

Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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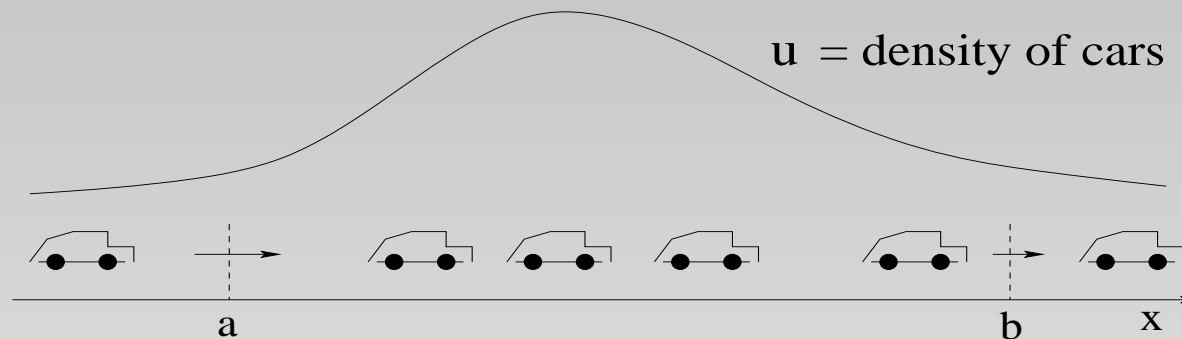
Outline

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- Other scaling invariant quantities (KdV equation)
- The Zakharov-Kuznetsov equation
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- Conclusions and future work

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Examples of Conservation Laws

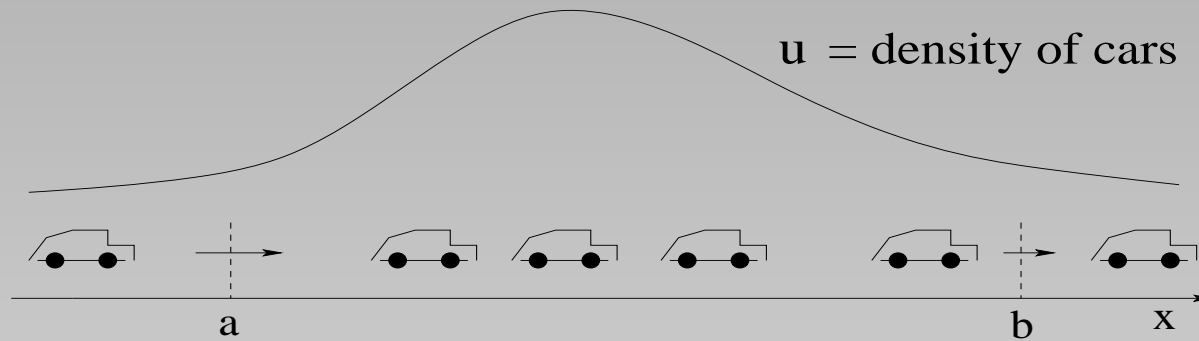
Example 1: Traffic Flow



Modeling the density of cars (Bressan, 2009)

$u(x, t)$ density of cars on a highway (e.g., number of cars per 100 meters).

$s(u)$ mean (equilibrium) speed of the cars (depends on the density).



Change in number of cars in segment $[a, b]$ equals the difference between cars entering at a and leaving at b during time interval $[t_1, t_2]$:

$$\int_a^b (u(x, t_2) - u(x, t_1)) dx = \int_{t_1}^{t_2} (J(a, t) - J(b, t)) dt$$

$$\int_a^b \left(\int_{t_1}^{t_2} u_t(x, t) dt \right) dx = - \int_{t_1}^{t_2} \left(\int_a^b J_x(x, t) dx \right) dt$$

where $J(x, t) = u(x, t)s(u(x, t))$ is the traffic flow (e.g., in cars per hour) at location x and time t .

Then, $\int_a^b \int_{t_1}^{t_2} (u_t + J_x) dt dx = 0$ holds $\forall (a, b, t_1, t_2)$

Yields the conservation law:

$$\boxed{u_t + [s(u) u]_x = 0} \quad \text{or} \quad \boxed{D_t \rho + D_x J = 0}$$

$\rho = u$ is the conserved density;

$J(u) = s(u) u$ is the associated flux.

A simple Lighthill-Whitham-Richards model:

$$s(u) = s_{\max} \left(1 - \frac{u}{u_{\max}} \right), \quad 0 \leq u \leq u_{\max}$$

s_{\max} is posted road speed, u_{\max} is the jam density.

Example 2: Fluid and Gas Dynamics

Euler equations for a compressible, non-viscous fluid:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + \nabla \cdot (\mathbf{u} \otimes (\rho \mathbf{u})) + \nabla p = 0$$

$$E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0$$

or, in components

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$(\rho u_i)_t + \nabla \cdot (\rho u_i \mathbf{u} + p \mathbf{e}_i) = 0 \quad (i = 1, 2, 3)$$

$$E_t + \nabla \cdot ((E + p) \mathbf{u}) = 0$$

Express conservation of mass, momentum, energy.

\otimes is the dyadic product.

ρ is the mass density.

$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3$ is the velocity.

p is the pressure $p(\rho, e)$.

E is the energy density per unit volume:

$$E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e.$$

e is internal energy density per unit of mass
(related to temperature).

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(N)})$
In examples: $\mathbf{u} = (u, v, \theta, h, \dots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kxly} = \frac{\partial^{k+l} u}{\partial x^k \partial y^l}$, etc.

Examples:

$$u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$$
$$u_{xxyyyy} = u_{2x4y} = \frac{\partial^6 u}{\partial x^2 \partial y^4}$$

- *Differential functions*

Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

- *Total derivatives:* D_t, D_x, D_y, \dots

Example: Let $f = uvv_x + x^2u_x^3v_x + u_xv_{xx}$

Then

$$\begin{aligned}
 D_x f &= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} \\
 &\quad + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}} \\
 &= 2xu_x^3v_x + u_x(vv_x) + u_{xx}(3x^2u_x^2v_x + v_{xx}) \\
 &\quad + v_x(uv_x) + v_{xx}(uv + x^2u_x^3) + v_{xxx}(u_x) \\
 &= 2xu_x^3v_x + vu_xv_x + 3x^2u_x^2v_xu_{xx} + u_{xx}v_{xx} \\
 &\quad + uv_x^2 + uvv_{xx} + x^2u_x^3v_{xx} + u_xv_{xxx}
 \end{aligned}$$

Conservation Laws for Nonlinear PDEs

- System of evolution equations of order M

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))$$

with $\mathbf{u} = (u, v, w, \dots)$ and $\mathbf{x} = (x, y, z)$.

- Conservation law in (1+1)-dimensions

$$D_t \rho + D_x J \doteq 0$$

where the dot means evaluated on the PDE.
Conserved density ρ and flux J .

$$P = \int_{-\infty}^{\infty} \rho dx = \text{constant in time}$$

if J vanishes at $\pm\infty$.

- Conservation law in (2+1)-dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$.

- Conservation law in (3+1)-dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 + D_z J_3 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$.

Reasons for Computing Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...).
- Testing of complete integrability and application of Inverse Scattering Transform.
- Testing of numerical integrators.
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, ...).
- Verify the closure of a model.

Examples of PDEs with Conservation Laws

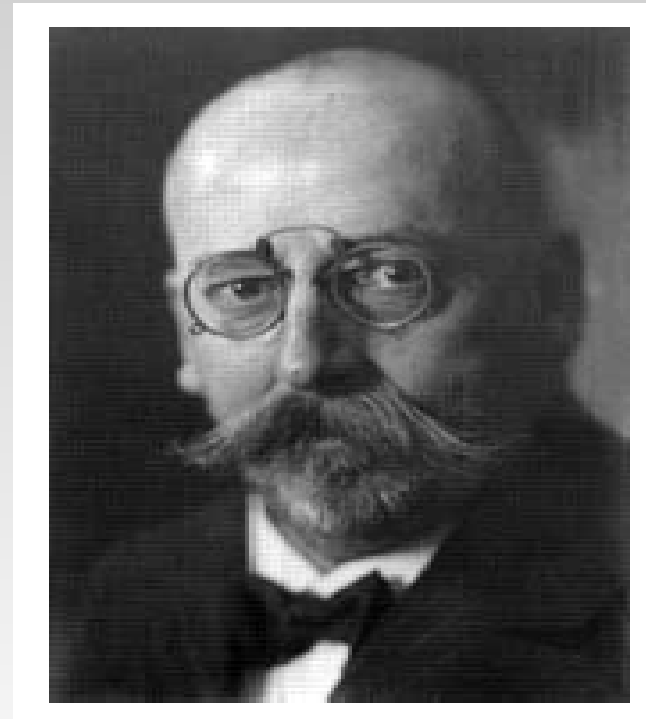
Example 1: KdV Equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad \boxed{u_t + 6uu_x + u_{xxx} = 0}$$

shallow water waves, ion-acoustic waves in plasmas



Diederik Korteweg



Gustav de Vries

Dilation Symmetry

$$u_t + 6uu_x + u_{xxx} = 0$$

has dilation (scaling) symmetry $(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$
 λ is an arbitrary parameter.

Notion of **weight**: $W(x) = -1$, thus, $W(D_x) = 1$.

$$W(t) = -3, \text{ hence, } W(D_t) = 3.$$

$$W(u) = 2.$$

Notion of **rank** (total weight of a monomial).

Examples: $\text{Rank}(u^3) = \text{Rank}(3u_x^2) = 6$.

$$\text{Rank}(u^3 u_{xx}) = 10.$$

Key Observation: Scaling Invariance

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

$$\boxed{D_t \rho + D_x J \doteq 0}$$

is uniform in rank.

Hence,

$$\text{Rank}(\rho) + \text{Rank}(D_t) = \text{Rank}(J) + \text{Rank}(D_x)$$

- First six (of infinitely many) conservation laws:

$$D_t(u) + D_x(3u^2 + u_{xx}) \doteq 0$$

$$D_t(u^2) + D_x(4u^3 - u_x^2 + 2uu_{xx}) \doteq 0$$

$$D_t\left(u^3 - \frac{1}{2}u_x^2\right)$$

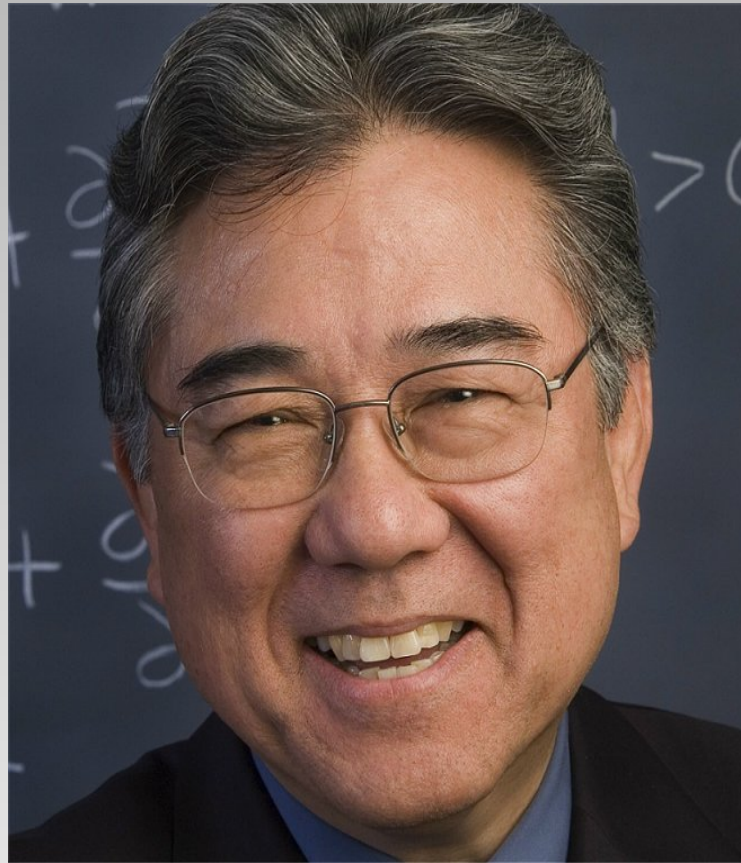
$$+ D_x\left(\frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}\right) \doteq 0$$

$$D_t\left(u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2\right) + D_x\left(\frac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx}\right)$$

$$+ 2u_x^2u_{xx} + \frac{16}{5}uu_{xx}^2 - 4uu_xu_{xxx} - \frac{1}{5}u_{xxx}^2 + \frac{2}{5}u_{xx}u_{4x}\right) \doteq 0$$

$$\begin{aligned}
& D_t \left(u^5 - 5 u^2 u_x^2 + u u_{xx}^2 - \frac{1}{14} u_{xxx}^2 \right) \\
& + D_x \left(5 u^6 - 40 u^3 u_x^2 - \dots - \frac{1}{7} u_{xxx} u_{5x} \right) \doteq 0 \\
& D_t \left(u^6 - 10 u^3 u_x^2 - \frac{5}{6} u_x^4 + 3 u^2 u_{xx}^2 \right. \\
& \quad \left. + \frac{10}{21} u_{xx}^3 - \frac{3}{7} u u_{xxx}^2 + \frac{1}{42} u_{4x}^2 \right) \\
& + D_x \left(\frac{36}{7} u^7 - 75 u^4 u_x^2 - \dots + \frac{1}{21} u_{4x} u_{6x} \right) \doteq 0
\end{aligned}$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966



Robert Miura

- Conservation law explicitly dependent on t and x :

$$D_t \left(tu^2 - \frac{1}{3}xu \right)$$

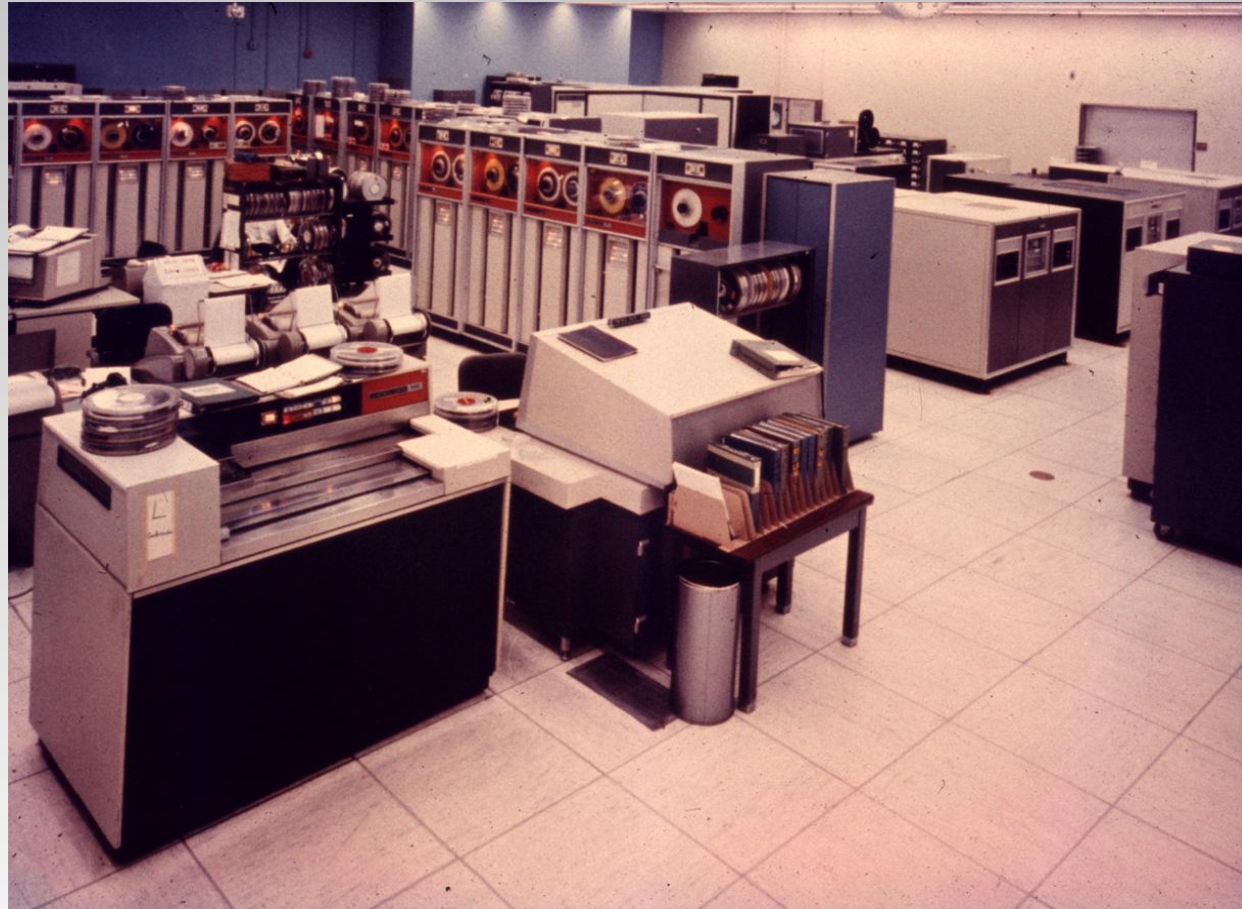
$$+ D_x \left(4tu^3 - xu^2 + \frac{1}{3}u_x - tu_x^2 + 2tuu_{xx} - \frac{1}{3}xu_{xx} \right) \doteq 0$$

- First five: IBM 7094 computer with FORMAC (1966) → storage space problem!



IBM 7094 Computer

- First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)
→ large integers problem!



Control Data CDC-6600

Other Scaling Invariant Quantities

- **Generalized Symmetries:**

$G(x, t, u^{(N)})$ is an N th order *generalized symmetry* iff it leaves $u_t = F(x, t, u^{(M)})$ invariant for the replacement $u \rightarrow u + \epsilon G$, $u_{ix} \rightarrow u_{ix} + \epsilon D_x^i G$, within order ϵ :

$$D_t(u + \epsilon G) \doteq F(u + \epsilon G)$$

must hold up to order ϵ .

- **Defining equation:**

$$D_t G \doteq F'(u)[G]$$

$F'(u)[G]$ is the Fréchet derivative of $F(u)$ in the

direction of G :

$$F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0} = \sum_{i=0}^M (D_x^i G) \frac{\partial F}{\partial u_{ix}}$$

- First 4 generalized symmetries of $u_t = 6uu_x + u_{3x}$

$$G^{(1)} = u_x$$

$$G^{(2)} = 6uu_x + u_{xxx}$$

$$G^{(3)} = 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{5x}$$

$$G^{(4)} = 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{xxx} \\ + 70u_{xx}u_{xxx} + 42u_xu_{4x} + 14uu_{5x} + u_{7x}$$

Generalized symmetries are invariant under the scaling symmetry.

- **Recursion Operator:**

A recursion operator \mathcal{R} connects symmetries

$$G^{(j+s)} = \mathcal{R}G^{(j)}, \quad j = 1, 2, \dots$$

s is seed ($s = 1$ in simplest case).

- **Defining equation:**

$$\boxed{D_t \mathcal{R} + [\mathcal{R}, F'(u)] \doteq 0}$$

Explicitly,

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[F] + \mathcal{R} \circ F'(u) - F'(u) \circ \mathcal{R} \doteq 0$$

where \circ stands for composition, and $\mathcal{R}'[F]$ is the Fréchet derivative of \mathcal{R} in the direction of F :

$$\mathcal{R}'[F] = \sum_{i=0}^n (D_x^i F) \frac{\partial \mathcal{R}}{\partial u_{ix}}$$

- Recursion operator (KdV equation):

$$\mathcal{R} = D_x^2 + 2uI + 2D_x u D_x^{-1} = D_x^2 + 4uI + 2u_x D_x^{-1}$$

- For example,

$$\mathcal{R}u_x = (D_x^2 + 2uI + 2D_x u D_x^{-1})u_x = 6uu_x + u_{3x}$$

$$\begin{aligned} \mathcal{R}(6uu_x + u_{3x}) &= (D_x^2 + 2uI + 2D_x u D_x^{-1})(6uu_x + u_{3x}) \\ &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x} \end{aligned}$$

Recursion operator is invariant under the scaling symmetry.

- **Lax Pair:**
- Key idea: Replace $u_t + 6uu_x + u_{xxx} = 0$ with a compatible linear system (Lax pair):

$$\psi_{xx} + (u - \lambda) \psi = 0$$

$$\psi_t + 4\psi_{xxx} + 6u\psi_x + 3u_x\psi = 0$$

ψ is eigenfunction; λ is constant eigenvalue
 ($\lambda_t = 0$) (isospectral)

- **Lax Pair $(\mathcal{L}, \mathcal{M})$ in Operator Form:**

$$\mathcal{L}\psi = \lambda\psi \quad \text{and} \quad \mathcal{D}_t\psi = \mathcal{M}\psi$$

- Require compatibility of both equations

$$\mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi \doteq 0$$

- **Defining Equation:**

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}$$

with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

Furthermore, $\mathcal{L}_t\psi = [\mathcal{D}_t, \mathcal{L}]\psi = \mathcal{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathcal{D}_t\psi$.

- **Lax pair for the KdV equation:**

$$\mathcal{L} = \mathcal{D}_x^2 + u\mathbf{I}$$

$$\mathcal{M} = - \left(4\mathcal{D}_x^3 + 6u\mathcal{D}_x + 3u_x\mathbf{I} \right)$$

Lax pair is invariant under the scaling symmetry.

Example 2:

The Zakharov-Kuznetsov Equation

$$u_t + \alpha u u_x + \beta (u_{xx} + u_{yy})_x = 0$$

models ion-sound solitons in a low pressure uniform magnetized plasma.

- Conservation laws:

$$D_t(u) + D_x\left(\frac{\alpha}{2}u^2 + \beta u_{xx}\right) + D_y(\beta u_{xy}) = 0$$

$$D_t(u^2) + D_x\left(\frac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) + D_y(-2\beta u_x u_y) = 0$$

- More conservation laws (ZK equation):

$$\begin{aligned}
& \mathbf{D}_t \left(u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + \mathbf{D}_x \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right. \\
& \left. + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{yy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\
& + \mathbf{D}_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0
\end{aligned}$$

$$\begin{aligned}
& \mathbf{D}_t \left(tu^2 - \frac{2}{\alpha} xu \right) + \mathbf{D}_x \left(t \left(\frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) \right. \\
& \left. - x \left(u^2 + \frac{2\beta}{\alpha} u_{xx} \right) + \frac{2\beta}{\alpha} u_x \right) + \mathbf{D}_y \left(-2\beta (tu_x u_y + \frac{1}{\alpha} x u_{xy}) \right) = 0
\end{aligned}$$

Methods for Computing Conservation Laws

- Use the Lax pair L and A , satisfying $[L, A] = 0$.

If $L = D_x + U$, $A = D_t + V$ then $V_x - U_t + [U, V] = 0$.

$\hat{L} = TLT^{-1}$ gives the densities, $\hat{A} = TAT^{-1}$ gives the fluxes.

- Use Noether's theorem (Lagrangian formulation) to generate conservation laws from symmetries (Ovsiannikov, Olver, Mahomed, Kara, etc.).
- Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Mason, Naz, etc.) require solving ODEs (or PDEs).

Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.
- Compute $D_t \rho$ with total derivative operator.
- Use variational derivative (Euler operator) to express exactness.
- Solve a (parametrized) linear system to find the undetermined coefficients.
- Use the homotopy operator to compute the flux (invert D_x or Div).

- Work with linearly independent pieces in finite dimensional spaces.
- Use linear algebra, calculus, and variational calculus (algorithmic).
- Implement the algorithm in **Mathematica**.

Tools from the Calculus of Variations

- Definition:

A differential function f is a **exact** iff $f = \text{Div } \mathbf{F}$.

Special case (1D): $f = D_x F$.

- Question: How can one test that $f = \text{Div } \mathbf{F}$?

- Theorem (exactness test):

$f = \text{Div } \mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \dots, N.$

N is the number of dependent variables.

The Euler operator annihilates divergences

- Euler operator in 1D (variable $u(x)$):

$$\begin{aligned}\mathcal{L}_{u(x)} &= \sum_{k=0}^M (-D_x)^k \frac{\partial}{\partial u_{kx}} \\ &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \dots\end{aligned}$$

- Euler operator in 2D (variable $u(x, y)$):

$$\begin{aligned}\mathcal{L}_{u(x,y)} &= \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial}{\partial u_{kx \ell y}} \\ &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} \\ &\quad + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} \dots\end{aligned}$$

- Question: How can one compute $\mathbf{F} = \text{Div}^{-1} f$?
- Theorem (integration by parts):
 - In 1D: If f is exact then

$$F = D_x^{-1} f = \int f dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \text{Div}^{-1} f = (\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f, \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f)$$

The homotopy operator inverts total derivatives and divergences!

- Homotopy Operator in 1D (variable x):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}$, $\mathbf{u}_x \rightarrow \lambda \mathbf{u}_x$, etc.

More general: $\mathbf{u} \rightarrow \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \rightarrow \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ etc.

- Homotopy Operator in 2D (variables x and y):

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(y)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where for dependent variable $u(x, y)$

$$\mathcal{I}_u^{(x)} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right. \\ \left. (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx \ell y}}$$

Application 1: The KdV Equation

$$u_t + 6uu_x + u_{xxx} = 0$$

- **Step 1: Compute the dilation symmetry**

$$\text{Set } (x, t, u) \rightarrow \left(\frac{x}{\lambda}, \frac{t}{\lambda^a}, \lambda^b u\right) = (\tilde{x}, \tilde{t}, \tilde{u})$$

Apply change of variables (chain rule)

$$\lambda^{-(a+b)} \tilde{u}_{\tilde{t}} + \lambda^{-(2b+1)} \tilde{u} \tilde{u}_{\tilde{x}} + \lambda^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0$$

Solve $a + b = 2b + 1 = b + 3$.

Solution: $a = 3$ and $b = 2$

$$(x, t, u) \rightarrow \left(\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u\right)$$

Compute the density of selected **rank**, say, 6.

- **Step 2: Determine the form of the density**

List powers of u , up to rank 6 : $[u, u^2, u^3]$

Differentiate with respect to x to increase the rank

u has weight 2 \rightarrow apply D_x^4

u^2 has weight 4 \rightarrow apply D_x^2

u^3 has weight 6 \rightarrow no derivatives needed

Apply the D_x derivatives

Remove total and highest derivative terms:

$$D_x^4 u \rightarrow \{u_{4x}\} \rightarrow \text{empty list}$$

$$D_x^2 u^2 \rightarrow \{u_x^2, uu_{xx}\} \rightarrow \{u_x^2\}$$

$$\text{since } uu_{xx} = (uu_x)_x - u_x^2$$

$$D_x^0 u^3 \rightarrow \{u^3\} \rightarrow \{u^3\}$$

Linearly combine the “building blocks”

Candidate density: $\rho = c_1 u^3 + c_2 u_x^2$

- **Step 3: Compute the coefficients c_i** Compute

$$\begin{aligned}
 D_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\
 &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^M \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t \\
 &= (3c_1 u^2 I + 2c_2 u_x D_x) u_t
 \end{aligned}$$

Substitute u_t by $-(6uu_x + u_{xxx})$

$$\begin{aligned}
 E &= -D_t \rho = (3c_1 u^2 I + 2c_2 u_x D_x)(6uu_x + u_{xxx}) \\
 &= 18c_1 u^3 u_x + 12c_2 u_x^3 + 12c_2 u u_x u_{xx} \\
 &\quad + 3c_1 u^2 u_{xxx} + 2c_2 u_x u_{4x}
 \end{aligned}$$

Apply the **Euler operator** (variational derivative)

$$\mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^m (-D_x)^k \frac{\partial}{\partial u_{kx}}$$

Here, E has order $m = 4$, thus

$$\begin{aligned}\mathcal{L}_u(x) E &= \frac{\partial E}{\partial u} - D_x \frac{\partial E}{\partial u_x} + D_x^2 \frac{\partial E}{\partial u_{xx}} - D_x^3 \frac{\partial E}{\partial u_{3x}} + D_x^4 \frac{\partial E}{\partial u_{4x}} \\ &= -18(c_1 + 2c_2)u_x u_{xx}\end{aligned}$$

This term must vanish!

So, $c_2 = -\frac{1}{2}c_1$. Set $c_1 = 1$ then $c_2 = -\frac{1}{2}$.

Hence, the **final form density** is

$$\rho = u^3 - \frac{1}{2}u_x^2$$

- **Step 4: Compute the flux J**

Method 1: Integrate by parts (simple cases)

Now,

$$E = 18u^3u_x + 3u^2u_{xxx} - 6u_x^3 - 6uu_xu_{xx} - u_xu_{xxxx}$$

Integration of $D_x J = E$ yields the **flux**

$$J = \frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}$$

Method 2: Use the homotopy operator

$$J = D_x^{-1} E = \int E dx = \mathcal{H}_{\mathbf{u}(x)} E = \int_0^1 (I_u E)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with integrand

$$I_u E = \sum_{k=1}^M \left(\sum_{i=0}^{k-1} u_{ix} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}$$

Here $M = 4$, thus

$$\begin{aligned} I_u E &= (uI)\left(\frac{\partial E}{\partial u_x}\right) + (u_x I - uD_x)\left(\frac{\partial E}{\partial u_{xx}}\right) \\ &\quad + (u_{xx}I - u_x D_x + uD_x^2)\left(\frac{\partial E}{\partial u_{xxx}}\right) \\ &\quad + (u_{xxx}I - u_{xx}D_x + u_x D_x^2 - uD_x^3)\left(\frac{\partial E}{\partial u_{4x}}\right) \\ &= (uI)(18u^3 + 18u_x^2 - 6uu_{xx} - u_{xxxx}) \\ &\quad + (u_x I - uD_x)(-6uu_x) \\ &\quad + (u_{xx}I - u_x D_x + uD_x^2)(3u^2) \\ &\quad + (u_{xxx}I - u_{xx}D_x + u_x D_x^2 - uD_x^3)(-u_x) \\ &= 18u^4 - 18uu_x^2 + 9u^2u_{xx} + u_{xx}^2 - 2u_xu_{xxx} \end{aligned}$$

Note: correct terms but incorrect coefficients!

Finally,

$$\begin{aligned} J &= \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(18\lambda^3 u^4 - 18\lambda^2 u u_x^2 + 9\lambda^2 u^2 u_{xx} + \lambda u_{xx}^2 \right. \\ &\quad \left. - 2\lambda u_x u_{xxx} \right) d\lambda \\ &= \frac{9}{2} u^4 - 6u u_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx} \end{aligned}$$

Final form of the flux:

$$J = \frac{9}{2} u^4 - 6u u_x^2 + 3u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx}$$

Application 2: Zakharov-Kuznetsov Equation

$$u_t + \alpha u u_x + \beta (u_{xx} + u_{yy})_x = 0$$

- Step 1: Compute the dilation invariance

ZK equation is invariant under scaling symmetry

$$(t, x, y, u) \rightarrow \left(\frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u \right) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})$$

λ is an arbitrary parameter.

- Hence, the weights of the variables are

$$W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.$$

- A conservation law is invariant under the scaling symmetry of the PDE.

$$W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.$$

For example,

$$\begin{aligned} & D_t \left(u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right. \\ & \left. + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{xy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\ & + D_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0 \end{aligned}$$

$$\text{Rank } (\rho) = 6, \quad \text{Rank } (J) = 8.$$

$$\text{Rank (conservation law)} = 9.$$

Compute the density of selected **rank**, say, 6.

- Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

$$\{u^3, u_x^2, uu_{xx}, u_y^2, uu_{yy}, u_xu_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}$$

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

$$\rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y$$

- Step 3: Compute the undetermined coefficients

Compute

$$\begin{aligned} D_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\ &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{l=0}^{M_y} \frac{\partial \rho}{\partial u_{kx \ ly}} D_x^k D_y^l u_t \\ &= \left(3c_1 u^2 I + 2c_2 u_x D_x + 2c_3 u_y D_y + c_4 (u_y D_x + u_x D_y) \right) u_t \end{aligned}$$

Substitute $u_t = -\left(\alpha u u_x + \beta (u_{xx} + u_{yy}) x \right)$.

$$\begin{aligned}
E = -D_t \rho = & 3c_1 u^2 (\alpha u u_x + \beta (u_{xx} + u_{xy})_x) \\
& + 2c_2 u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + 2c_3 u_y (\alpha u u_x \\
& + \beta (u_{xx} + u_{yy})_x)_y + c_4 (u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x \\
& + u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_y)
\end{aligned}$$

Apply the **Euler operator** (variational derivative)

$$\begin{aligned}
\mathcal{L}_{u(x,y)} E &= \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial E}{\partial u_{kx \ell y}} \\
&= -2 \left((3c_1 \beta + c_3 \alpha) u_x u_{yy} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} \right. \\
&\quad \left. + 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{xx} + 3(3c_1 \beta + c_2 \alpha) u_x u_{xx} \right) \\
&\equiv 0
\end{aligned}$$

Solve a parameterized **linear system** for the c_i :

$$3c_1\beta + c_3\alpha = 0, \quad c_4\alpha = 0, \quad 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \quad c_2 = -\frac{3\beta}{\alpha}, \quad c_3 = -\frac{3\beta}{\alpha}, \quad c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y$$

Final density of rank 6:

$$\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)$$

- Step 4: Compute the flux

Use the **homotopy operator** to invert **Div**:

$$\mathbf{J} = \text{Div}^{-1} E = \left(\mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right)$$

where

$$\mathcal{H}_{u(x,y)}^{(x)} E = \int_0^1 (I_u^{(x)} E) [\lambda u] \frac{d\lambda}{\lambda}$$

with

$$\mathcal{I}_u^{(x)} E = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right. \\ \left. (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial E}{\partial u_{kx \ell y}}$$

Similar formulas for $\mathcal{H}_{u(x,y)}^{(y)} E$ and $\mathcal{I}_u^{(y)} E$.

Let $A = \alpha u u_x + \beta(u_{xxx} + u_{xyy})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

Then,

$$\begin{aligned} \mathbf{J} &= \left(\mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right) \\ &= \left(\frac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u (3u_x^2 + u_y^2) \right. \\ &\quad + \frac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x \left(\frac{7}{2} u_{xyy} + 6u_{xxx} \right) \\ &\quad - \frac{\beta^2}{\alpha} u_y \left(4u_{xxy} + \frac{3}{2} u_{yyy} \right) + \frac{\beta^2}{\alpha} \left(3u_{xx}^2 + \frac{5}{2} u_{xy}^2 + \frac{3}{4} u_{yy}^2 \right) \\ &\quad + \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_{xy} - 4\beta u u_x u_y \\ &\quad - \frac{3\beta^2}{4\alpha} u (u_x 3y + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x (13u_{xxy} + 3u_{yyy}) \\ &\quad \left. - \frac{5\beta^2}{4\alpha} u_y (u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy} (u_{xx} + u_{yy}) \right) \end{aligned}$$

However, $\text{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (D_y\theta, -D_x\theta)$ is a **curl term**.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha} \left(3u(u_{xxy} + u_{yyy}) + 10u_x u_{xy} + 5u_y(3u_{yy} + u_{xx}) \right)$$

Shorter flux:

$$\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{K}$$

$$\begin{aligned} &= \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha} \left(u_{xx}^2 - u_{yy}^2 \right) \right. \\ &\quad \left. - \frac{6\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})), \right. \\ &\quad \left. 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right) \end{aligned}$$

Software Demonstration

Software packages in *Mathematica*

Codes are available via the Internet:

URL: <http://inside.mines.edu/~whereman/>

Additional Examples

- Manakov-Santini system

$$u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y = 0$$

$$v_{tx} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0$$

- Conservation laws for Manakov-Santini system:

$$\begin{aligned} & \mathbf{D}_t \left(f u_x v_x \right) + \mathbf{D}_x \left(f (u u_x v_x - u_x v_x v_y - u_y v_y) \right. \\ & \quad \left. - f' y (u_t + u u_x - u_x v_y) \right) + \mathbf{D}_y \left(f (u_x v_y + u_y v_x + u_x v_x^2) \right. \\ & \quad \left. + f' (u - y u_y - y u_x v_x) \right) = 0 \end{aligned}$$

where $f = f(t)$ is arbitrary.

Conservation laws – continued:

$$\begin{aligned}
 & \mathbf{D}_t \left(f(2u + v_x^2 - yu_x v_x) \right) + \mathbf{D}_x \left(f(u^2 + uv_x^2 + u_y v \right. \\
 & \quad \left. - v_y^2 - v_x^2 v_y - y(uu_x v_x - u_x v_x v_y - u_y v_y)) \right. \\
 & \quad \left. - f' y(v_t + uv_x - v_x v_y) + (f' - 2fx)y^2(u_t + uu_x - u_x v_y) \right) \\
 & + \mathbf{D}_y \left(f(v_x^3 + 2v_x v_y - u_x v - y(u_x v_x^2 + u_x v_y + u_y v_x)) \right. \\
 & \quad \left. + f'(v - y(2u + v_y + v_x^2)) + (f' y^2 - 2fx)(u_x v_x + u_y) \right) = 0
 \end{aligned}$$

where $f = f(t)$ is arbitrary.

There are three additional conservation laws.

- (2+1)-dimensional Camassa-Holm equation

$$(\alpha u_t + \kappa u_x - u_{txx} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{yy} = 0$$

Interchange t with y

$$(\alpha u_y + \kappa u_x - u_{xxy} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{tt} = 0$$

Set $v = u_t$ to get

$$u_t = v$$

$$v_t = -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta u u_{xx} + 2u_{xx}^2 \\ + 3u_x u_{xxx} + u u_{4x}$$

- Conservation laws for the Camassa-Holm equation

$$\begin{aligned}
& \mathbf{D}_t \left(f u \right) + \mathbf{D}_x \left(\frac{1}{\alpha} f \left(\frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} \right) \right. \\
& \left. + \left(\frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) \left(\alpha u_t + \kappa u_x + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx} \right. \right. \\
& \left. \left. - u_{txx} \right) \right) + \mathbf{D}_y \left(\left(\frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) u_y - f' y u \right) = 0
\end{aligned}$$

$$\begin{aligned}
& \mathbf{D}_t \left(f y u \right) + \mathbf{D}_x \left(\frac{1}{\alpha} f y \left(\frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} \right) \right. \\
& \left. + y \left(\frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) \left(\alpha u_t + \kappa u_x + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx} \right. \right. \\
& \left. \left. - u_{txx} \right) \right) + \mathbf{D}_y \left(y \left(\frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) u_y + \left(\frac{1}{\alpha} f x - \frac{1}{2} f' y^2 \right) u \right) = 0
\end{aligned}$$

where $f = f(t)$ is an arbitrary function.

- Khoklov-Zabolotskaya equation

describes e.g., sound waves in nonlinear media

$$(u_t - uu_x)_x - u_{yy} - u_{zz} = 0$$

Conservation law:

$$\begin{aligned} & \mathbf{D}_t \left(fu \right) - \mathbf{D}_x \left(\frac{1}{2} fu^2 + (fx + g)(u_t - uu_x) \right) \\ & + \mathbf{D}_y \left((fx + g)u_y - (f_yx + g_y)u \right) \\ & + \mathbf{D}_z \left((fx + g)u_z - (f_zx + g_z)u \right) = 0 \end{aligned}$$

under the constraints $\Delta f = 0$ and $\Delta g = f_t$
where $f = f(t, y, z)$ and $g = g(t, y, z)$.

- Shallow water wave model (atmosphere)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \boldsymbol{\Omega} \times \mathbf{u} + \nabla(\theta h) - \frac{1}{2} h \nabla \theta = \mathbf{0}$$

$$\theta_t + \mathbf{u} \cdot (\nabla \theta) = 0$$

$$h_t + \nabla \cdot (\mathbf{u} h) = 0$$

where $\mathbf{u}(x, y, t)$, $\theta(x, y, t)$ and $h(x, y, t)$.

- In components:

$$u_t + uu_x + vu_y - 2\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- First few conservation laws of SWW model:

$$\begin{aligned}
 \rho_{(1)} &= h & \mathbf{J}^{(1)} &= h \begin{pmatrix} u \\ v \end{pmatrix} \\
 \rho_{(2)} &= h\theta & \mathbf{J}^{(2)} &= h\theta \begin{pmatrix} u \\ v \end{pmatrix} \\
 \rho_{(3)} &= h\theta^2 & \mathbf{J}^{(3)} &= h\theta^2 \begin{pmatrix} u \\ v \end{pmatrix} \\
 \rho_{(4)} &= h(u^2 + v^2 + h\theta) & \mathbf{J}^{(4)} &= h \begin{pmatrix} u(u^2 + v^2 + 2h\theta) \\ v(v^2 + u^2 + 2h\theta) \end{pmatrix} \\
 \rho_{(5)} &= \theta(2\Omega + v_x - u_y) & \mathbf{J}^{(5)} &= \frac{1}{2}\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}
 \end{aligned}$$

- More general conservation laws for SWW model:

$$D_t \left(f(\theta) h \right) + D_x \left(f(\theta) h u \right) + D_y \left(f(\theta) h v \right) = 0$$

$$\begin{aligned} & D_t \left(g(\theta) (2\Omega + v_x - u_x) \right) \\ & + D_x \left(\frac{1}{2} g(\theta) (4\Omega u - 2u u_y + 2u v_x - h \theta_y) \right) \\ & + D_y \left(\frac{1}{2} g(\theta) (4\Omega v - 2u_y v + 2v v_x + h \theta_x) \right) = 0 \end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$.

- Kadomtsev-Petviashvili (KP) equation

$$(u_t + \alpha u u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

Equation be written as a conservation law

$$D_t(u_x) + D_x(\alpha u u_x + u_{xxx}) + D_y(\sigma^2 u_y) = 0.$$

Exchange y and t and set $u_t = v$

$$u_t = v$$

$$v_t = -\frac{1}{\sigma^2} (u_{xy} + \alpha u_x^2 + \alpha u u_{xx} + u_{xxxx})$$

- Examples of conservation laws for KP equation (explicitly dependent on t , x , and y)

$$D_t(xu_x) + D_x(3u^2 - u_{xx} - 6xu_x + xu_{xxx}) + D_y(\alpha xu_y) = 0$$

$$D_t(yu_x) + D_x(y(\alpha uu_x + u_{xxx})) + D_y(\sigma^2(yu_y - u)) = 0$$

$$D_t(\sqrt{t}u) + D_x\left(\frac{\alpha}{2}\sqrt{t}u^2 + \sqrt{t}u_{xx} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{xxx} + \frac{\alpha\sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{xxx}\right) + D_y\left(x\sqrt{t}u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}}\right) = 0$$

- More general conservation laws for KP equation:

$$\begin{aligned}
& D_t \left(f u \right) + D_x \left(f \left(\frac{\alpha}{2} u^2 + u_{xx} \right) \right. \\
& \left. + \left(\frac{\sigma^2}{2} f' y^2 - f x \right) (u_t + \alpha u u_x + u_{3x}) \right) \\
& + D_y \left(\left(\frac{1}{2} f' y^2 - \sigma^2 f x \right) u_y - f' y u \right) = 0
\end{aligned}$$

$$\begin{aligned}
& D_t \left(f y u \right) + D_x \left(f y \left(\frac{\alpha}{2} u^2 + u_{xx} \right) \right. \\
& \left. + y \left(\frac{\sigma^2}{6} f' y^2 - f x \right) (u_t + \alpha u u_x + u_{3x}) \right) \\
& + D_y \left(y \left(\frac{1}{6} f' y^2 - \sigma^2 f x \right) u_y + \left(\sigma^2 f x - \frac{1}{2} f' y^2 \right) u \right) = 0
\end{aligned}$$

where $f(t)$ is arbitrary function.

- Potential KP equation

Replace u by u_x and integrate with respect to x .

$$u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0$$

- Examples of conservation laws

(not explicitly dependent on x, y, t):

$$D_t(u_x) + D_x\left(\frac{\alpha}{2}u_x^2 + u_{xxx}\right) + D_y(\sigma^2 u_y) = 0$$

$$D_t(u_x^2) + D_x\left(\frac{2\alpha}{3}u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy}\right) \\ + D_y(2\sigma^2 u_x u_y) = 0$$

Conservation laws for pKP equation – continued:

$$\begin{aligned} & D_t \left(u_x u_y \right) + D_x \left(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx} u_y - 2u_{xx} u_{xy} \right) \\ & + D_y \left(\sigma^2 u_y^2 - \frac{1}{3} u_x^3 - u_t u_x + u_{xx}^2 \right) = 0 \end{aligned}$$

$$\begin{aligned} & D_t \left(2\alpha u u_x u_{xx} + 3u u_{4x} - 3\sigma^2 u_y^2 \right) + D_x \left(2\alpha u_t u_x^2 + 3u_t^2 \right. \\ & \quad \left. - 2\alpha u u_x u_{tx} - 3u_{tx} u_{xx} + 3u_t u_{xxx} + 3u_x u_{txx} - 3u u_{txxx} \right) \\ & + D_y \left(6\sigma^2 u_t u_y \right) = 0 \end{aligned}$$

Various generalizations exist.

- Generalized Zakharov-Kuznetsov equation

$$u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0$$

where n is rational, $n \neq 0$.

Conservation laws:

$$D_t(u) + D_x\left(\frac{\alpha}{n+1}u^{n+1} + \beta u_{xx}\right) + D_y(\beta u_{xy}) = 0$$

$$D_t(u^2) + D_x\left(\frac{2\alpha}{n+2}u^{n+2} - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) \\ + D_y(-2\beta u_x u_y) = 0$$

- Third conservation law for gZK equation:

$$\begin{aligned}
& \mathbf{D}_t \left(u^{n+2} - \frac{(n+1)(n+2)\beta}{2\alpha} (u_x^2 + u_y^2) \right) \\
& + \mathbf{D}_x \left(\frac{(n+2)\alpha}{2(n+1)} u^{2(n+1)} + (n+2)\beta u^{n+1} u_{xx} \right. \\
& - (n+1)(n+2)\beta u^n (u_x^2 + u_y^2) + \frac{(n+1)(n+2)\beta^2}{2\alpha} (u_{xx}^2 - u_{yy}^2) \\
& \left. - \frac{(n+1)(n+2)\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})) \right) \\
& + \mathbf{D}_y \left((n+2)\beta u^{n+1} u_{xy} + \frac{(n+1)(n+2)\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0.
\end{aligned}$$

Conclusions and Future Work

- The power of Euler and homotopy operators:
 - ▶ Testing exactness
 - ▶ Integration by parts: D_x^{-1} and Div^{-1}

- Integration of non-exact expressions

Example: $f = u_x v + u v_x + u^2 u_{xx}$

$$\int f dx = uv + \int u^2 u_{xx} dx$$

- Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)

- Broader class of PDEs (beyond evolution type)

Example: short pulse equation (nonlinear optics)

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx}$$

with non-polynomial conservation law

$$D_t \left(\sqrt{1 + 6u_x^2} \right) - D_x \left(3u^2 \sqrt{1 + 6u_x^2} \right) = 0$$

- Continue the implementation in *Mathematica*
- Software: <http://inside.mines.edu/~whereman>

Thank You

Publications

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