetry of space and time of all known fundamental laws of nature.

For example, the following laws, equations, and theories respect the Lorentz transformation (a.k.a. Lorentz covariance):

$$
\begin{aligned}
t' &= \frac{t - \frac{v\,x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x' = \frac{x - v\,t}{\sqrt{1 - \frac{v^2}{c^2}}}, \\
y' &= y, \quad z' = z
\end{aligned}
$$

- The kinematical laws of special relativity
- Maxwell's field equations in the theory of electromagnetism


IEEE MILESTONE IN ELECTRICAL ENGINEERING AND COMPUTING
Maxwell's Equations, 1860–1871
Between 1860 and 1871, at his family home Glenlair and at King's College London, where he was Professor of Natural Philosophy, James Clerk Maxwell conceived and developed his unified theory of electricity, magnetism and light. A cornerstone of classical physics, the Theory of Electromagnetism is summarized in four key equations that now bear his name. Maxwell's equations today underpin all modern information and communication technologies.
$\nabla \cdot D = \rho \quad \nabla \cdot B = 0 \quad \nabla \times E = -\frac{\partial B}{\partial t} \quad \nabla \times H = \frac{\partial D}{\partial t} + J$
August 2009
IEEE

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  - The basic theory of elementary particles and the physical laws that govern them.
  - The marriage of the electroweak theory (which describes the electromagnetic and weak forces responsible for radioactive decay) with quantum chromodynamics (which describes the strong forces which holds protons and neutrons tightly bound together in the atomic nucleus).

- Symmetry considerations lead to the prediction of new particles: the electrically charged W particle and the neutral Z particle (later observed at CERN in 1983-84).

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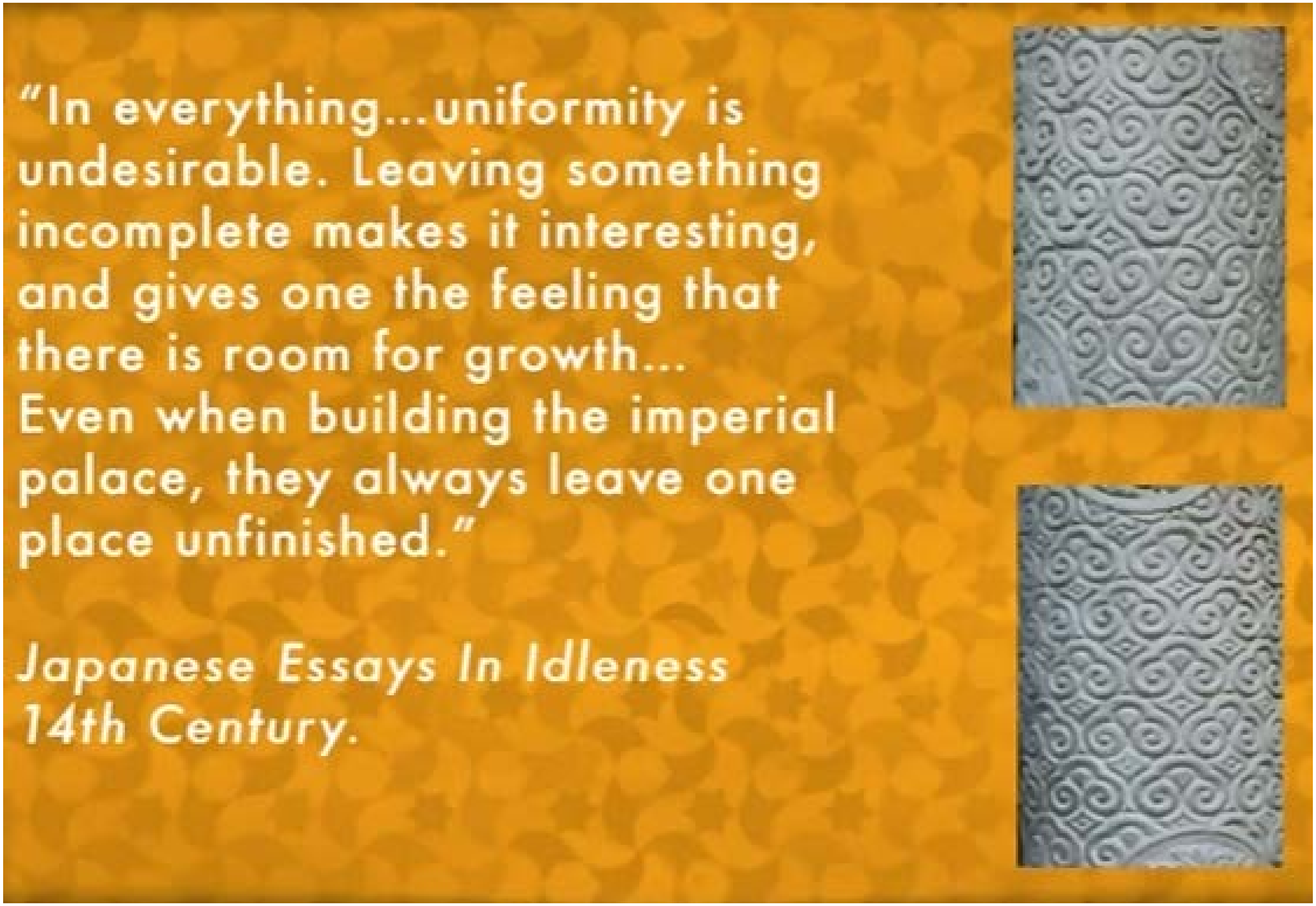
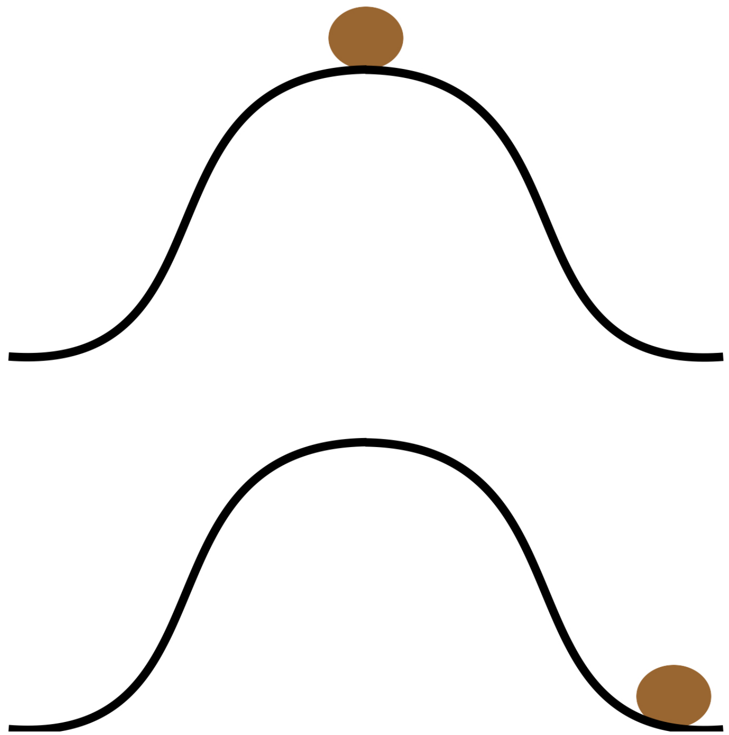
- The group structure of the standard model is isomorphic with the product of three Lie groups (the special unitary groups $\mathrm{U}(1), \mathrm{U}(2)$, and $\mathrm{U}(3)$).

Broken Symmetry



Symmetry Breaking



Ball is sitting on top of a hill in symmetric state.

That state is unstable: a slight disturbance will cause the ball to roll down the hill in some particular direction. Symmetry has been broken!

Broken Symmetry

Broken Symmetry

Other Examples of Symmetry Breaking

- Fugues – Bach: Play fugue2.mp3
- Ferromagnetic materials:

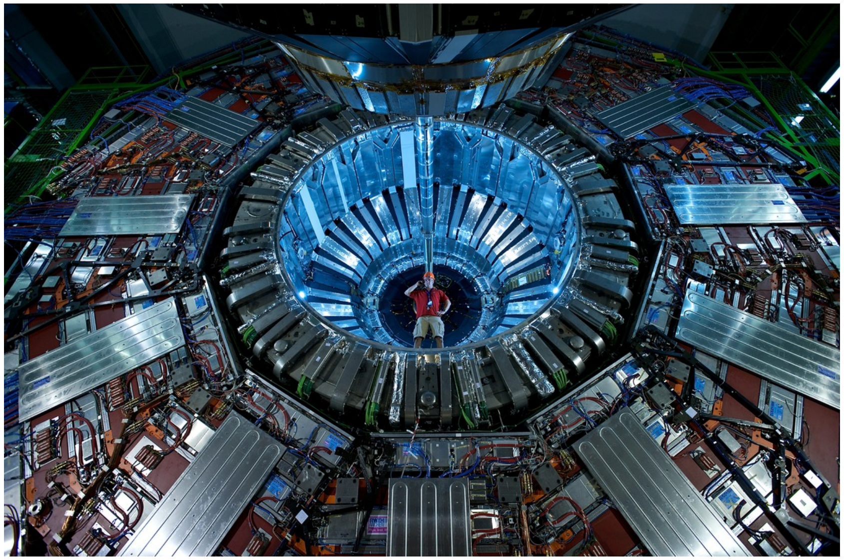
The laws describing it are invariant under spatial rotations. Here, the order parameter is the magnetization, which measures the magnetic dipole density.

Above the Curie temperature, the order parameter is zero, which is spatially invariant and there is no symmetry breaking.

Below the Curie temperature, however, the magnetization acquires a constant nonzero value which points in a certain direction. The residual rotational symmetries which leaves the orientation of this vector invariant remain unbroken but the other rotations get spontaneously broken.

Other Examples of Symmetry Breaking

Large Hadron Collider (≈ 17 miles circumference)



Higgs boson – spontaneous symmetry breaking.

Peter Higgs (Nobel prize 2013).

Modern Applications of Symmetry

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- New concept: String theory, supersymmetric string theory. Replace pointlike particles with tiny loops of vibrating strings. Consider pairing of particles with spin $\frac{1}{2}$ (fermions) and spin 1 and 2 (bosons).
- Holy grail: A theory for everything (include gravitational force with all other forces).

Symmetry in Mathematics

- Symmetry reduces complexity: puzzles
- Use of symmetry in proofs of (un-)solvability
- Paraphrasing Wigner: "... the unreasonable effectiveness of using symmetries in mathematics... ."

Father – Daughter Puzzle

Today, the ages of a father and his daughter add up to 40 years. Five years from now, the father will be 4 times older than his daughter. How old is each today?

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$$\begin{aligned} F + D &= 40 \\ F + 5 &= 4\,(D + 5) \end{aligned}$$

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$$F + D = 40$$
$$F + 5 = 4\,(D + 5)$$

Eliminate $F = 40 - D$ and solve for D (smaller number):

$$45 - D = 4\,(D + 5) \quad \text{or} \quad 5D = 25$$

Hence, $D = 5$ and $F = 40 - D = 35$.

Symmetry Reduces Complexity

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Hence, $x = 15$.

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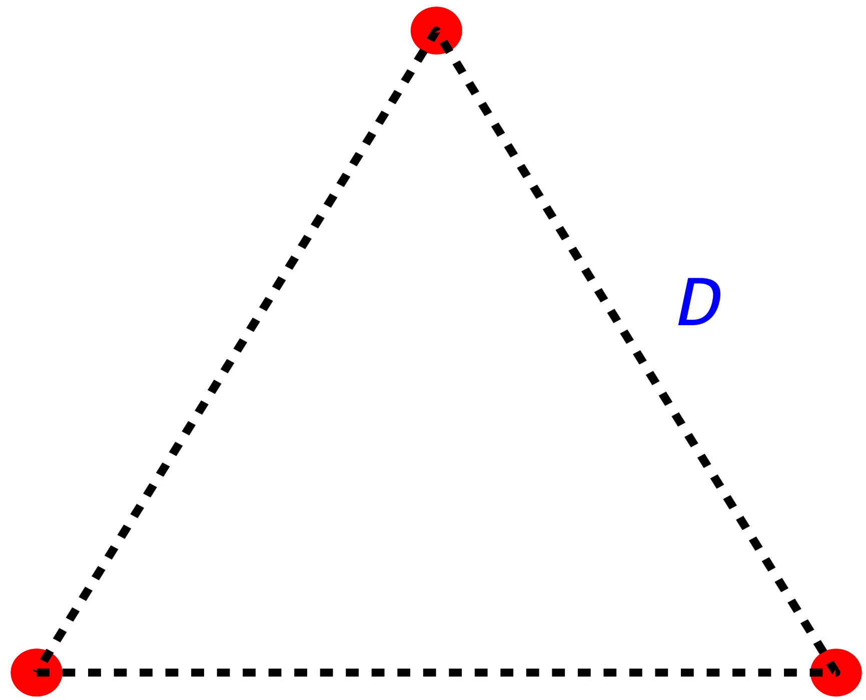
$$25 + x = 4\,(25 - x) \qquad \text{or} \quad 5x = 75$$

Hence, $x = 15$.

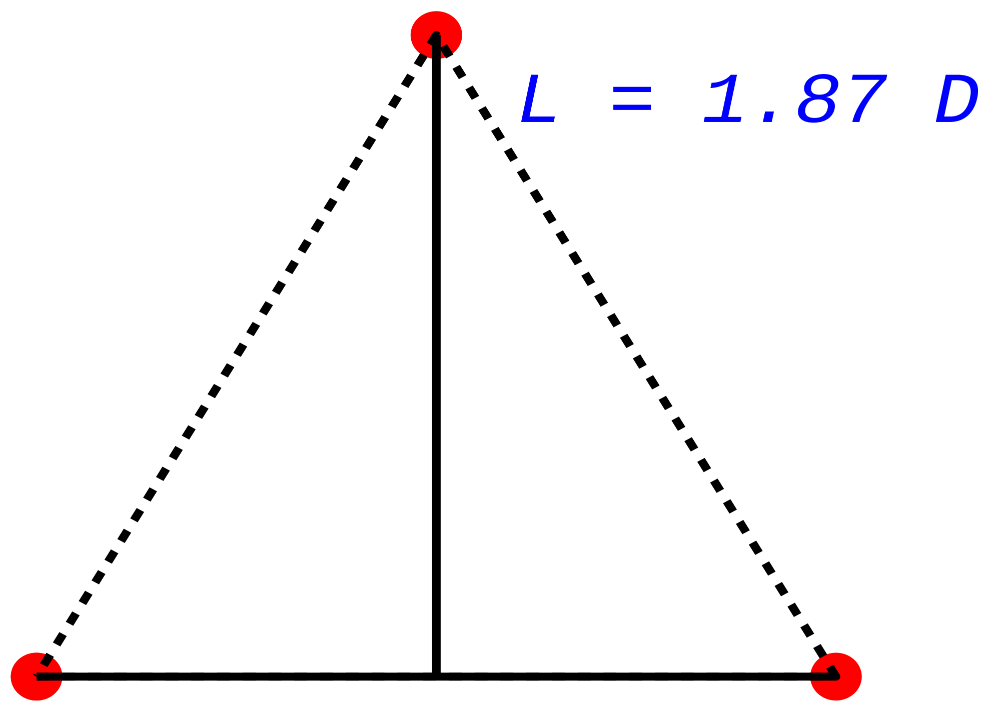
Father is $20 + x = 35$. Daughter is $20 - x = 5$.

Shortest path puzzle

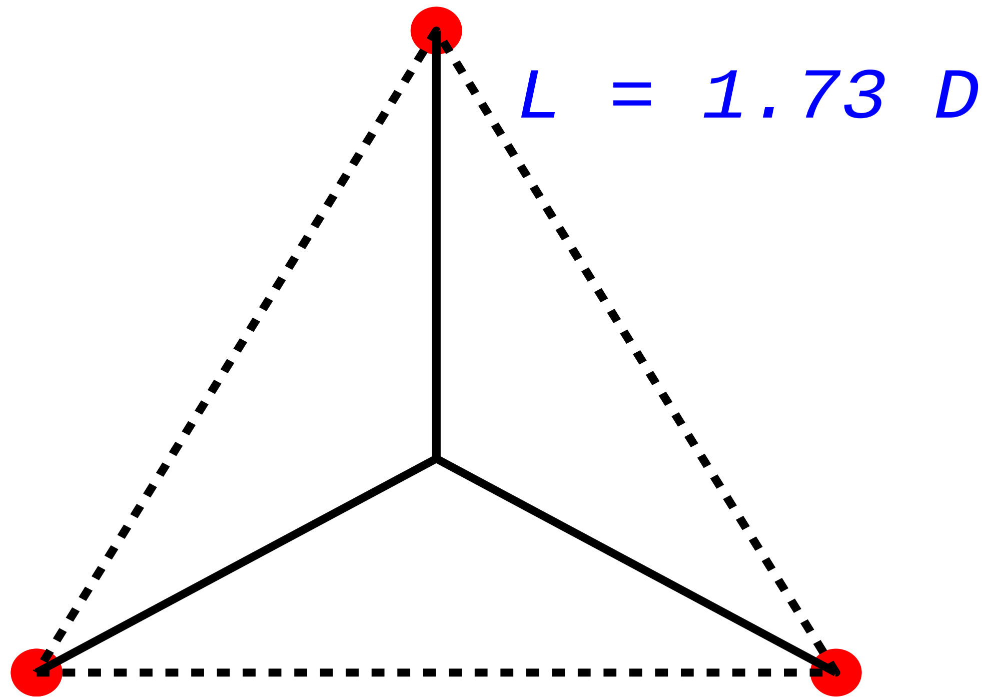
Connecting Cities with Shortest Road System



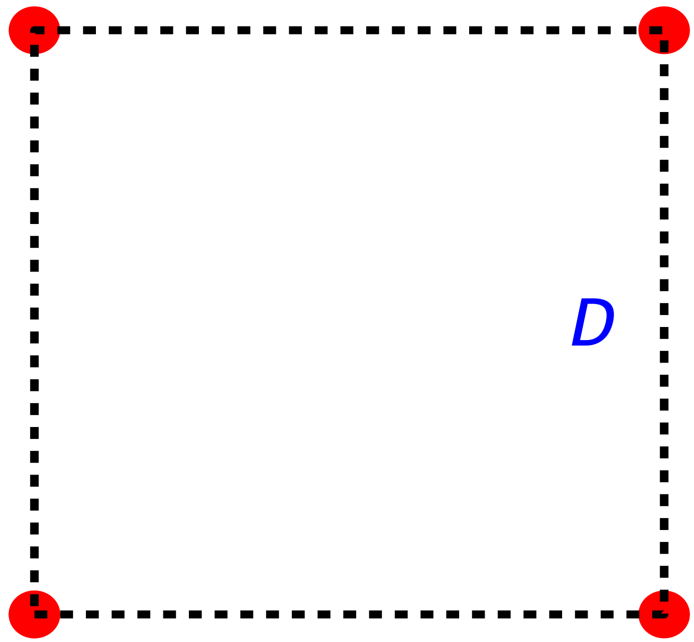
Three cities in equilateral triangle



One choice of a road system

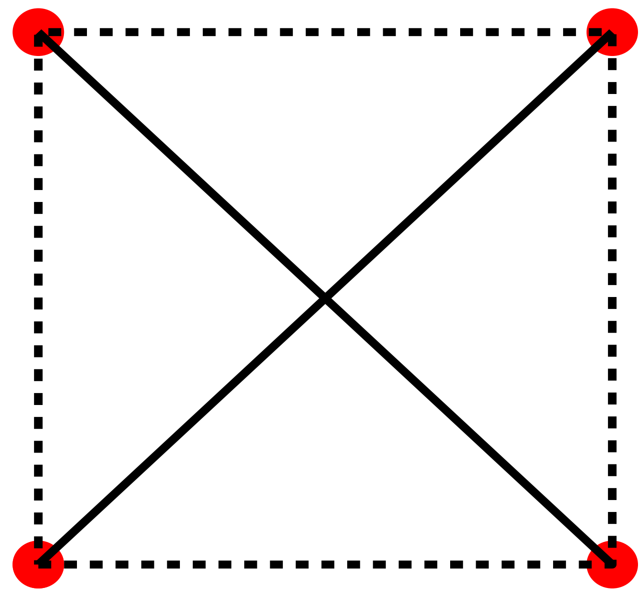


Shortest road system connecting three cities



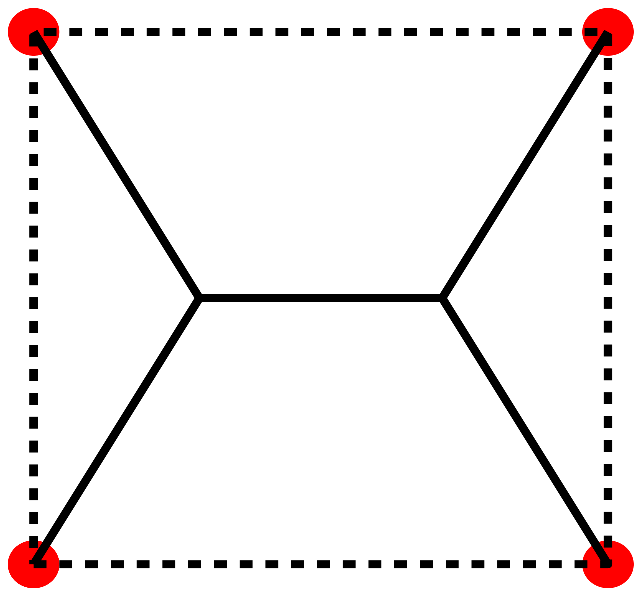
Four cities in a square

$L = 2.83\ D$

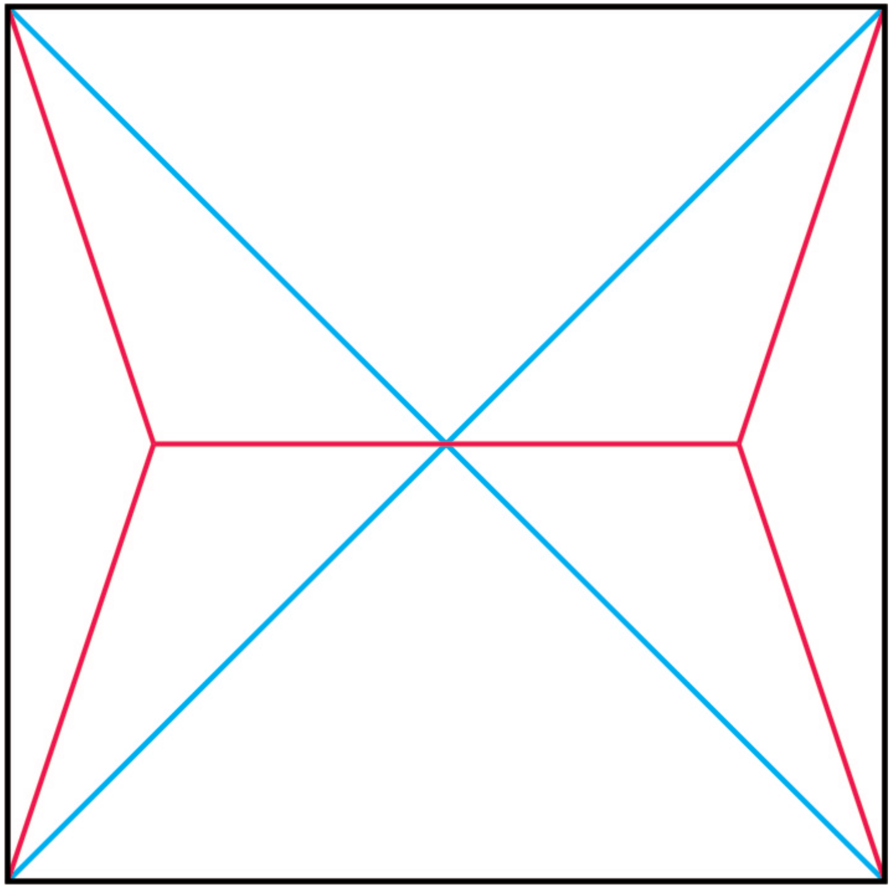


One choice of a road system

$L = 2.73\ D$



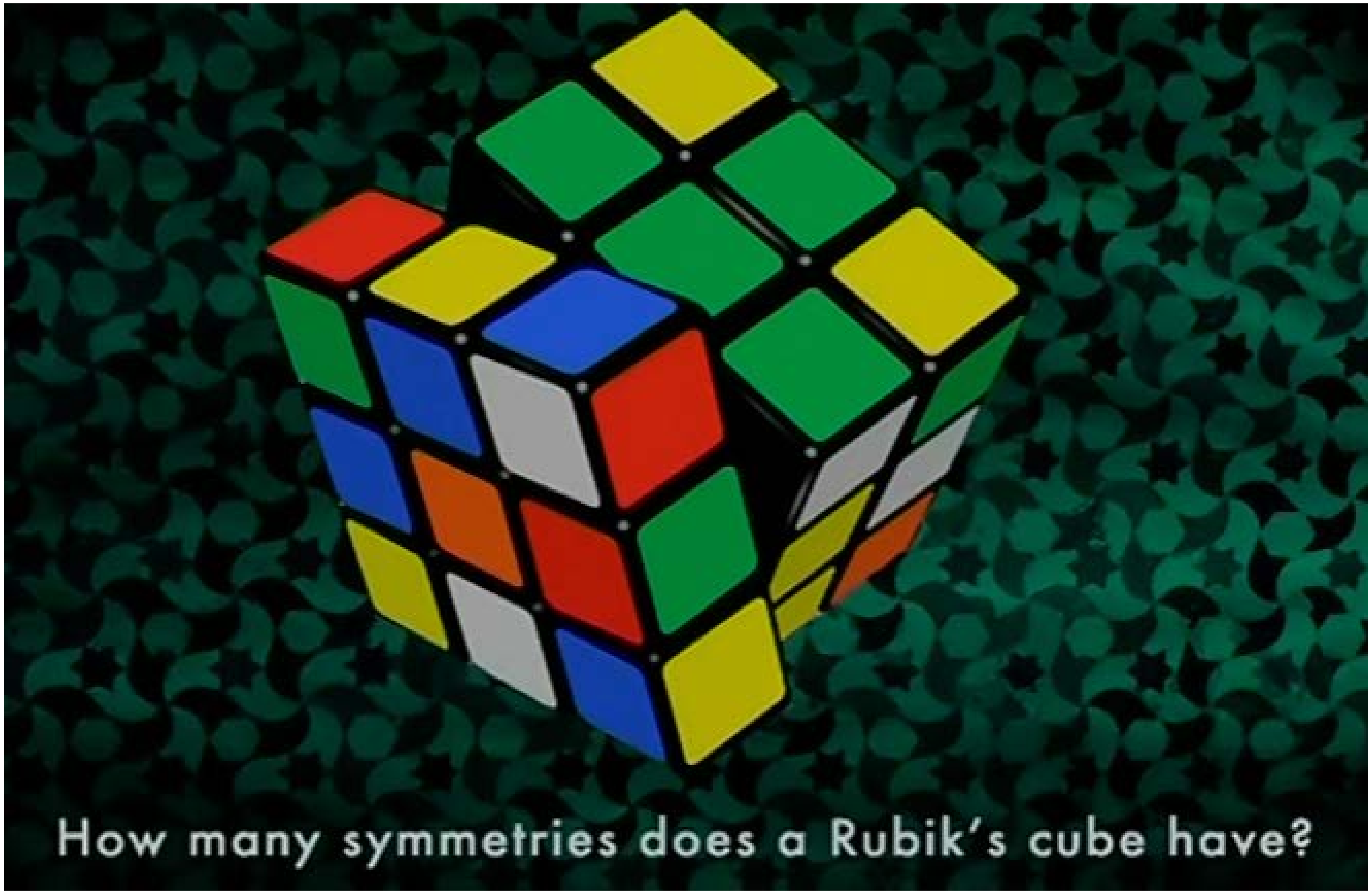
Shortest road system connecting four cities



Add the triangles on the square!

Rubik's Cube

Ernö Rubik (1944-   )



Symmetries of Rubik's Cube

Rubik's cube can be solved in fewer than 20 moves.

The solution strategy is based on group theory!

An old card trick

Gilles-Edme Giuot (1769)

An old card trick

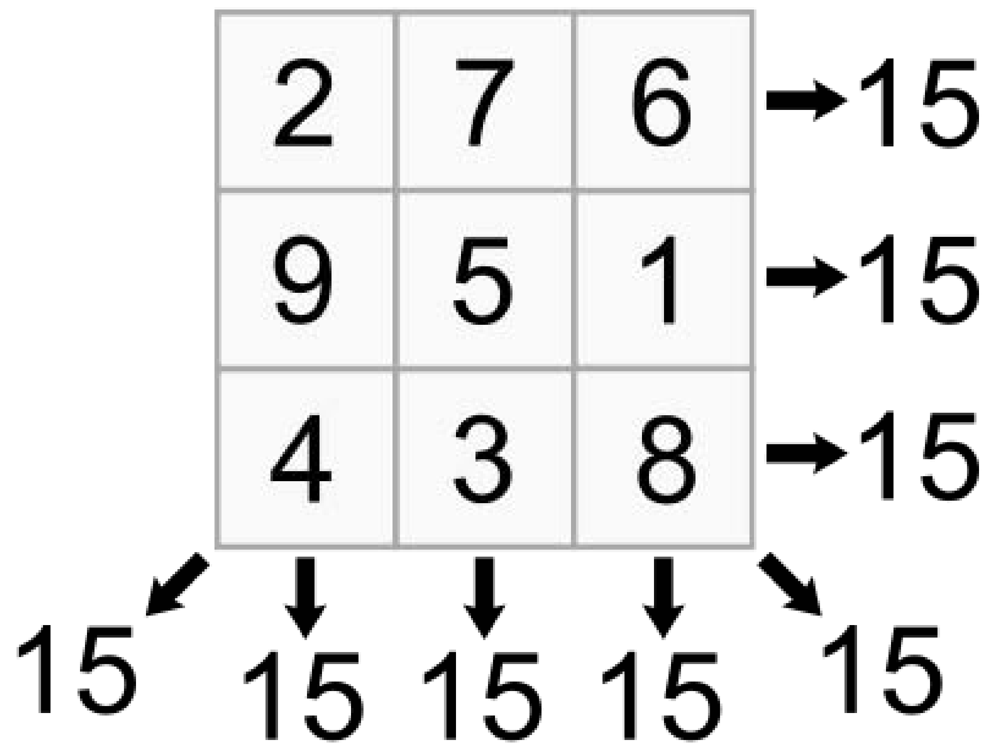
Gilles-Edme Giuot (1769)

M	U	C	U	T
D	E	D	I	T
N	O	M	E	N
C	O	S	I	S

Variants (if you show the trick several times):

- put words in different order
- change words: mutus, dedit, nomen, cocis

modeled on ancient magic squares



Solution is only unique up to rotation and reflection

Compare: Sudoku can also have multiple solutions.

Symmetry in Mathematics

Symmetric matrices, tensors, functions,...

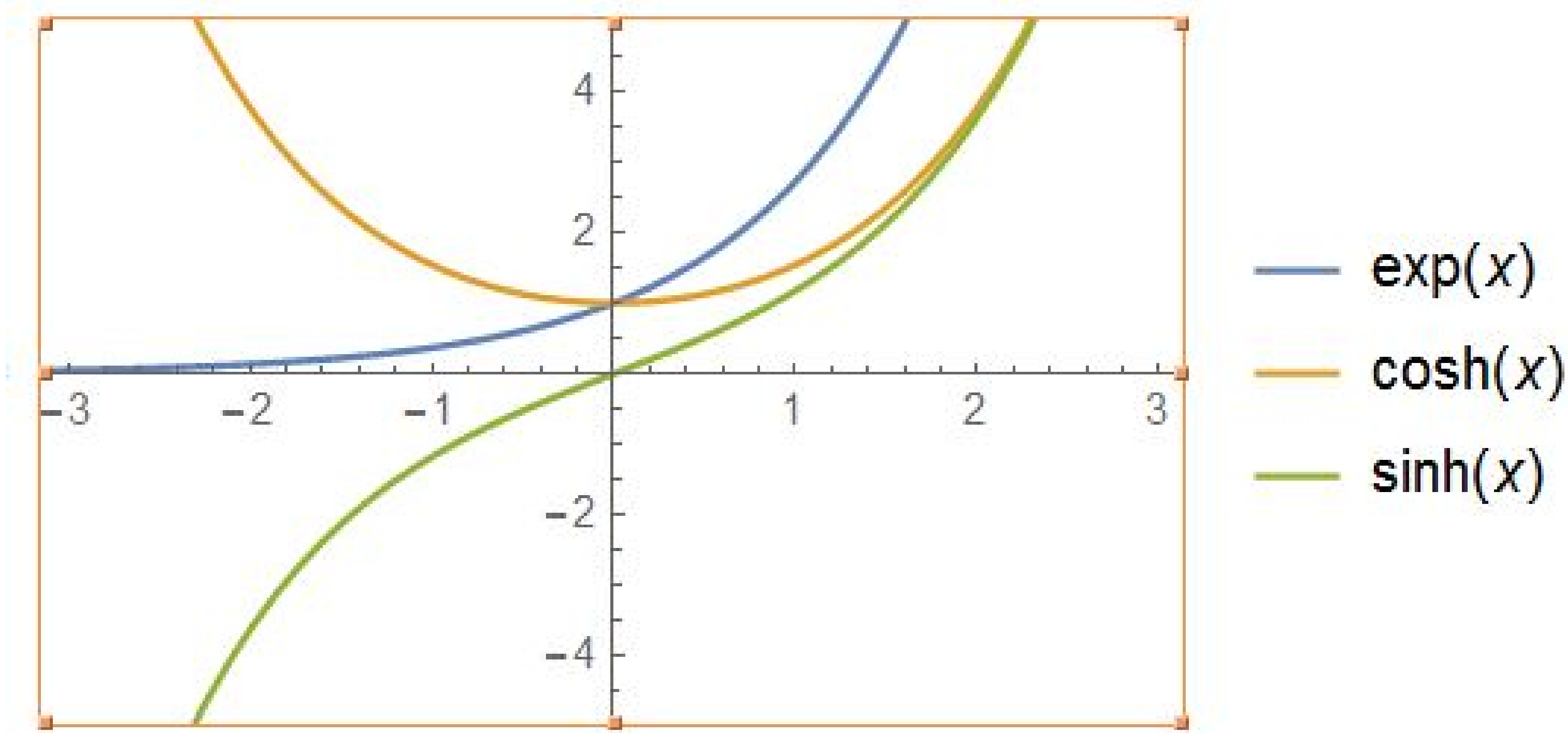
- Arbitrary square matrices can be split in the sum of symmetric and skew-symmetric matrices.

$$\begin{pmatrix} 1 & -1 & 3 \\ -5 & 4 & 7 \\ 11 & 19 & -9 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 7 \\ -3 & 4 & 13 \\ 7 & 13 & -9 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -4 \\ -2 & 0 & -6 \\ 4 & 6 & 0 \end{pmatrix}$$

- Symmetric matrices: real eigenvalues and eigenvectors, orthogonally diagonalizable, etc.

- Arbitrary functions can be split in the sum of even and odd functions.

$$e^x = \cosh x + \sinh x$$



Symmetry in Mathematics

Solving Polynomial Equations

- Linear equation:

$$a\,x + b = 0$$

Solving Polynomial Equations

- Linear equation:

$$a\,x + b = 0$$

- Solution (ancient times):

$$x = -\frac{b}{a}$$

Solving Polynomial Equations

- Quadratic equation:

$$a\,x^2 + b\,x + c = 0$$

Solving Polynomial Equations

- Quadratric equation:

$$a\,x^2 + b\,x + c = 0$$

- Solution (Babylonians, 400 BC):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Using $x_1 + x_2 = -\frac{b}{a}$ and $x_1 x_2 = \frac{c}{a}$

$$\frac{1}{2}[(x_1 + x_2) \pm \sqrt{(x_1 + x_2) - 4x_1 x_2}]$$

gives x_1 (for the "+" sign) and x_2 (for the "-" sign). The formula is symmetric in x_1 and x_2.

Solving Polynomial Equations

- Cubic equation:

$$a\,x^3 + b\,x^2 + c\,x + d = 0$$

- Solution: Italians dal Ferro & Fior (1525-1535)
  Formulas of Tartaglia (1535) and Cardano (1545)

Solving Polynomial Equations

- Quartic equation:

$$a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e = 0$$

- Solution: Formulas of Ferrari and Cardano (1545)

Solving Polynomial Equations

- General case:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

- For example, the quintic equation

$$a x^5 + b x^4 + c x^3 + d x^2 + e x + f = 0$$

- Solution: For $n \geq 5$, solution can not be obtained algebraically (by the four arithmetic operations and the taking of roots).

Niels Henrik Abel (1802–1829)

For $n = 5$, an algebraic solution is impossible.

Paolo Ruffini (1765-1822) has claimed earlier that the general cubic could not be solved.

Father of Group Theory

Evariste Galois (1811–1832)

- Old approach:

  If you want to know whether an equation is solvable, simply try to solve it!

  The method of "trial and error" failed.

- New idea:

  - Associate to every equation a “genetic code” called the Galois group of the equation.

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  - The Galois group is the “permutation group” of its roots.

  - The properties of the Galois group determine whether or not the equation is solvable by a formula.

- Method:
  - The maximum number of permutations of n roots is $n!$ These permutations form the group S_n.

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  - Find the maximal normal subgroup (largest size).

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  - Investigate the properties of subgroups (with a subset of the roots).
  - Find all “normal” subgroups (sandwich an element of the subgroup by an element and its inverse from original group).
  - Find the maximal normal subgroup (largest size).
  - Create a genealogy of maximal subgroups.

- Compute the composition factors (order of the parent group divided by that of the maximal normal subgroup).

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- Investigate the “solvability” of the Galois group (composite factors must all be prime numbers).

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- Conclusions:

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- Investigate the “solvability” of the Galois group (composite factors must all be prime numbers).

- Conclusions:
  - The algebraic equation can be solved with a formula if the corresponding Galois group is solvable!
  - In essence, the solution can then be found by solving equations of lower degree.

- For $n \geq 5$, no algebraic solution is possible (one of the composite factors is 60).

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- For $n \leq 4$, an algebraic solution is always possible (all composite factors are prime numbers).

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- For $n \leq 4$, an algebraic solution is always possible (all composite factors are prime numbers).

- Groups/symmetries unified the different approaches for solving polynomial equations.

A Sample of Galois' Notes

Three Classical Problems

Impossible with a Straightedge and Compass

Doubling a cube, trisecting an angle, and squaring a circle are impossible.

- Duplicating a cube: given a side of a cube s, can you construct a side $s^\star$ of a cube whose volume is exactly twice that of the first cube?
- Squaring a circle: given a circle, can you construct a square with the same area?

Trisecting an angle: given an angle α, can you construct $\frac{\alpha}{3}$?

Using Galois theory, Pierre Wantzel (1837) proved that it is impossible to trisect an angle (with a straightedge and compass).

That does not stop people from claiming they have found a solution!

Teenager tackles ancient math problem

Boy solves geometric puzzle, passes first test on way to recognition

By CAROL CHOREY
Camera Staff Writer

Arian Hample of Gunbarrel has never taken a geometry course.

But that didn't stop the 15-year-old student at Boulder Valley's New High School from putting his mind to a math problem proclaimed impossible 2,000 years ago by Euclid, the father of geometry.

It even may have helped him, according to his mother, Patricia Hample, who said one of the first things taught in geometry classes is that you can't trisect an angle using only a compass and a straight-edge.

"I'm glad Arian didn't hear that," she said. "I don't think I've ever told Arian he couldn't do something."

He started working on the problem last spring, and after several months of work he trisected the angle and wrote a mathematical proof to show his answer was right.

He spent Wednesday and Thursday in Washington, D.C., meeting with government and education officials interested in his work.

The meeting was arranged by John Kettling, a computer lab volunteer at Casey Middle School where Arian went last year. Kettling worked with Arian on the proof and accompanied him to Washington, his mother said.

Neither Arian nor Kettling could be

(See TEEN, Page 3C)

Teen tackles math problem

(From Page 1C)

reached Thursday, but Arian's mother, who spoke to them, said everything went well. Kettling told her they had passed the first test and could now proceed to "the next level" of trying to interest other mathematicians in the proof.

"There have been other people who have tried it and it has been disproved," she explained. "But everybody who has looked at (Arian's proof) thinks there's something fascinating about it."

"I think it's fine no matter what happens. Just that he got this far is great."

A spokeswoman for U.S. Sen. Hank Brown, R-Colo., who helped Arian make the link with education officials, explained that the proof basically "establishes a proportion at point zero and sends it out in all directions."

"They also had pi on both sides of the equation, which hadn't been done before," she added.

U.S. Rep. David Skaggs, D-Colo., for whom Arian also demonstrated the trisection, said it was convincing, though admittedly he is not a mathematician.

"It made enough sense to me. My judgment is he was able to make his case," Skaggs said.

"I just think it's great when especially a young person hits this sort of thing out of the park," he said about wanting to meet Arian and add a layer of recognition to the others received in Washington.

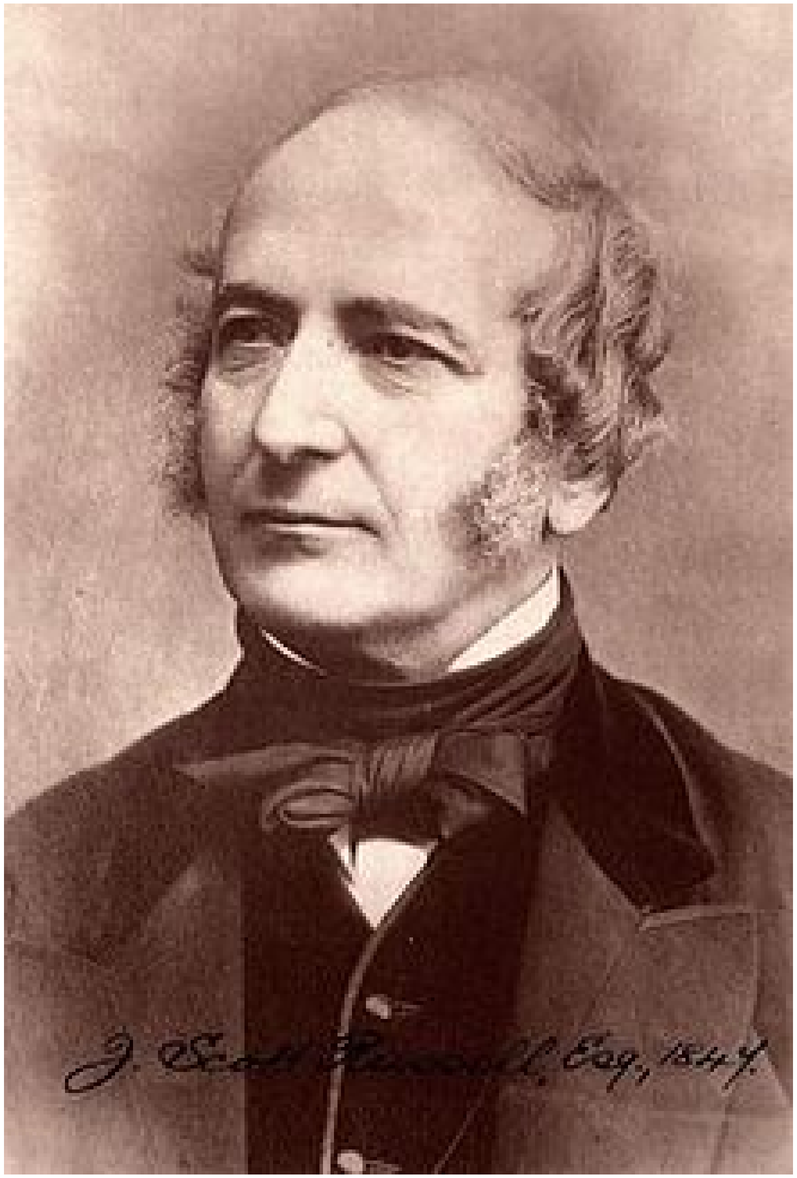
"I have the (drawing) he did for me in my briefcase to take home and show my kids," Skaggs said.

Use of Symmetries in my Research

- Study of wave phenomena (solitons and wavelets).
- Compute Lie point symmetries and conservation laws.
- Find exact solutions of nonlinear differential equations.
- Make Mathematica software to do the computations as one would with pen on paper.

Two illustrations:

- Symmetries and conservation laws
- Solitons



John Scott Russell (1808-1882)

Scott Russell was observing the motion of a boat which was drawn along a narrow channel by a pair of horses, when it suddenly stopped.

A wave formed under the boat, moved to the prow, and then rolled forward with great velocity, assuming the form of a large solitary elevation.

He followed it on horseback and was astounded to see that the wave kept going at the same size and pace for a couple of miles. He later called it "the wave of translation" and described the event as "the happiest day of my life."

The Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Diederik Korteweg (1848-1941)

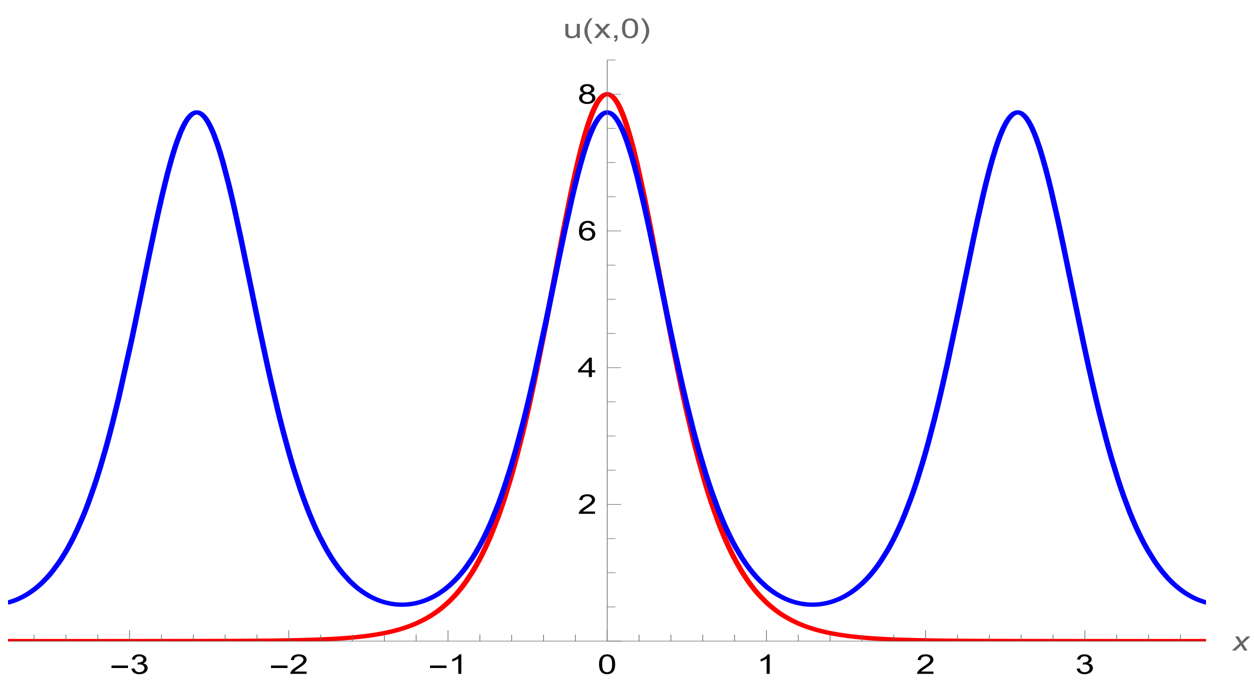
Gustav de Vries (1866-1934)

Solitary wave and periodic solutions

$$u(x,t) = 2k^2 \operatorname{sech}^2(kx - 4k^3 t + \delta) \quad \text{and}$$

$$u(x,t) = \tfrac{4}{3}k^2(1-m) + 2k^2 m \operatorname{cn}^2(kx - 4k^3 t + \delta; m)$$

Graphs of the solitary wave (red) and cnoidal (blue) wave solutions for $k = 2$, $m = \frac{9}{10}$, $\delta = 0$.



Symmetries Applied to Differential Equations

Marius “Sophus” Lie (1842–1899)

Scaling symmetry of the KdV equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

has dilation (scaling) symmetry

$$(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u) \; = (\tilde{x}, \tilde{t}, \tilde{u})$$

where λ is an arbitrary parameter.

Replace x, t, u in terms of $\tilde{x} = \frac{x}{\lambda},\ \tilde{t} = \frac{t}{\lambda^3},\ \tilde{u} = \lambda^2 u$ then

$$\frac{1}{\lambda^5}\Big(\tilde{u}_{\tilde{t}} + 6\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}\Big) = 0$$

Making new solutions of the KdV equation

If $u = F(x,t)$ is a solution of the KdV equation, so are

$u = F(x - \varepsilon, t)$	space translation
$u = F(x, t - \varepsilon)$	time translation
$u = F(x - \varepsilon t, t) + \varepsilon$	Galilean boost
$u = \frac{1}{\lambda^2} F(\frac{x}{\lambda}, \frac{t}{\lambda^3})$	scaling (dilation)

Symmetries and Conservation Laws

Emmy Noether (1882–1935)

Symmetry	Conserved Quantity
Translation in time	Energy
Translation in space	Linear Momentum
Rotation in space	Angular Momentum
Scaling symmetry	more conservation laws, Lax pair, recursion operator, etc.

Conservation laws of the KdV equation

- First six (of infinitely many) conservation laws:

$$\mathrm{D}_t\Big(u\Big) + \mathrm{D}_x\Big(3u^2 + u_{xx}\Big) \doteq 0$$

$$\mathrm{D}_t\Big(u^2\Big) + \mathrm{D}_x\Big(4u^3 - {u_x}^2 + 2uu_{xx}\Big) \doteq 0$$

$$\mathrm{D}_t\Big(u^3 - \frac{1}{2}u_x^2\Big)$$
$$+ \mathrm{D}_x\Big(\tfrac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_x u_{xxx}\Big) \doteq 0$$

$$\mathrm{D}_t\Big(u^4 - 2uu_x^2 + \tfrac{1}{5}u_{xx}^2\Big) + \mathrm{D}_x\Big(\tfrac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx}$$
$$+2u_x^2u_{xx} + \tfrac{16}{5}uu_{xx}^2 - 4uu_xu_{xxx} - \tfrac{1}{5}u_{xxx}^2 + \tfrac{2}{5}u_{xx}u_{4x}\Big) \doteq 0$$

$$\mathrm{D}_t\left(u^5 - 5\,u^2u_x^2 + \,uu_{xx}^2 - \tfrac{1}{14}u_{xxx}^2\right)$$

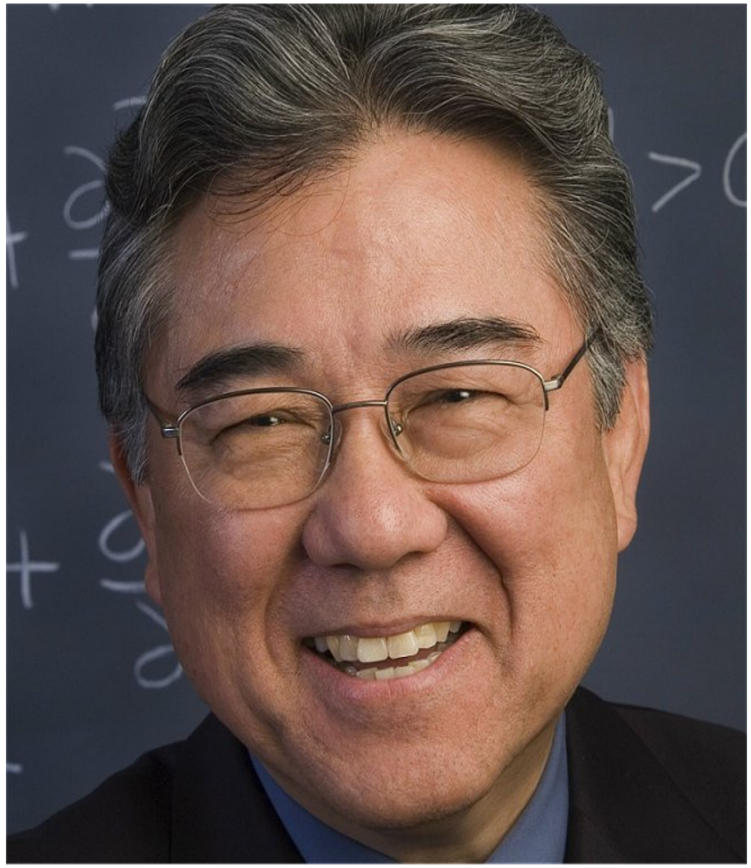
$$+\,\mathrm{D}_x\left(5u^6 - 40u^3u_x^2 - \,\ldots\, - \tfrac{1}{7}u_{xxx}u_{5x}\right) \doteq 0$$

$$\mathrm{D}_t\left(u^6 - 10\,u^3u_x^2 - \frac{5}{6}\,u_x^4 + 3\,u^2u_{xx}^2\right.$$

$$\left. + \tfrac{10}{21}\,u_{xx}^3 - \tfrac{3}{7}\,uu_{xxx}^2 + \tfrac{1}{42}\,u_{4x}^2\right)$$

$$+\,\mathrm{D}_x\left(\tfrac{36}{7}u^7 - 75u^4u_x^2 - \,\ldots\, + \tfrac{1}{21}u_{4x}u_{6x}\right) \doteq 0$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966

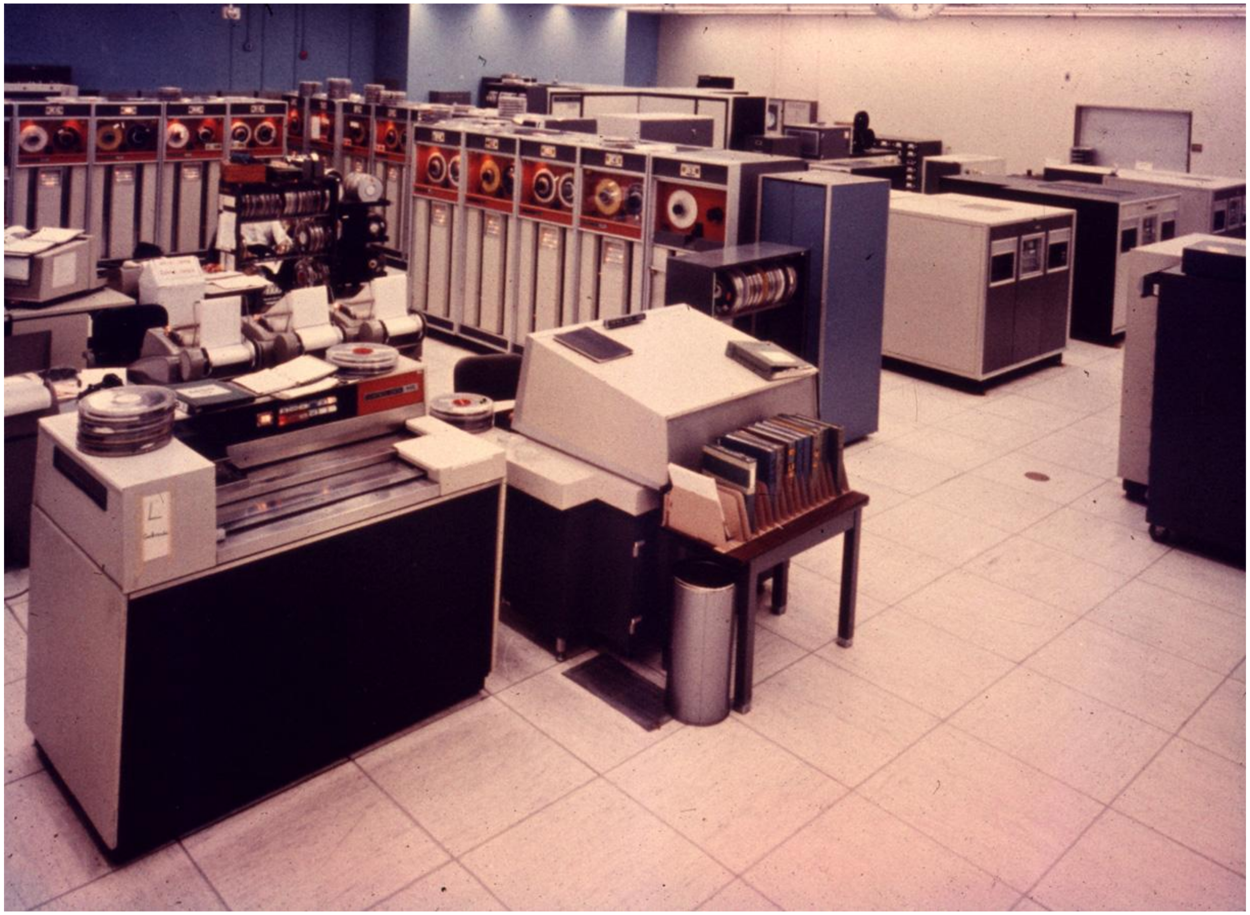


Robert Miura

- First five: IBM 7094 computer with FORMAC (1966) $\longrightarrow$ storage space problem!

IBM 7094 Computer

- First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)
  $\longrightarrow$ large integers problem!

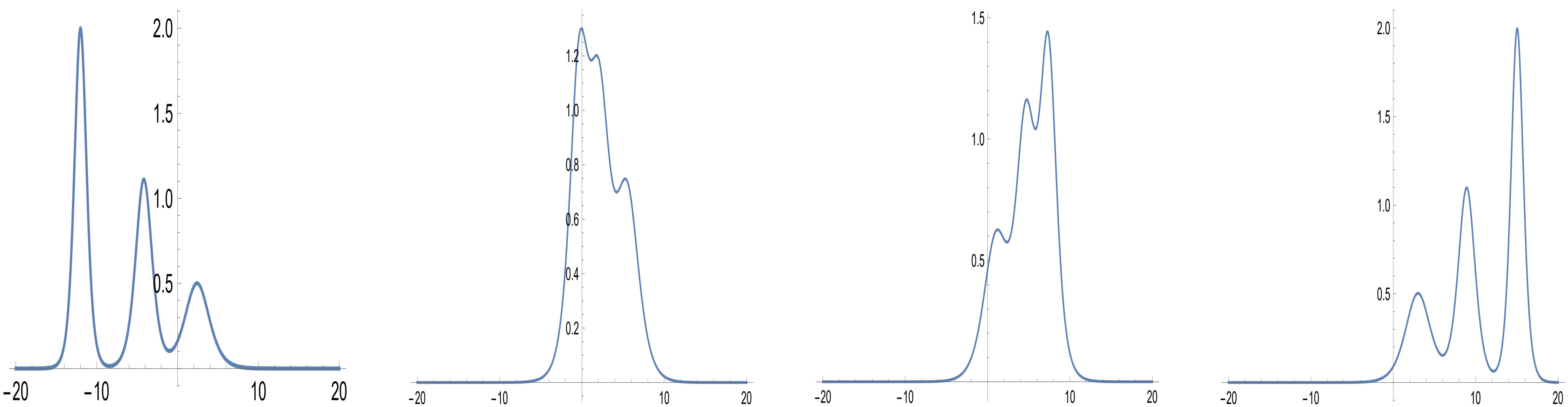


Control Data CDC-6600

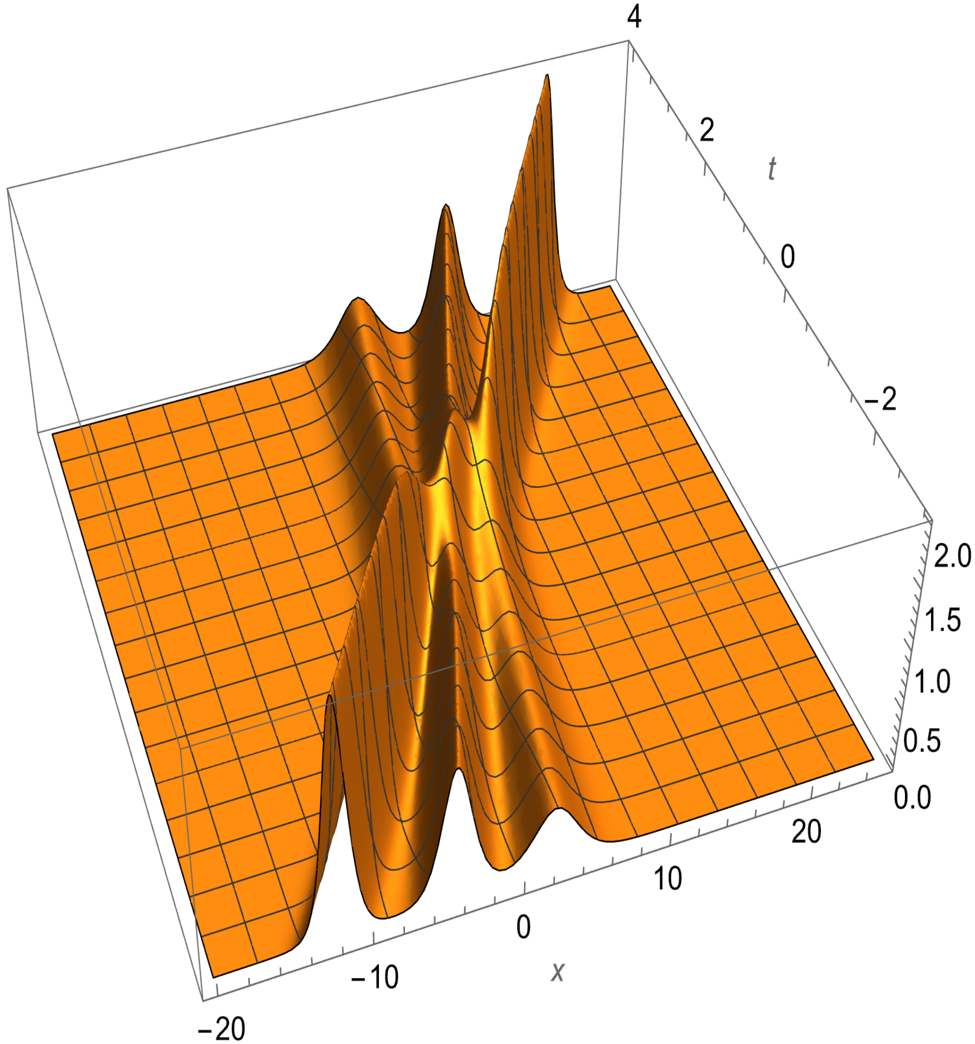
Solitons

Norman Zabusky and Martin Kruskal (1965)

Collision of three-solitons for the KdV equation



Bird's eye view of a 3-soliton collision for the KdV equation. Notice the phase shift.



Solitons in optical fibers

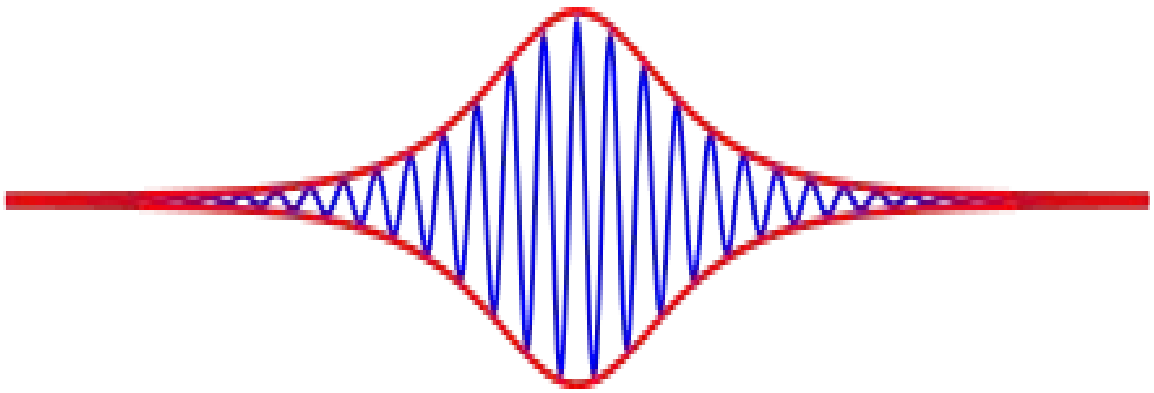
The nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + |u|^2\, u = 0$$

for complex function $u(x,t)$

A soliton solution

$$u(x,t) = \sqrt{2}\,\mathrm{e}^{i(vx-(v^2-1)t)}\,\mathrm{sech}(x-2vt)$$



Demonstrations with Mathematica

Thank You

Solution "Magic Eye" Stereogram