

Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

Willy Hereman

Department of Applied Mathematics and Statistics

Colorado School of Mines

Golden, Colorado

whereman@mines.edu

<http://inside.mines.edu/~whereman/>

Department of Mathematics and Statistics

Jordan University of Science and Technology, Irbid, Jordan

Tuesday, October 22, 2019, 13:30

Acknowledgements

Douglas Poole (Chadron State College, Nebraska)

Ünal Gökteş (Turgut Özal University, Turkey)

Mark Hickman (Univ. of Canterbury, New Zealand)

Bernard Deconinck (Univ. of Washington, Seattle)

Several undergraduate and graduate students

Research supported in part by NSF
under Grant No. CCF-0830783

This presentation was made in TeXpower

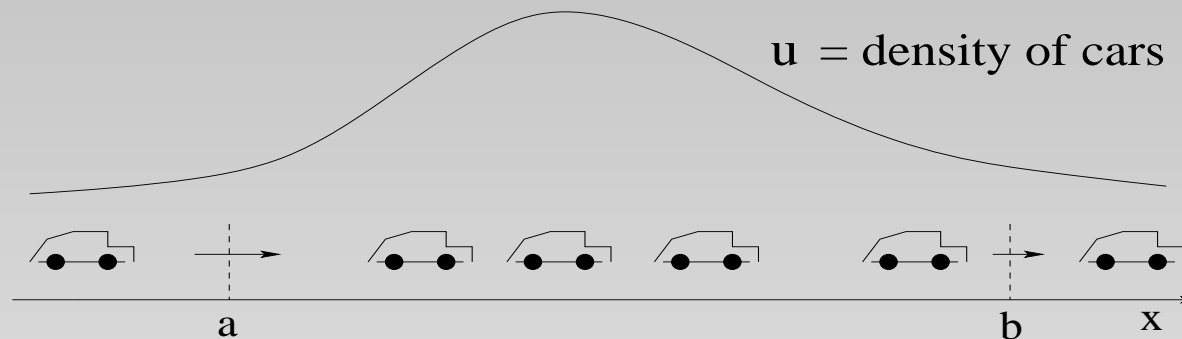
Outline

- Examples of conservation laws
- The Korteweg-de Vries equation
- Other scaling invariant quantities (KdV equation)
- The Zakharov-Kuznetsov equation
- Methods for computing conservation laws
- Tools from the calculus of variations
 - The variational derivative (testing exactness)
 - The homotopy operator (inverting D_x and Div)
- Application 1: The KdV equation
- Application 2: The Zakharov-Kuznetsov equation
- Demonstration of *ConservationLawsMD.m*
- Conclusions and future work

- Additional examples
 - ▶ Manakov-Santini system
 - ▶ Camassa-Holm equation in $(2+1)$ dimensions
 - ▶ Khoklov-Zabolotskaya equation
 - ▶ Shallow water wave model for atmosphere
 - ▶ Kadomtsev-Petviashvili equation
 - ▶ Potential Kadomtsev-Petviashvili equation
 - ▶ Generalized Zakharov-Kuznetsov equation

Examples of Conservation Laws

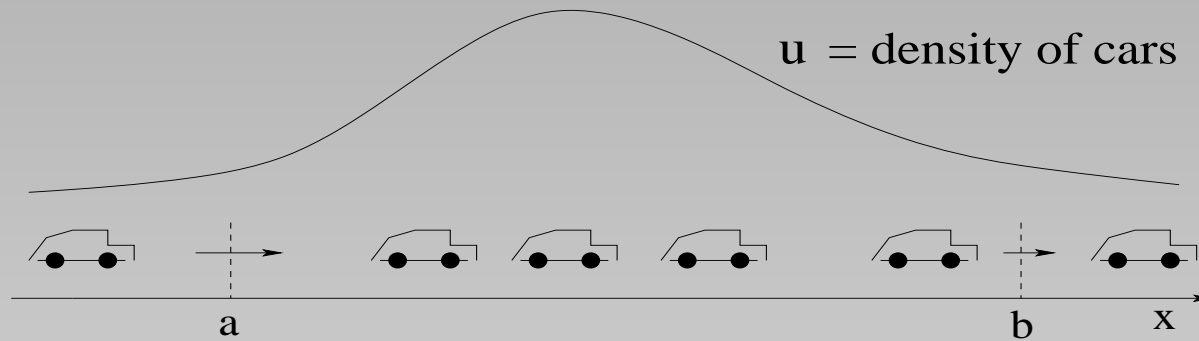
Example 1: Traffic Flow



Modeling the density of cars (Bressan, 2009)

$u(x, t)$ density of cars on a highway (e.g., number of cars per 100 meters).

$s(u)$ mean (equilibrium) speed of the cars (depends on the density).



Change in number of cars in segment $[a, b]$ equals the difference between cars entering at a and leaving at b during time interval $[t_1, t_2]$:

$$\int_a^b (u(x, t_2) - u(x, t_1)) dx = \int_{t_1}^{t_2} (J(a, t) - J(b, t)) dt$$

$$\int_a^b \left(\int_{t_1}^{t_2} u_t(x, t) dt \right) dx = - \int_{t_1}^{t_2} \left(\int_a^b J_x(x, t) dx \right) dt$$

where $J(x, t) = u(x, t)s(u(x, t))$ is the traffic flow (e.g., in cars per hour) at location x and time t .

Then, $\int_a^b \int_{t_1}^{t_2} (u_t + J_x) dt dx = 0$ holds $\forall (a, b, t_1, t_2)$

Yields the conservation law:

$$\boxed{u_t + [s(u) u]_x = 0} \quad \text{or} \quad \boxed{D_t \rho + D_x J = 0}$$

$\rho = u$ is the conserved density;

$J(u) = s(u) u$ is the associated flux.

A simple Lighthill-Whitham-Richards model:

$$s(u) = s_{\max} \left(1 - \frac{u}{u_{\max}} \right), \quad 0 \leq u \leq u_{\max}$$

s_{\max} is posted road speed, u_{\max} is the jam density.

Example 2: Fluid and Gas Dynamics

Euler equations for a compressible, non-viscous fluid:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + \nabla \cdot (\mathbf{u} \otimes (\rho \mathbf{u})) + \nabla p = 0$$

$$E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0$$

or, in components

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$(\rho u_i)_t + \nabla \cdot (\rho u_i \mathbf{u} + p \mathbf{e}_i) = 0 \quad (i = 1, 2, 3)$$

$$E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0$$

Express conservation of mass, momentum, energy.

\otimes is the dyadic product.

ρ is the mass density.

$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3$ is the velocity.

p is the pressure $p(\rho, e)$.

E is the energy density per unit volume:

$$E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e.$$

e is internal energy density per unit of mass
(related to temperature).

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(N)})$
In examples: $\mathbf{u} = (u, v, \theta, h, \dots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kxly} = \frac{\partial^{k+l} u}{\partial x^k \partial y^l}$, etc.

Examples: $u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$

$$u_{xxyyyy} = u_{2x4y} = \frac{\partial^6 u}{\partial x^2 \partial y^4}$$

- *Differential functions*

Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

- *Total derivatives:* D_t, D_x, D_y, \dots

Example: Let $f = uvv_x + x^2u_x^3v_x + u_xv_{xx}$

Then

$$\begin{aligned}
 D_x f &= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} \\
 &\quad + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}} \\
 &= 2xu_x^3v_x + u_x(vv_x) + u_{xx}(3x^2u_x^2v_x + v_{xx}) \\
 &\quad + v_x(uv_x) + v_{xx}(uv + x^2u_x^3) + v_{xxx}(u_x) \\
 &= 2xu_x^3v_x + vu_xv_x + 3x^2u_x^2v_xu_{xx} + u_{xx}v_{xx} \\
 &\quad + uv_x^2 + uvv_{xx} + x^2u_x^3v_{xx} + u_xv_{xxx}
 \end{aligned}$$

Conservation Laws for Nonlinear PDEs

- System of evolution equations of order M

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))$$

with $\mathbf{u} = (u, v, w, \dots)$ and $\mathbf{x} = (x, y, z)$.

- Conservation law in (1+1)-dimensions

$$D_t \rho + D_x J \doteq 0$$

where the dot means evaluated on the PDE.
Conserved density ρ and flux J .

$$P = \int_{-\infty}^{\infty} \rho dx = \text{constant in time}$$

if J vanishes at $\pm\infty$.

- Conservation law in (2+1)-dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$.

- Conservation law in (3+1)-dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 + D_z J_3 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$.

Reasons for Computing Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...).
- Testing of complete integrability and application of Inverse Scattering Transform.
- Testing of numerical integrators.
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, ...).
- Verify the closure of a model.

Examples of PDEs with Conservation Laws

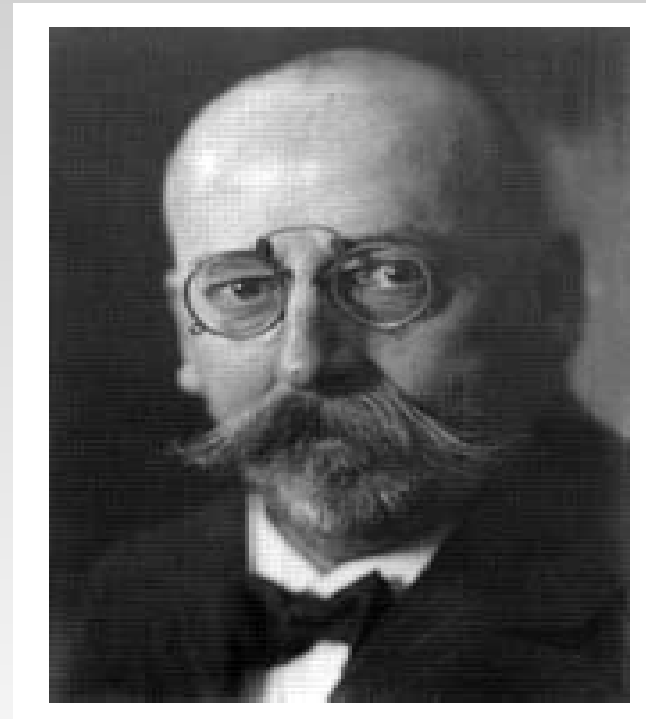
Example 1: KdV Equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad \boxed{u_t + 6uu_x + u_{xxx} = 0}$$

shallow water waves, ion-acoustic waves in plasmas



Diederik Korteweg



Gustav de Vries

Dilation Symmetry

$$u_t + 6uu_x + u_{xxx} = 0$$

has dilation (scaling) symmetry $(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$
 λ is an arbitrary parameter.

Notion of **weight**: $W(x) = -1$, thus, $W(D_x) = 1$.

$$W(t) = -3, \text{ hence, } W(D_t) = 3.$$

$$W(u) = 2.$$

Notion of **rank** (total weight of a monomial).

Examples: $\text{Rank}(u^3) = \text{Rank}(3u_x^2) = 6$.

$$\text{Rank}(u^3 u_{xx}) = 10.$$

Key Observation: Scaling Invariance

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

$$\boxed{D_t \rho + D_x J \doteq 0}$$

is uniform in rank.

Hence,

$$\text{Rank}(\rho) + \text{Rank}(D_t) = \text{Rank}(J) + \text{Rank}(D_x)$$

- First six (of infinitely many) conservation laws:

$$D_t(u) + D_x(3u^2 + u_{xx}) \doteq 0$$

$$D_t(u^2) + D_x(4u^3 - u_x^2 + 2uu_{xx}) \doteq 0$$

$$D_t\left(u^3 - \frac{1}{2}u_x^2\right)$$

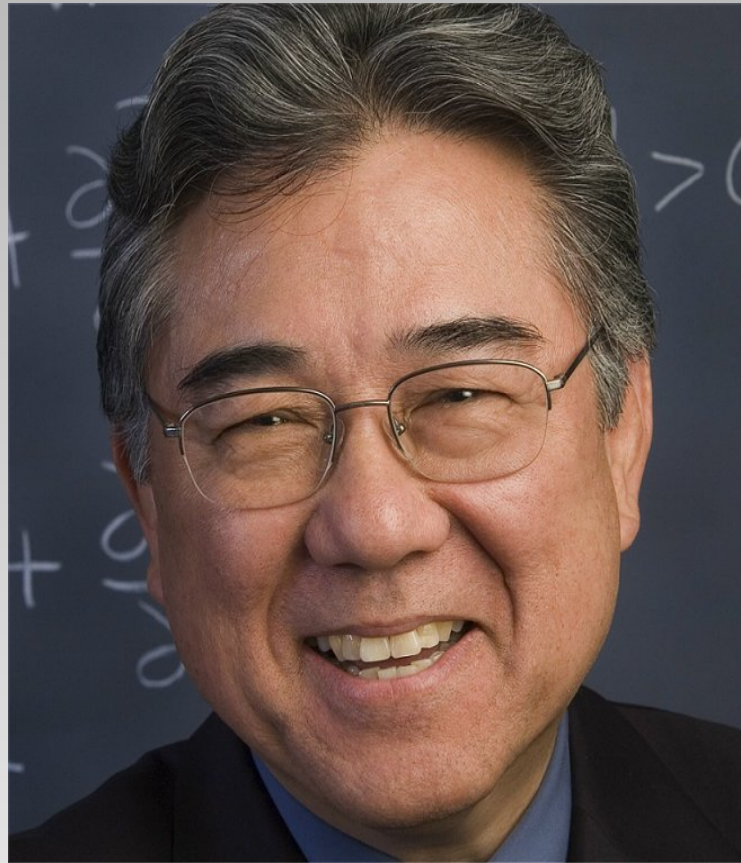
$$+ D_x\left(\frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}\right) \doteq 0$$

$$D_t\left(u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2\right) + D_x\left(\frac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx}\right)$$

$$+ 2u_x^2u_{xx} + \frac{16}{5}uu_{xx}^2 - 4uu_xu_{xxx} - \frac{1}{5}u_{xxx}^2 + \frac{2}{5}u_{xx}u_{4x}) \doteq 0$$

$$\begin{aligned}
& D_t \left(u^5 - 5 u^2 u_x^2 + u u_{xx}^2 - \frac{1}{14} u_{xxx}^2 \right) \\
& + D_x \left(5 u^6 - 40 u^3 u_x^2 - \dots - \frac{1}{7} u_{xxx} u_{5x} \right) \doteq 0 \\
& D_t \left(u^6 - 10 u^3 u_x^2 - \frac{5}{6} u_x^4 + 3 u^2 u_{xx}^2 \right. \\
& \quad \left. + \frac{10}{21} u_{xx}^3 - \frac{3}{7} u u_{xxx}^2 + \frac{1}{42} u_{4x}^2 \right) \\
& + D_x \left(\frac{36}{7} u^7 - 75 u^4 u_x^2 - \dots + \frac{1}{21} u_{4x} u_{6x} \right) \doteq 0
\end{aligned}$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966



Robert Miura

- Conservation law explicitly dependent on t and x :

$$D_t \left(tu^2 - \frac{1}{3}xu \right)$$

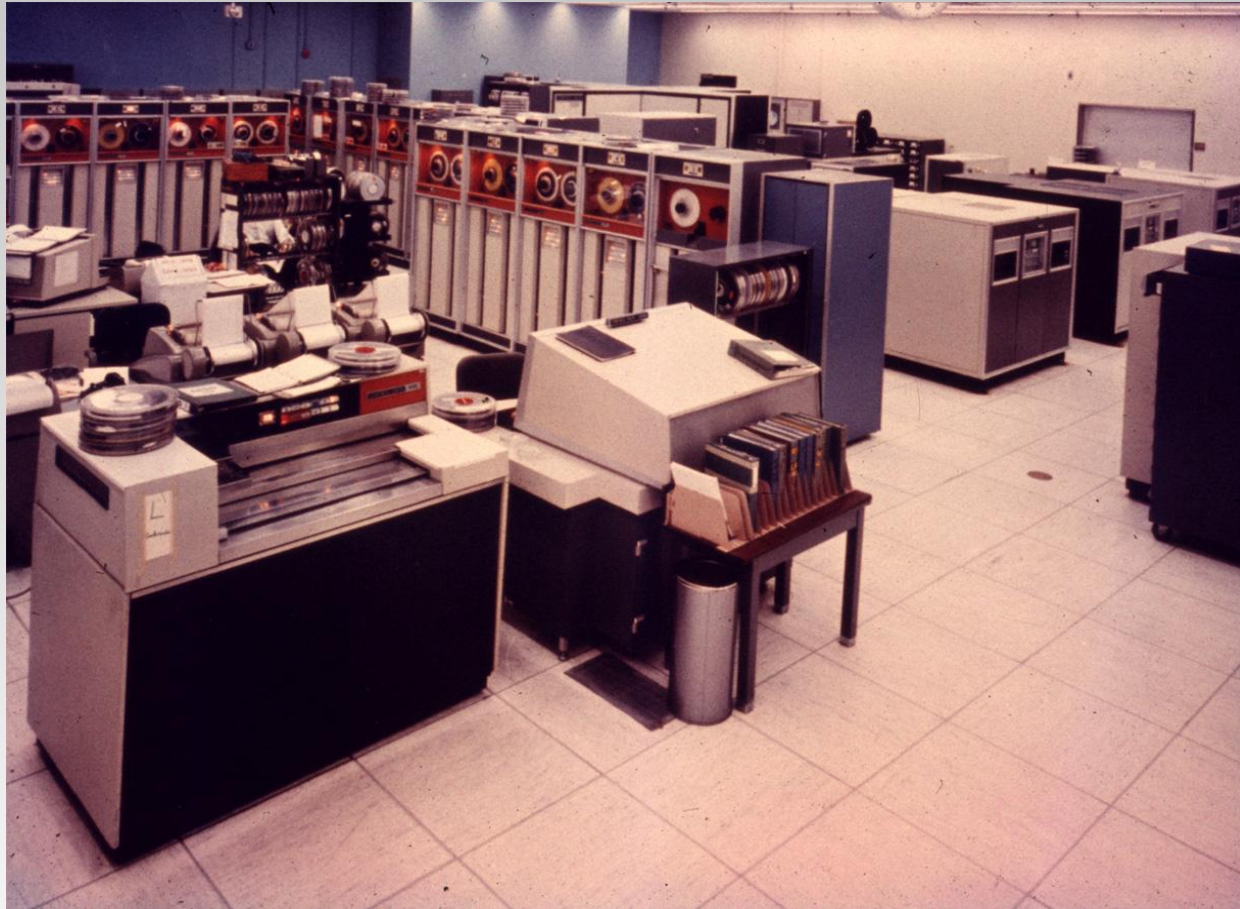
$$+ D_x \left(4tu^3 - xu^2 + \frac{1}{3}u_x - tu_x^2 + 2tuu_{xx} - \frac{1}{3}xu_{xx} \right) \doteq 0$$

- First five: IBM 7094 computer with FORMAC (1966) → storage space problem!



IBM 7094 Computer

- First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)
→ large integers problem!



Control Data CDC-6600

Other Scaling Invariant Quantities

- **Generalized Symmetries:**

$G(x, t, u^{(N)})$ is an N th order *generalized symmetry* iff it leaves $u_t = F(x, t, u^{(M)})$ invariant for the replacement $u \rightarrow u + \epsilon G$, $u_{ix} \rightarrow u_{ix} + \epsilon D_x^i G$, within order ϵ :

$$D_t(u + \epsilon G) \doteq F(u + \epsilon G)$$

must hold up to order ϵ .

- **Defining equation:**

$$D_t G \doteq F'(u)[G]$$

$F'(u)[G]$ is the Fréchet derivative of $F(u)$ in the

direction of G :

$$F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0} = \sum_{i=0}^M (D_x^i G) \frac{\partial F}{\partial u_{ix}}$$

- First 4 generalized symmetries of $u_t = 6uu_x + u_{3x}$

$$G^{(1)} = u_x$$

$$G^{(2)} = 6uu_x + u_{xxx}$$

$$G^{(3)} = 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x}$$

$$G^{(4)} = 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{xxx} \\ + 70u_{xx}u_{xxx} + 42u_xu_{4x} + 14uu_{5x} + u_{7x}$$

Generalized symmetries are invariant under the scaling symmetry.

- **Recursion Operator:**

A recursion operator \mathcal{R} connects symmetries

$$G^{(j+s)} = \mathcal{R}G^{(j)}, \quad j = 1, 2, \dots$$

s is seed ($s = 1$ in simplest case).

- **Defining equation:**

$$\boxed{D_t \mathcal{R} + [\mathcal{R}, F'(u)] \doteq 0}$$

Explicitly,

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[F] + \mathcal{R} \circ F'(u) - F'(u) \circ \mathcal{R} \doteq 0$$

where \circ stands for composition, and $\mathcal{R}'[F]$ is the Fréchet derivative of \mathcal{R} in the direction of F :

$$\mathcal{R}'[F] = \sum_{i=0}^n (D_x^i F) \frac{\partial \mathcal{R}}{\partial u_{ix}}$$

- Recursion operator (KdV equation):

$$\mathcal{R} = D_x^2 + 2uI + 2D_x u D_x^{-1} = D_x^2 + 4uI + 2u_x D_x^{-1}$$

- For example,

$$\mathcal{R}u_x = (D_x^2 + 2uI + 2D_x u D_x^{-1})u_x = 6uu_x + u_{3x}$$

$$\begin{aligned} \mathcal{R}(6uu_x + u_{3x}) &= (D_x^2 + 2uI + 2D_x u D_x^{-1})(6uu_x + u_{3x}) \\ &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x} \end{aligned}$$

Recursion operator is invariant under the scaling symmetry.

- **Lax Pair:**
- Key idea: Replace $u_t + 6uu_x + u_{xxx} = 0$ with a compatible linear system (Lax pair):

$$\psi_{xx} + (u - \lambda) \psi = 0$$

$$\psi_t + 4\psi_{xxx} + 6u\psi_x + 3u_x\psi = 0$$

ψ is eigenfunction; λ is constant eigenvalue
 ($\lambda_t = 0$) (isospectral)

- **Lax Pair $(\mathcal{L}, \mathcal{M})$ in Operator Form:**

$$\mathcal{L}\psi = \lambda\psi \quad \text{and} \quad \mathcal{D}_t\psi = \mathcal{M}\psi$$

- Require compatibility of both equations

$$\mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi \doteq 0$$

- **Defining Equation:**

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}$$

with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

Furthermore, $\mathcal{L}_t\psi = [\mathcal{D}_t, \mathcal{L}]\psi = \mathcal{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathcal{D}_t\psi$.

- **Lax pair for the KdV equation:**

$$\mathcal{L} = \mathcal{D}_x^2 + u\mathbf{I}$$

$$\mathcal{M} = - \left(4\mathcal{D}_x^3 + 6u\mathcal{D}_x + 3u_x\mathbf{I} \right)$$

Lax pair is invariant under the scaling symmetry.

Example 2:

The Zakharov-Kuznetsov Equation

$$u_t + \alpha u u_x + \beta (u_{xx} + u_{yy})_x = 0$$

models ion-sound solitons in a low pressure uniform magnetized plasma.

- Conservation laws:

$$D_t(u) + D_x\left(\frac{\alpha}{2}u^2 + \beta u_{xx}\right) + D_y(\beta u_{xy}) = 0$$

$$D_t(u^2) + D_x\left(\frac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) + D_y(-2\beta u_x u_y) = 0$$

- More conservation laws (ZK equation):

$$\begin{aligned}
& \mathbf{D}_t \left(u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + \mathbf{D}_x \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right. \\
& \left. + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{yy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\
& + \mathbf{D}_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0
\end{aligned}$$

$$\begin{aligned}
& \mathbf{D}_t \left(tu^2 - \frac{2}{\alpha} xu \right) + \mathbf{D}_x \left(t \left(\frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) \right. \\
& \left. - x \left(u^2 + \frac{2\beta}{\alpha} u_{xx} \right) + \frac{2\beta}{\alpha} u_x \right) + \mathbf{D}_y \left(-2\beta (tu_x u_y + \frac{1}{\alpha} x u_{xy}) \right) = 0
\end{aligned}$$

Methods for Computing Conservation Laws

- Use the Lax pair L and A , satisfying $[L, A] = 0$.

If $L = D_x + U$, $A = D_t + V$ then $V_x - U_t + [U, V] = 0$.

$\hat{L} = TLT^{-1}$ gives the densities, $\hat{A} = TAT^{-1}$ gives the fluxes.

- Use Noether's theorem (Lagrangian formulation) to generate conservation laws from symmetries (Ovsiannikov, Olver, Mahomed, Kara, etc.).
- Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Mason, Naz, etc.) require solving ODEs (or PDEs).

Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.
- Compute $D_t \rho$ with total derivative operator.
- Use variational derivative (Euler operator) to express exactness.
- Solve a (parametrized) linear system to find the undetermined coefficients.
- Use the homotopy operator to compute the flux (invert D_x or Div).

- Work with linearly independent pieces in finite dimensional spaces.
- Use linear algebra, calculus, and variational calculus (algorithmic).
- Implement the algorithm in **Mathematica**.

Tools from the Calculus of Variations

- Definition:

A differential function f is a **exact** iff $f = \text{Div } \mathbf{F}$.

Special case (1D): $f = D_x F$.

- Question: How can one test that $f = \text{Div } \mathbf{F}$?

- Theorem (exactness test):

$f = \text{Div } \mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \dots, N.$

N is the number of dependent variables.

The Euler operator annihilates divergences

- Euler operator in 1D (variable $u(x)$):

$$\begin{aligned}\mathcal{L}_{u(x)} &= \sum_{k=0}^M (-D_x)^k \frac{\partial}{\partial u_{kx}} \\ &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \dots\end{aligned}$$

- Euler operator in 2D (variable $u(x, y)$):

$$\begin{aligned}\mathcal{L}_{u(x,y)} &= \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial}{\partial u_{kx \ell y}} \\ &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} \\ &\quad + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} \dots\end{aligned}$$

- Question: How can one compute $\mathbf{F} = \text{Div}^{-1} f$?
- Theorem (integration by parts):
 - In 1D: If f is exact then

$$F = D_x^{-1} f = \int f dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \text{Div}^{-1} f = (\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f, \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f)$$

The homotopy operator inverts total derivatives and divergences!

- Homotopy Operator in 1D (variable x):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}$, $\mathbf{u}_x \rightarrow \lambda \mathbf{u}_x$, *etc.*

More general: $\mathbf{u} \rightarrow \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \rightarrow \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ *etc.*

- Homotopy Operator in 2D (variables x and y):

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(y)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where for dependent variable $u(x, y)$

$$\mathcal{I}_u^{(x)} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right. \\ \left. (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx \ell y}}$$

Application 1: The KdV Equation

$$u_t + 6uu_x + u_{xxx} = 0$$

- **Step 1: Compute the dilation symmetry**

$$\text{Set } (x, t, u) \rightarrow \left(\frac{x}{\lambda}, \frac{t}{\lambda^a}, \lambda^b u\right) = (\tilde{x}, \tilde{t}, \tilde{u})$$

Apply change of variables (chain rule)

$$\lambda^{-(a+b)} \tilde{u}_{\tilde{t}} + \lambda^{-(2b+1)} \tilde{u} \tilde{u}_{\tilde{x}} + \lambda^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0$$

Solve $a + b = 2b + 1 = b + 3$.

Solution: $a = 3$ and $b = 2$

$$(x, t, u) \rightarrow \left(\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u\right)$$

Compute the density of selected **rank**, say, 6.

- **Step 2: Determine the form of the density**

List powers of u , up to rank 6 : $[u, u^2, u^3]$

Differentiate with respect to x to increase the rank

u has weight 2 \rightarrow apply D_x^4

u^2 has weight 4 \rightarrow apply D_x^2

u^3 has weight 6 \rightarrow no derivatives needed

Apply the D_x derivatives

Remove total and highest derivative terms:

$$D_x^4 u \rightarrow \{u_{4x}\} \rightarrow \text{empty list}$$

$$D_x^2 u^2 \rightarrow \{u_x^2, uu_{xx}\} \rightarrow \{u_x^2\}$$

$$\text{since } uu_{xx} = (uu_x)_x - u_x^2$$

$$D_x^0 u^3 \rightarrow \{u^3\} \rightarrow \{u^3\}$$

Linearly combine the “building blocks”

Candidate density: $\rho = c_1 u^3 + c_2 u_x^2$

- **Step 3: Compute the coefficients c_i** Compute

$$\begin{aligned}
 D_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\
 &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^M \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t \\
 &= (3c_1 u^2 I + 2c_2 u_x D_x) u_t
 \end{aligned}$$

Substitute u_t by $-(6uu_x + u_{xxx})$

$$\begin{aligned}
 E &= -D_t \rho = (3c_1 u^2 I + 2c_2 u_x D_x)(6uu_x + u_{xxx}) \\
 &= 18c_1 u^3 u_x + 12c_2 u_x^3 + 12c_2 u u_x u_{xx} \\
 &\quad + 3c_1 u^2 u_{xxx} + 2c_2 u_x u_{4x}
 \end{aligned}$$

Apply the **Euler operator** (variational derivative)

$$\mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^m (-D_x)^k \frac{\partial}{\partial u_{kx}}$$

Here, E has order $m = 4$, thus

$$\begin{aligned}\mathcal{L}_u(x) E &= \frac{\partial E}{\partial u} - D_x \frac{\partial E}{\partial u_x} + D_x^2 \frac{\partial E}{\partial u_{xx}} - D_x^3 \frac{\partial E}{\partial u_{3x}} + D_x^4 \frac{\partial E}{\partial u_{4x}} \\ &= -18(c_1 + 2c_2)u_x u_{xx}\end{aligned}$$

This term must vanish!

So, $c_2 = -\frac{1}{2}c_1$. Set $c_1 = 1$ then $c_2 = -\frac{1}{2}$.

Hence, the **final form density** is

$$\rho = u^3 - \frac{1}{2}u_x^2$$

- **Step 4: Compute the flux J**

Method 1: Integrate by parts (simple cases)

Now,

$$E = 18u^3u_x + 3u^2u_{xxx} - 6u_x^3 - 6uu_xu_{xx} - u_xu_{xxxx}$$

Integration of $D_x J = E$ yields the **flux**

$$J = \frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}$$

Method 2: Use the homotopy operator

$$J = D_x^{-1} E = \int E dx = \mathcal{H}_{\mathbf{u}(x)} E = \int_0^1 (I_u E)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with integrand

$$I_u E = \sum_{k=1}^M \left(\sum_{i=0}^{k-1} u_{ix} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}$$

Here $M = 4$, thus

$$\begin{aligned} I_u E &= (uI)\left(\frac{\partial E}{\partial u_x}\right) + (u_x I - uD_x)\left(\frac{\partial E}{\partial u_{xx}}\right) \\ &\quad + (u_{xx}I - u_x D_x + uD_x^2)\left(\frac{\partial E}{\partial u_{xxx}}\right) \\ &\quad + (u_{xxx}I - u_{xx}D_x + u_x D_x^2 - uD_x^3)\left(\frac{\partial E}{\partial u_{4x}}\right) \\ &= (uI)(18u^3 + 18u_x^2 - 6uu_{xx} - u_{xxxx}) \\ &\quad + (u_x I - uD_x)(-6uu_x) \\ &\quad + (u_{xx}I - u_x D_x + uD_x^2)(3u^2) \\ &\quad + (u_{xxx}I - u_{xx}D_x + u_x D_x^2 - uD_x^3)(-u_x) \\ &= 18u^4 - 18uu_x^2 + 9u^2u_{xx} + u_{xx}^2 - 2u_xu_{xxx} \end{aligned}$$

Note: correct terms but incorrect coefficients!

Finally,

$$\begin{aligned} J &= \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(18\lambda^3 u^4 - 18\lambda^2 u u_x^2 + 9\lambda^2 u^2 u_{xx} + \lambda u_{xx}^2 \right. \\ &\quad \left. - 2\lambda u_x u_{xxx} \right) d\lambda \\ &= \frac{9}{2} u^4 - 6u u_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx} \end{aligned}$$

Final form of the flux:

$$J = \frac{9}{2} u^4 - 6u u_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx}$$

Application 2: Zakharov-Kuznetsov Equation

$$u_t + \alpha u u_x + \beta (u_{xx} + u_{yy})_x = 0$$

- Step 1: Compute the dilation invariance

ZK equation is invariant under scaling symmetry

$$(t, x, y, u) \rightarrow \left(\frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u \right) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})$$

λ is an arbitrary parameter.

- Hence, the weights of the variables are

$$W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.$$

- A conservation law is invariant under the scaling symmetry of the PDE.

$$W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.$$

For example,

$$\begin{aligned} & D_t \left(u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right. \\ & \left. + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{xy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\ & + D_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0 \end{aligned}$$

$$\text{Rank } (\rho) = 6, \quad \text{Rank } (J) = 8.$$

$$\text{Rank (conservation law)} = 9.$$

Compute the density of selected **rank**, say, 6.

- Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

$$\{u^3, u_x^2, uu_{xx}, u_y^2, uu_{yy}, u_xu_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}$$

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

$$\rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y$$

- Step 3: Compute the undetermined coefficients

Compute

$$\begin{aligned}
 D_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\
 &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{l=0}^{M_y} \frac{\partial \rho}{\partial u_{kx \ ly}} D_x^k D_y^l u_t \\
 &= \left(3c_1 u^2 I + 2c_2 u_x D_x + 2c_3 u_y D_y + c_4 (u_y D_x + u_x D_y) \right) u_t
 \end{aligned}$$

Substitute $u_t = -\left(\alpha u u_x + \beta (u_{xx} + u_{yy}) x \right)$.

$$\begin{aligned}
E = -D_t \rho = & 3c_1 u^2 (\alpha u u_x + \beta (u_{xx} + u_{xy})_x) \\
& + 2c_2 u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + 2c_3 u_y (\alpha u u_x \\
& + \beta (u_{xx} + u_{yy})_x)_y + c_4 (u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x \\
& + u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_y)
\end{aligned}$$

Apply the **Euler operator** (variational derivative)

$$\begin{aligned}
\mathcal{L}_{u(x,y)} E &= \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial E}{\partial u_{kx \ell y}} \\
&= -2 \left((3c_1 \beta + c_3 \alpha) u_x u_{yy} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} \right. \\
&\quad \left. + 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{xx} + 3(3c_1 \beta + c_2 \alpha) u_x u_{xx} \right) \\
&\equiv 0
\end{aligned}$$

Solve a parameterized **linear system** for the c_i :

$$3c_1\beta + c_3\alpha = 0, \quad c_4\alpha = 0, \quad 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \quad c_2 = -\frac{3\beta}{\alpha}, \quad c_3 = -\frac{3\beta}{\alpha}, \quad c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y$$

Final density of rank 6:

$$\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)$$

- Step 4: Compute the flux

Use the **homotopy operator** to invert **Div**:

$$\mathbf{J} = \text{Div}^{-1} E = \left(\mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right)$$

where

$$\mathcal{H}_{u(x,y)}^{(x)} E = \int_0^1 (I_u^{(x)} E) [\lambda u] \frac{d\lambda}{\lambda}$$

with

$$\begin{aligned} \mathcal{I}_u^{(x)} E = & \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right. \\ & \left. (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial E}{\partial u_{kx \ell y}} \end{aligned}$$

Similar formulas for $\mathcal{H}_{u(x,y)}^{(y)} E$ and $\mathcal{I}_u^{(y)} E$.

Let $A = \alpha u u_x + \beta(u_{xxx} + u_{xyy})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

Then,

$$\begin{aligned} \mathbf{J} &= \left(\mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right) \\ &= \left(\frac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u (3u_x^2 + u_y^2) \right. \\ &\quad + \frac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x \left(\frac{7}{2} u_{xyy} + 6u_{xxx} \right) \\ &\quad - \frac{\beta^2}{\alpha} u_y \left(4u_{xxy} + \frac{3}{2} u_{yyy} \right) + \frac{\beta^2}{\alpha} \left(3u_{xx}^2 + \frac{5}{2} u_{xy}^2 + \frac{3}{4} u_{yy}^2 \right) \\ &\quad + \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_{xy} - 4\beta u u_x u_y \\ &\quad - \frac{3\beta^2}{4\alpha} u (u_x 3y + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x (13u_{xxy} + 3u_{yyy}) \\ &\quad \left. - \frac{5\beta^2}{4\alpha} u_y (u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy} (u_{xx} + u_{yy}) \right) \end{aligned}$$

However, $\text{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (D_y\theta, -D_x\theta)$ is a **curl term**.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha} \left(3u(u_{xxy} + u_{yyy}) + 10u_x u_{xy} + 5u_y(3u_{yy} + u_{xx}) \right)$$

Shorter flux:

$$\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{K}$$

$$\begin{aligned} &= \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha} \left(u_{xx}^2 - u_{yy}^2 \right) \right. \\ &\quad \left. - \frac{6\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})), \right. \\ &\quad \left. 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right) \end{aligned}$$

Software Demonstration

Software packages in *Mathematica*

Codes are available via the Internet:

URL: <http://inside.mines.edu/~whereman/>

Additional Examples

- Manakov-Santini system

$$u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y = 0$$

$$v_{tx} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0$$

- Conservation laws for Manakov-Santini system:

$$\begin{aligned} & \mathbf{D}_t \left(f u_x v_x \right) + \mathbf{D}_x \left(f (u u_x v_x - u_x v_x v_y - u_y v_y) \right. \\ & \quad \left. - f' y (u_t + u u_x - u_x v_y) \right) + \mathbf{D}_y \left(f (u_x v_y + u_y v_x + u_x v_x^2) \right. \\ & \quad \left. + f' (u - y u_y - y u_x v_x) \right) = 0 \end{aligned}$$

where $f = f(t)$ is arbitrary.

Conservation laws – continued:

$$\begin{aligned}
 & \mathbf{D}_t \left(f(2u + v_x^2 - yu_x v_x) \right) + \mathbf{D}_x \left(f(u^2 + uv_x^2 + u_y v \right. \\
 & \quad \left. - v_y^2 - v_x^2 v_y - y(uu_x v_x - u_x v_x v_y - u_y v_y)) \right. \\
 & \quad \left. - f' y(v_t + uv_x - v_x v_y) + (f' - 2fx)y^2(u_t + uu_x - u_x v_y) \right) \\
 & + \mathbf{D}_y \left(f(v_x^3 + 2v_x v_y - u_x v - y(u_x v_x^2 + u_x v_y + u_y v_x)) \right. \\
 & \quad \left. + f'(v - y(2u + v_y + v_x^2)) + (f' y^2 - 2fx)(u_x v_x + u_y) \right) = 0
 \end{aligned}$$

where $f = f(t)$ is arbitrary.

There are three additional conservation laws.

- (2+1)-dimensional Camassa-Holm equation

$$(\alpha u_t + \kappa u_x - u_{txx} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{yy} = 0$$

Interchange t with y

$$(\alpha u_y + \kappa u_x - u_{xxy} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{tt} = 0$$

Set $v = u_t$ to get

$$u_t = v$$

$$v_t = -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta u u_{xx} + 2u_{xx}^2 \\ + 3u_x u_{xxx} + u u_{4x}$$

- Conservation laws for the Camassa-Holm equation

$$\begin{aligned}
& \mathbf{D}_t \left(f u \right) + \mathbf{D}_x \left(\frac{1}{\alpha} f \left(\frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} \right) \right. \\
& \left. + \left(\frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) \left(\alpha u_t + \kappa u_x + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx} \right. \right. \\
& \left. \left. - u_{txx} \right) \right) + \mathbf{D}_y \left(\left(\frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) u_y - f' y u \right) = 0
\end{aligned}$$

$$\begin{aligned}
& \mathbf{D}_t \left(f y u \right) + \mathbf{D}_x \left(\frac{1}{\alpha} f y \left(\frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} \right) \right. \\
& \left. + y \left(\frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) \left(\alpha u_t + \kappa u_x + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx} \right. \right. \\
& \left. \left. - u_{txx} \right) \right) + \mathbf{D}_y \left(y \left(\frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) u_y + \left(\frac{1}{\alpha} f x - \frac{1}{2} f' y^2 \right) u \right) = 0
\end{aligned}$$

where $f = f(t)$ is an arbitrary function.

- Khoklov-Zabolotskaya equation

describes e.g., sound waves in nonlinear media

$$(u_t - uu_x)_x - u_{yy} - u_{zz} = 0$$

Conservation law:

$$\begin{aligned} & \mathbf{D}_t \left(fu \right) - \mathbf{D}_x \left(\frac{1}{2} fu^2 + (fx + g)(u_t - uu_x) \right) \\ & + \mathbf{D}_y \left((fx + g)u_y - (f_yx + g_y)u \right) \\ & + \mathbf{D}_z \left((fx + g)u_z - (f_zx + g_z)u \right) = 0 \end{aligned}$$

under the constraints $\Delta f = 0$ and $\Delta g = f_t$
where $f = f(t, y, z)$ and $g = g(t, y, z)$.

- Shallow water wave model (atmosphere)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \boldsymbol{\Omega} \times \mathbf{u} + \nabla(\theta h) - \frac{1}{2} h \nabla \theta = \mathbf{0}$$

$$\theta_t + \mathbf{u} \cdot (\nabla \theta) = 0$$

$$h_t + \nabla \cdot (\mathbf{u} h) = 0$$

where $\mathbf{u}(x, y, t)$, $\theta(x, y, t)$ and $h(x, y, t)$.

- In components:

$$u_t + uu_x + vu_y - 2\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- First few conservation laws of SWW model:

$$\begin{aligned}
 \rho_{(1)} &= h & \mathbf{J}^{(1)} &= h \begin{pmatrix} u \\ v \end{pmatrix} \\
 \rho_{(2)} &= h \theta & \mathbf{J}^{(2)} &= h \theta \begin{pmatrix} u \\ v \end{pmatrix} \\
 \rho_{(3)} &= h \theta^2 & \mathbf{J}^{(3)} &= h \theta^2 \begin{pmatrix} u \\ v \end{pmatrix} \\
 \rho_{(4)} &= h (u^2 + v^2 + h\theta) & \mathbf{J}^{(4)} &= h \begin{pmatrix} u (u^2 + v^2 + 2h\theta) \\ v (v^2 + u^2 + 2h\theta) \end{pmatrix} \\
 \rho_{(5)} &= \theta (2\Omega + v_x - u_y) & \mathbf{J}^{(5)} &= \frac{1}{2} \theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}
 \end{aligned}$$

- More general conservation laws for SWW model:

$$D_t \left(f(\theta) h \right) + D_x \left(f(\theta) h u \right) + D_y \left(f(\theta) h v \right) = 0$$

$$\begin{aligned} & D_t \left(g(\theta) (2\Omega + v_x - u_x) \right) \\ & + D_x \left(\frac{1}{2} g(\theta) (4\Omega u - 2u u_y + 2u v_x - h \theta_y) \right) \\ & + D_y \left(\frac{1}{2} g(\theta) (4\Omega v - 2u_y v + 2v v_x + h \theta_x) \right) = 0 \end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$.

- Kadomtsev-Petviashvili (KP) equation

$$(u_t + \alpha u u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

Equation be written as a conservation law

$$D_t(u_x) + D_x(\alpha u u_x + u_{xxx}) + D_y(\sigma^2 u_y) = 0.$$

Exchange y and t and set $u_t = v$

$$u_t = v$$

$$v_t = -\frac{1}{\sigma^2} (u_{xy} + \alpha u_x^2 + \alpha u u_{xx} + u_{xxx})$$

- Examples of conservation laws for KP equation (explicitly dependent on t , x , and y)

$$D_t(xu_x) + D_x(3u^2 - u_{xx} - 6xu u_x + xu_{xxx}) + D_y(\alpha x u_y) = 0$$

$$D_t(yu_x) + D_x(y(\alpha u u_x + u_{xxx})) + D_y(\sigma^2(yu_y - u)) = 0$$

$$D_t(\sqrt{t}u) + D_x\left(\frac{\alpha}{2}\sqrt{t}u^2 + \sqrt{t}u_{xx} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{xxx} + \frac{\alpha\sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{xxx}\right) + D_y\left(x\sqrt{t}u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}}\right) = 0$$

- More general conservation laws for KP equation:

$$\begin{aligned}
& D_t \left(f u \right) + D_x \left(f \left(\frac{\alpha}{2} u^2 + u_{xx} \right) \right. \\
& \left. + \left(\frac{\sigma^2}{2} f' y^2 - f x \right) (u_t + \alpha u u_x + u_{3x}) \right) \\
& + D_y \left(\left(\frac{1}{2} f' y^2 - \sigma^2 f x \right) u_y - f' y u \right) = 0
\end{aligned}$$

$$\begin{aligned}
& D_t \left(f y u \right) + D_x \left(f y \left(\frac{\alpha}{2} u^2 + u_{xx} \right) \right. \\
& \left. + y \left(\frac{\sigma^2}{6} f' y^2 - f x \right) (u_t + \alpha u u_x + u_{3x}) \right) \\
& + D_y \left(y \left(\frac{1}{6} f' y^2 - \sigma^2 f x \right) u_y + \left(\sigma^2 f x - \frac{1}{2} f' y^2 \right) u \right) = 0
\end{aligned}$$

where $f(t)$ is arbitrary function.

- Potential KP equation

Replace u by u_x and integrate with respect to x .

$$u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0$$

- Examples of conservation laws

(not explicitly dependent on x, y, t):

$$D_t(u_x) + D_x\left(\frac{\alpha}{2}u_x^2 + u_{xxx}\right) + D_y(\sigma^2 u_y) = 0$$

$$D_t(u_x^2) + D_x\left(\frac{2\alpha}{3}u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy}\right) \\ + D_y(2\sigma^2 u_x u_y) = 0$$

Conservation laws for pKP equation – continued:

$$\begin{aligned} & D_t \left(u_x u_y \right) + D_x \left(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx} u_y - 2u_{xx} u_{xy} \right) \\ & + D_y \left(\sigma^2 u_y^2 - \frac{1}{3} u_x^3 - u_t u_x + u_{xx}^2 \right) = 0 \end{aligned}$$

$$\begin{aligned} & D_t \left(2\alpha u u_x u_{xx} + 3u u_{4x} - 3\sigma^2 u_y^2 \right) + D_x \left(2\alpha u_t u_x^2 + 3u_t^2 \right. \\ & \quad \left. - 2\alpha u u_x u_{tx} - 3u_{tx} u_{xx} + 3u_t u_{xxx} + 3u_x u_{txx} - 3u u_{txxx} \right) \\ & + D_y \left(6\sigma^2 u_t u_y \right) = 0 \end{aligned}$$

Various generalizations exist.

- Generalized Zakharov-Kuznetsov equation

$$u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0$$

where n is rational, $n \neq 0$.

Conservation laws:

$$D_t(u) + D_x\left(\frac{\alpha}{n+1}u^{n+1} + \beta u_{xx}\right) + D_y(\beta u_{xy}) = 0$$

$$D_t(u^2) + D_x\left(\frac{2\alpha}{n+2}u^{n+2} - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) \\ + D_y(-2\beta u_x u_y) = 0$$

- Third conservation law for gZK equation:

$$\begin{aligned}
& \mathbf{D}_t \left(u^{n+2} - \frac{(n+1)(n+2)\beta}{2\alpha} (u_x^2 + u_y^2) \right) \\
& + \mathbf{D}_x \left(\frac{(n+2)\alpha}{2(n+1)} u^{2(n+1)} + (n+2)\beta u^{n+1} u_{xx} \right. \\
& - (n+1)(n+2)\beta u^n (u_x^2 + u_y^2) + \frac{(n+1)(n+2)\beta^2}{2\alpha} (u_{xx}^2 - u_{yy}^2) \\
& \left. - \frac{(n+1)(n+2)\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})) \right) \\
& + \mathbf{D}_y \left((n+2)\beta u^{n+1} u_{xy} + \frac{(n+1)(n+2)\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0.
\end{aligned}$$

Conclusions and Future Work

- The power of Euler and homotopy operators:
 - ▶ Testing exactness
 - ▶ Integration by parts: D_x^{-1} and Div^{-1}

- Integration of non-exact expressions

Example: $f = u_x v + u v_x + u^2 u_{xx}$

$$\int f dx = uv + \int u^2 u_{xx} dx$$

- Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)

- Broader class of PDEs (beyond evolution type)

Example: short pulse equation (nonlinear optics)

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx}$$

with non-polynomial conservation law

$$D_t \left(\sqrt{1 + 6u_x^2} \right) - D_x \left(3u^2 \sqrt{1 + 6u_x^2} \right) = 0$$

- Continue the implementation in *Mathematica*
- Software: <http://inside.mines.edu/~whereman>

Thank You

Publications

1. D. Poole and W. Hereman, Symbolic computation of conservation laws for nonlinear partial differential equations in multiple space dimensions, *Journal of Symbolic Computation* (2011), 26 pages, in press.
2. W. Hereman, P. J. Adams, H. L. Eklund, M. S. Hickman, and B. M. Herbst, Direct Methods and Symbolic Software for Conservation Laws of Nonlinear Equations, In: *Advances of Nonlinear Waves and Symbolic Computation*, Ed.: Z. Yan, Nova Science Publishers, New York (2009), Chapter 2, pp. 19-79.

3. W. Hereman, M. Colagrosso, R. Sayers, A. Ringler, B. Deconinck, M. Nivala, and M. S. Hickman, Continuous and Discrete Homotopy Operators and the Computation of Conservation Laws. In: Differential Equations with Symbolic Computation, Eds.: D. Wang and Z. Zheng, Birkhäuser Verlag, Basel (2005), Chapter 15, pp. 249-285.
4. W. Hereman, B. Deconinck, and L. D. Poole, Continuous and discrete homotopy operators: A theoretical approach made concrete, Math. Comput. Simul. **74**(4-5), 352-360 (2007).

5. W. Hereman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quan. Chem. **106**(1), 278-299 (2006).