Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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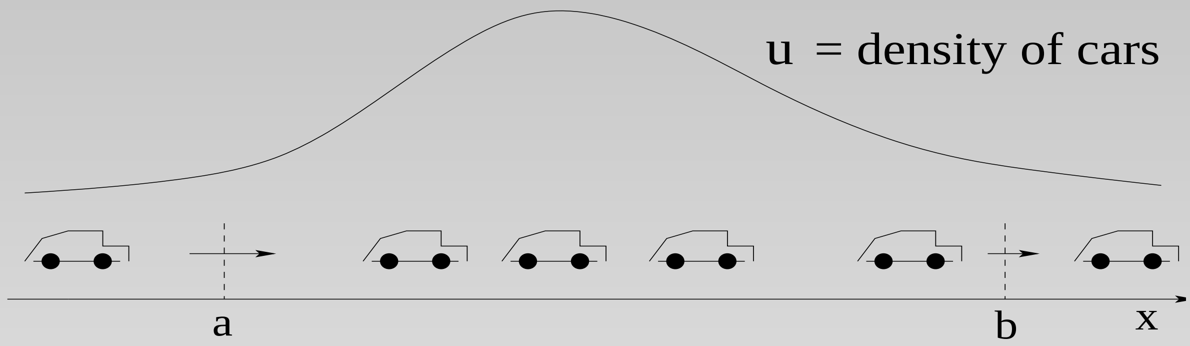
Examples of Conservation Laws

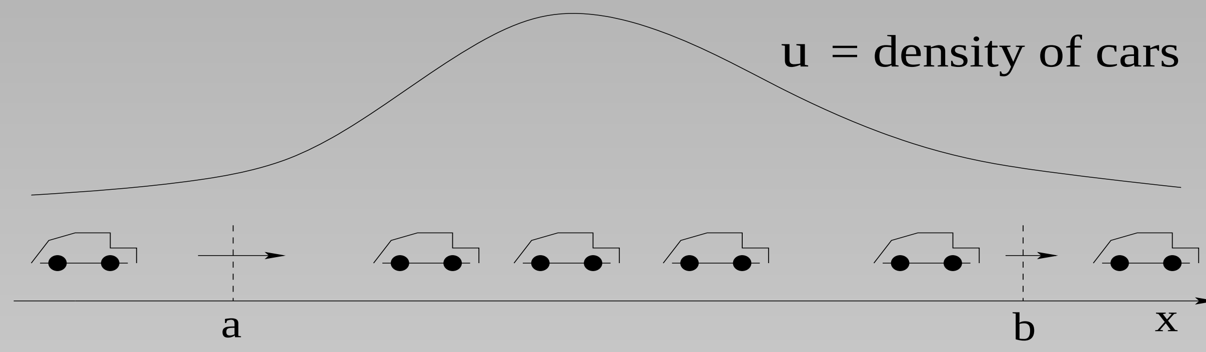
Example 1: Traffic Flow



Modeling the density of cars (Bressan, 2009)

$u(x,t)$ density of cars on a highway (e.g., number of cars per 100 meters).

$s(u)$ mean (equilibrium) speed of the cars (depends on the density).



Change in number of cars in segment $[a,b]$ equals the difference between cars entering at a and leaving at b during time interval $[t_1,t_2]$:

$$\int_a^b \left(u(x,t_2)-u(x,t_1)\right)\,dx \;=\; \int_{t_1}^{t_2}\left(J(a,t)-J(b,t)\right)\,dt$$

$$\int_a^b \left(\int_{t_1}^{t_2} u_t(x,t)\,dt\right)\,dx \;=\; -\int_{t_1}^{t_2}\left(\int_a^b J_x(x,t)\,dx\right)\,dt$$

where $J(x,t)=u(x,t)s(u(x,t))$ is the traffic flow (e.g., in cars per hour) at location x and time t.

Then, $\int_a^b \int_{t_1}^{t_2} (u_t + J_x)\, dt\, dx = 0$ holds $\forall (a, b, t_1, t_2)$

Yields the conservation law:

$$\boxed{u_t + [s(u)\, u]_x = 0} \quad \text{or} \quad \boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0}$$

$\rho = u$ is the conserved density;

$J(u) = s(u)\, u$ is the associated flux.

A simple Lighthill-Whitham-Richards model:

$$s(u) = s_{\max} \left(1 - \frac{u}{u_{\max}} \right), \qquad 0 \leq u \leq u_{\max}$$

$s_{\max}$ is posted road speed, $u_{\max}$ is the jam density.

Example 2: Fluid and Gas Dynamics

Euler equations for a compressible, non-viscous fluid:

$$\rho_t + \nabla.(\rho\,\mathbf{u}) = 0$$
$$(\rho\,\mathbf{u})_t + \nabla.\left(\mathbf{u} \otimes (\rho\mathbf{u})\right) + \nabla p = 0$$
$$E_t + \nabla.\left((E+p)\,\mathbf{u}\right) = 0$$

or, in components

$$\rho_t + \nabla.(\rho\,\mathbf{u}) = 0$$
$$(\rho\,u_i)_t + \nabla.(\rho\,u_i\,\mathbf{u} + p\,\mathbf{e}_i) = 0 \qquad (i = 1,2,3)$$
$$E_t + \nabla.\left((E+p)\,\mathbf{u}\right) = 0$$

Express conservation of mass, momentum, energy.

$\otimes$ is the dyadic product.

ρ is the mass density.

$\mathbf{u} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$ is the velocity.

p is the pressure $p(\rho, e)$.

E is the energy density per unit volume:
$E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e$.

e is internal energy density per unit of mass (related to temperature).

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$

- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$

  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}, \; u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.

  Examples: $u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$

  $u_{xx\,yyyy} = u_{2x\,4y} = \frac{\partial^6 u}{\partial x^2 y^4}$

- *Differential functions*

  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

- *Total derivatives*: $\mathrm{D}_t,\ \mathrm{D}_x, \mathrm{D}_y, \ldots$

Example: Let $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

Then

$$\begin{aligned}
\mathrm{D}_x f &= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} \\
&\quad + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}} \\
&= 2x u_x^3 v_x + u_x (v v_x) + u_{xx} (3x^2 u_x^2 v_x + v_{xx}) \\
&\quad + v_x (u v_x) + v_{xx} (uv + x^2 u_x^3) + v_{xxx} (u_x) \\
&= 2x u_x^3 v_x + v u_x v_x + 3x^2 u_x^2 v_x u_{xx} + u_{xx} v_{xx} \\
&\quad + u v_x^2 + u v v_{xx} + x^2 u_x^3 v_{xx} + u_x v_{xxx}
\end{aligned}$$

Conservation Laws for Nonlinear PDEs

- System of evolution equations of order M

$$\boxed{\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))}$$

  with $\mathbf{u} = (u, v, w, \ldots)$ and $\mathbf{x} = (x, y, z)$.
- Conservation law in (1+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J \doteq 0}$$

  where the dot means evaluated on the PDE.
  Conserved density ρ and flux J.

$$P = \int_{-\infty}^{\infty} \rho \, dx = \text{constant in time}$$

  if J vanishes at $\pm\infty$.

- Conservation law in (2+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$.

- Conservation law in (3+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$.

Reasons for Computing Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...).
- Testing of complete integrability and application of Inverse Scattering Transform.
- Testing of numerical integrators.
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, . . .).
- Verify the closure of a model.

Examples of PDEs with Conservation Laws

Example 1: KdV Equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad \boxed{u_t + 6uu_x + u_{xxx} = 0}$$

shallow water waves, ion-acoustic waves in plasmas

Diederik Korteweg

Gustav de Vries

Dilation Symmetry

$$\boxed{u_t + 6uu_x + u_{xxx} = 0}$$

has dilation (scaling) symmetry $(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$

λ is an arbitrary parameter.

Notion of weight: $W(x) = -1$, thus, $W(\mathrm{D}_x) = 1$.

$W(t) = -3$, hence, $W(\mathrm{D}_t) = 3$.

$W(u) = 2$.

Notion of rank (total weight of a monomial).

Examples: $\mathrm{Rank}(u^3) = \mathrm{Rank}(3u_x^2) = 6$.

$\mathrm{Rank}(u^3 u_{xx}) = 10$.

Key Observation: Scaling Invariance

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J \doteq 0}$$

is uniform in rank.

Hence,

$$\mathrm{Rank}(\rho) + \mathrm{Rank}(\mathrm{D}_t) = \mathrm{Rank}(J) + \mathrm{Rank}(\mathrm{D}_x)$$

- First six (of infinitely many) conservation laws:

$$\mathrm{D}_t\Big(u\Big) + \mathrm{D}_x\Big(3u^2 + u_{xx}\Big) \doteq 0$$

$$\mathrm{D}_t\Big(u^2\Big) + \mathrm{D}_x\Big(4u^3 - {u_x}^2 + 2uu_{xx}\Big) \doteq 0$$

$$\mathrm{D}_t\Big(u^3 - \frac{1}{2}u_x^2\Big)$$

$$+\,\mathrm{D}_x\Big(\tfrac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}\Big) \doteq 0$$

$$\mathrm{D}_t\Big(u^4 - 2uu_x^2 + \tfrac{1}{5}u_{xx}^2\Big) + \mathrm{D}_x\Big(\tfrac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx}$$

$$+2u_x^2u_{xx} + \tfrac{16}{5}uu_{xx}^2 - 4uu_xu_{xxx} - \tfrac{1}{5}u_{xxx}^2 + \tfrac{2}{5}u_{xx}u_{4x}\Big) \doteq 0$$

$$\mathrm{D}_t\left(u^5 - 5\,u^2u_x^2 + \,uu_{xx}^2 - \tfrac{1}{14}u_{xxx}^2\right)$$

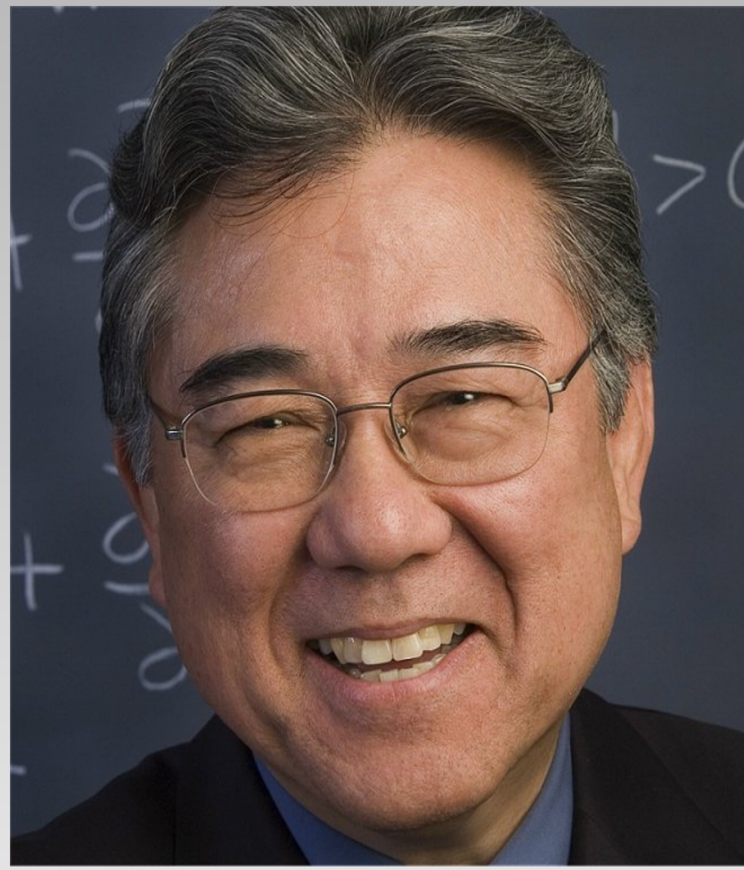
$$+\,\mathrm{D}_x\left(5u^6 - 40u^3u_x^2 - \,\ldots\, - \tfrac{1}{7}u_{xxx}u_{5x}\right) \doteq 0$$

$$\mathrm{D}_t\left(u^6 - 10\,u^3u_x^2 - \frac{5}{6}\,u_x^4 + 3\,u^2u_{xx}^2\right.$$

$$\left. + \tfrac{10}{21}\,u_{xx}^3 - \tfrac{3}{7}\,uu_{xxx}^2 + \tfrac{1}{42}\,u_{4x}^2\right)$$

$$+\,\mathrm{D}_x\left(\tfrac{36}{7}u^7 - 75u^4u_x^2 - \,\ldots\, + \tfrac{1}{21}u_{4x}u_{6x}\right) \doteq 0$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966



Robert Miura

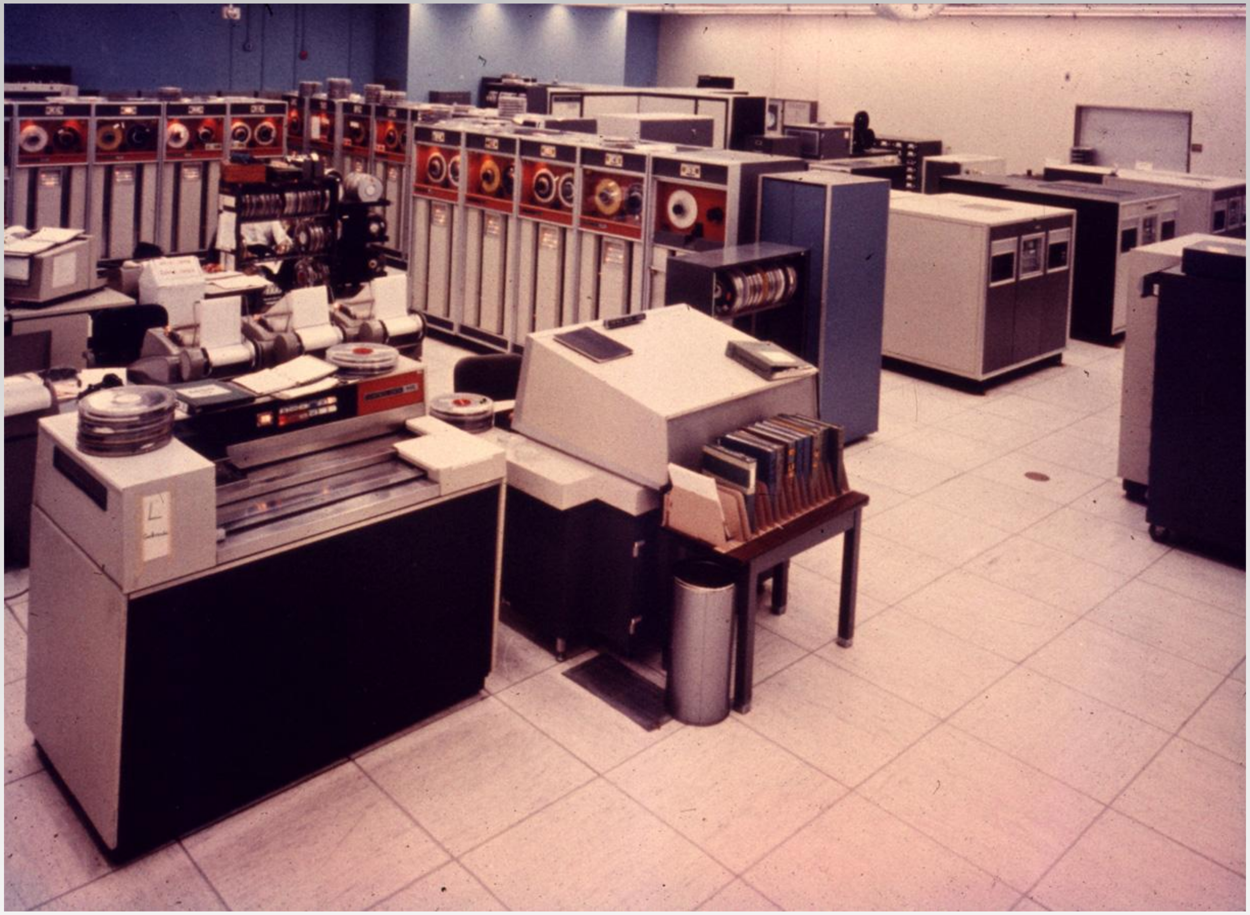
- Conservation law explicitly dependent on t and x:

$$\mathrm{D}_t\Big(tu^2 - \frac{1}{3}xu\Big)$$
$$+\mathrm{D}_x\Big(4tu^3 - xu^2 + \frac{1}{3}u_x - tu_x^2 + 2tuu_{xx} - \frac{1}{3}xu_{xx}\Big) \doteq 0$$

- First five: IBM 7094 computer with FORMAC (1966) $\longrightarrow$ storage space problem!

IBM 7094 Computer

- First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)
  $\longrightarrow$ large integers problem!



Control Data CDC-6600

Other Scaling Invariant Quantities

- **Generalized Symmetries:**

  $G(x, t, u^{(N)})$ is an $N\text{th}$ order *generalized symmetry* iff it leaves $u_t = F(x, t, u^{(M)})$ invariant for the replacement $u \to u + \epsilon G$, $u_{ix} \to u_{ix} + \epsilon \mathrm{D}_x^i G$, within order ϵ:

$$\mathrm{D}_t(u + \epsilon G) \doteq F(u + \epsilon G)$$

  must hold up to order ϵ.

- **Defining equation:**

$$\boxed{\mathrm{D}_t G \doteq F'(u)[G]}$$

  $F(u)'[G]$ is the Fréchet derivative of $F(u)$ in the direction of G:

$$F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0} = \sum_{i=0}^{M} (\mathrm{D}_x^i G) \frac{\partial F}{\partial u_{ix}}$$

- First 4 generalized symmetries of $u_t = 6uu_x + u_{3x}$

$$\begin{aligned}
G^{(1)} &= u_x \\
G^{(2)} &= 6uu_x + u_{xxx} \\
G^{(3)} &= 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x} \\
G^{(4)} &= 140u^3 u_x + 70u_x^3 + 280uu_x u_{xx} + 70u^2 u_{xxx} \\
&\quad +70u_{xx}u_{xxx} + 42u_x u_{4x} + 14uu_{5x} + u_{7x}
\end{aligned}$$

Generalized symmetries are invariant under the scaling symmetry.

- **Recursion Operator:**

  A *recursion operator* $\mathcal{R}$ connects symmetries

$$G^{(j+s)} = \mathcal{R}G^{(j)}, \; j = 1, 2, \ldots$$

  s is seed ($s = 1$ in simplest case).

- **Defining equation:**

$$\boxed{\mathrm{D}_t\mathcal{R} + [\mathcal{R}, F'(u)] \; \doteq \; 0}$$

  Explicitly,

$$\frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[F] + \mathcal{R} \circ F'(u) - F'(u) \circ \mathcal{R} \; \doteq \; 0$$

  where $\circ$ stands for composition, and $\mathcal{R}'[F]$ is the

Fréchet derivative of $\mathcal{R}$ in the direction of F:

$$\mathcal{R}'[F] = \sum_{i=0}^{n} (\mathrm{D}_x^i F) \, \frac{\partial \mathcal{R}}{\partial u_{ix}}$$

- Recursion operator (KdV equation):

$$\mathcal{R} = \mathrm{D}_x^2 + 2u\mathrm{I} + 2\mathrm{D}_x u \mathrm{D}_x^{-1} = \mathrm{D}_x^2 + 4u\mathrm{I} + 2u_x \mathrm{D}_x^{-1}$$

- For example,

$$\mathcal{R} u_x = (\mathrm{D}_x^2 + 2u\mathrm{I} + 2\mathrm{D}_x u \mathrm{D}_x^{-1}) u_x = 6uu_x + u_{3x}$$

$$\mathcal{R}(6uu_x + u_{3x}) = (\mathrm{D}_x^2 + 2u\mathrm{I} + 2\mathrm{D}_x u \mathrm{D}_x^{-1})(6uu_x + u_{3x})$$

$$= 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}$$

Recursion operator is invariant under the scaling symmetry.

- **Lax Pair:** Key idea: Replace $u_t + 6uu_x + u_{xxx} = 0$ with a compatible linear system (Lax pair):

$$\psi_{xx} + (u - \lambda)\,\psi = 0$$

$$\psi_t + 4\psi_{xxx} + 6u\psi_x + 3u_x\psi = 0$$

  ψ is eigenfunction; λ is constant eigenvalue $(\lambda_t = 0)$ (isospectral)

- **Lax Pair** $(\mathcal{L}, \mathcal{M})$ **in Operator Form**:

$$\mathcal{L}\psi = \lambda\psi \qquad \text{and} \qquad \mathrm{D}_t\psi = \mathcal{M}\psi$$

- Require compatibility of both equations

$$\mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi \doteq 0$$

- **Defining Equation**: $$\boxed{\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}}$$

  with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

  Furthermore, $\mathcal{L}_t \psi = [\mathrm{D}_t, \mathcal{L}]\psi = \mathrm{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathrm{D}_t\psi$.

- **Lax pair for the KdV equation:**

$$\mathcal{L} = \mathrm{D}_x^2 + u\,\mathbf{I}$$

$$\mathcal{M} = -\left(4\,\mathrm{D}_x^3 + 6u\mathrm{D}_x + 3u_x\,\mathbf{I}\right)$$

Lax pair is invariant under the scaling symmetry.

Example 2:
The Zakharov-Kuznetsov Equation

$$\boxed{u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0}$$

models ion-sound solitons in a low pressure uniform magnetized plasma.

- Conservation laws:

$$\mathrm{D}_t\Big(u\Big) + \mathrm{D}_x\Big(\tfrac{\alpha}{2}u^2 + \beta u_{xx}\Big) + \mathrm{D}_y\Big(\beta u_{xy}\Big) = 0$$

$$\mathrm{D}_t\Big(u^2\Big) + \mathrm{D}_x\Big(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\Big)$$
$$+ \mathrm{D}_y\Big(-2\beta u_x u_y\Big) = 0$$

- More conservation laws (ZK equation):

$$\mathrm{D}_t\Big(u^3 - \tfrac{3\beta}{\alpha}(u_x^2 + u_y^2)\Big) + \mathrm{D}_x\Big(3u^2(\tfrac{\alpha}{4}u^2 + \beta u_{xx}) - 6\beta u(u_x^2 + u_y^2)$$
$$+ \tfrac{3\beta^2}{\alpha}(u_{xx}^2 - u_{yy}^2) - \tfrac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}))\Big)$$
$$+ \mathrm{D}_y\Big(3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big) = 0$$

$$\mathrm{D}_t\Big(tu^2 - \tfrac{2}{\alpha}xu\Big) + \mathrm{D}_x\Big(t(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy}))$$
$$- x(u^2 + \tfrac{2\beta}{\alpha}u_{xx}) + \tfrac{2\beta}{\alpha}u_x\Big) + \mathrm{D}_y\Big(-2\beta(tu_x u_y + \tfrac{1}{\alpha}xu_{xy})\Big) = 0$$

Methods for Computing Conservation Laws

- Use the Lax pair L and A, satisfying $[L, A] = 0$.

  If $L = \mathrm{D}_x + U$, $A = \mathrm{D}_t + V$ then $V_x - U_t + [U, V] = 0$.

  $\hat{L} = TLT^{-1}$ gives the densities, $\hat{A} = TAT^{-1}$ gives the fluxes.

- Use Noether's theorem (Lagrangian formulation) to generate conservation laws from symmetries (Ovsiannikov, Olver, Mahomed, Kara, etc.).

- Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Mason, Naz, etc.) require solving ODEs (or PDEs).

Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.
- Compute $\mathrm{D}_t \rho$ with total derivative operator.
- Use variational derivative (Euler operator) to express exactness.
- Solve a (parametrized) linear system to find the undetermined coefficients.
- Use the homotopy operator to compute the flux (invert D_x or Div).

- Work with linearly independent pieces in finite dimensional spaces.
- Use linear algebra, calculus, and variational calculus (algorithmic).
- Implement the algorithm in Mathematica.

Tools from the Calculus of Variations

Differential Topology and Differential Geometry

- Definition:

  A differential function f is a exact iff $f = \mathrm{Div}\,\mathbf{F}$.

  Special case (1D): $f = \mathrm{D}_x\,F$.

- Question: How can one test that $f = \mathrm{Div}\,\mathbf{F}\,$?

- Theorem (exactness test):

  $f = \mathrm{Div}\,\mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \ldots, N.$

  N is the number of dependent variables.

The Euler operator annihilates divergences

- Euler operator in 1D (variable $u(x)$):

$$\mathcal{L}_{u(x)} = \sum_{k=0}^{M} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$

$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots$$

- Euler operator in 2D (variable $u(x,y)$):

$$\mathcal{L}_{u(x,y)} = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-\mathrm{D}_x)^k (-\mathrm{D}_y)^\ell \frac{\partial}{\partial u_{kx\,\ell y}}$$

$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} - \mathrm{D}_y \frac{\partial}{\partial u_y}$$

$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u_{yy}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} \cdots$$

Application: Testing Exactness

Example:

$$f = 8 v_x v_{xx} - u_x^3 \sin u + 2 u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

where $u(x)$ and $v(x)$

- f is exact
- After integration by parts (by hand):

$$F = \int f \, dx = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

- Exactness test with Euler operator:

$$f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

$$\mathcal{L}_{u(x)}\, f = \frac{\partial f}{\partial u} - \mathrm{D}_x \frac{\partial f}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial u_{xx}} \equiv 0$$

$$\mathcal{L}_{v(x)}\, f = \frac{\partial f}{\partial v} - \mathrm{D}_x \frac{\partial f}{\partial v_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial v_{xx}} \equiv 0$$

- Question: How can one compute $\mathbf{F} = \mathrm{Div}^{-1} f$?
- Theorem (integration by parts):
  - In 1D: If f is exact then

$$\boxed{F = \mathrm{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{\mathbf{u}(x)} f}$$

  - In 2D: If f is a divergence then

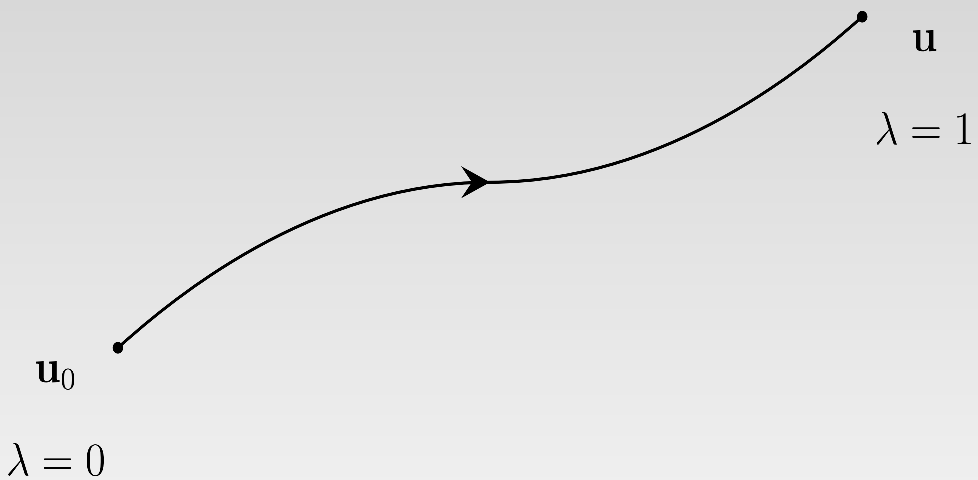
$$\boxed{\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f, \, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f)}$$

The homotopy operator inverts total derivatives and divergences!

Concept from (differential) Topology

Homotopic & Homotopy

Two continuous functions are called homotopic if one can be “continuously deformed” into the other. Such a deformation is called a homotopy between the two functions.



$$T(\mathbf{u}_0, \mathbf{u}) = \mathbf{u}_0 + \lambda(\mathbf{u} - \mathbf{u}_0) = (1 - \lambda)\mathbf{u}_0 + \lambda \mathbf{u}$$

Demonstration with Mathematica

- Homotopy Operator in 1D (variable x):

$$\boxed{\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \to \lambda \mathbf{u}, \, \mathbf{u}_x \to \lambda \mathbf{u}_x,$ *etc.*

More general: $\mathbf{u} \to \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \to \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ etc.

- Homotopy Operator in 2D (variables x and y):

$$\boxed{\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

$$\boxed{\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(y)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

where for dependent variable $u(x,y)$

$$\mathcal{I}^{(x)}_{u} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix\,jy} \frac{\binom{i+j}{i}\binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} (-\mathrm{D}_x)^{k-i-1} (-\mathrm{D}_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx\,\ell y}}$$

Application 1: The KdV Equation

$$\boxed{u_t + 6uu_x + u_{xxx} = 0}$$

- **Step 1**: **Compute the dilation symmetry**

  Set $(x, t, u) \to (\frac{x}{\lambda}, \frac{t}{\lambda^a}, \lambda^b u) = (\tilde{x}, \tilde{t}, \tilde{u})$

  Apply change of variables (chain rule)

$$\lambda^{-(a+b)} \tilde{u}_{\tilde{t}} + \lambda^{-(2b+1)} \tilde{u}\tilde{u}_{\tilde{x}} + \lambda^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0$$

  Solve $a + b = 2b + 1 = b + 3.$

  Solution: $a = 3$ and $b = 2$

$$(x, t, u) \to (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$$

Compute the density of selected **rank**, say, 6.

- **Step 2**: **Determine the form of the density**

List powers of u, up to rank $6:\quad [u, u^2, u^3]$

Differentiate with respect to x to increase the rank

u has weight 2 $\rightarrow$ apply D_x^4

u^2 has weight 4 $\rightarrow$ apply D_x^2

u^3 has weight 6 $\rightarrow$ no derivatives needed

Apply the D_x derivatives

Remove total and highest derivative terms:

$$\mathrm{D}_x^4 u \;\rightarrow\; \{u_{4x}\} \;\rightarrow\; \text{empty list}$$

$$\mathrm{D}_x^2 u^2 \;\rightarrow\; \{{u_x}^2, u u_{xx}\} \;\rightarrow\; \{{u_x}^2\}$$

$$\text{since } u u_{xx} = (u u_x)_x - {u_x}^2$$

$$\mathrm{D}_x^0 u^3 \;\rightarrow\; \{u^3\} \;\rightarrow\; \{u^3\}$$

Linearly combine the “building blocks”

Candidate density: $\boxed{\rho = c_1 u^3 + c_2 {u_x}^2}$

- **Step 3**: **Compute the coefficients** c_i Compute

$$\begin{aligned} \mathrm{D}_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\ &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_{kx}} \mathrm{D}_x^k u_t \\ &= (3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x) u_t \end{aligned}$$

Substitute u_t by $-(6uu_x + u_{xxx})$

$$\begin{aligned} E &= -\mathrm{D}_t \rho = (3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x)(6uu_x + u_{xxx}) \\ &= 18c_1 u^3 u_x + 12c_2 u_x^3 + 12c_2 u u_x u_{xx} \\ &\quad +3\, c_1 u^2 u_{xxx} + 2c_2 u_x u_{4x} \end{aligned}$$

Apply the Euler operator (variational derivative)

$$\mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^{m} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$

Here, E has order $m = 4$, thus

$$\begin{aligned}\mathcal{L}_u(x)\, E &= \frac{\partial E}{\partial u} - \mathrm{D}_x \frac{\partial E}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial E}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial E}{\partial u_{3x}} + \mathrm{D}_x^4 \frac{\partial E}{\partial u_{4x}} \\ &= -18(c_1 + 2c_2) u_x u_{xx}\end{aligned}$$

This term must vanish!

So, $c_2 = -\frac{1}{2} c_1$. Set $c_1 = 1$ then $c_2 = -\frac{1}{2}$.

Hence, the final form density is $\boxed{\rho = u^3 - \frac{1}{2} {u_x}^2}$

- **Step 4**: **Compute the flux** J

Method 1: Integrate by parts (simple cases)

Now,

$$E = 18u^3 u_x + 3u^2 u_{xxx} - 6u_x^3 - 6u u_x u_{xx} - u_x u_{xxxx}$$

Integration of $\mathrm{D}_x J = E$ yields the flux

$$\boxed{J = \frac{9}{2}u^4 - 6u{u_x}^2 + 3u^2 u_{xx} + \frac{1}{2}u_{xx}^2 - u_x u_{xxx}}$$

Method 2: Use the homotopy operator

$$J = \mathrm{D}_x^{-1} E = \int E \, dx = \mathcal{H}_{\mathbf{u}(x)} E = \int_0^1 (I_u E)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with integrand

$$I_u \, E = \sum_{k=1}^{M} \left(\sum_{i=0}^{k-1} u_{ix} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}$$

Here $M = 4$, thus

$$
\begin{aligned}
I_u E &= (u\mathrm{I})(\frac{\partial E}{\partial u_x}) + (u_x\mathrm{I} - u\mathrm{D}_x)(\frac{\partial E}{\partial u_{xx}}) \\
&\quad + (u_{xx}\mathrm{I} - u_x\mathrm{D}_x + u\mathrm{D}_x^2)(\frac{\partial E}{\partial u_{xxx}}) \\
&\quad + (u_{xxx}\mathrm{I} - u_{xx}\mathrm{D}_x + u_x\mathrm{D}_x^2 - u\mathrm{D}_x^3)(\frac{\partial E}{\partial u_{4x}}) \\
&= (u\mathrm{I})(18u^3 + 18u_x^2 - 6uu_{xx} - u_{xxxx}) \\
&\quad + (u_x\mathrm{I} - u\mathrm{D}_x)(-6uu_x) \\
&\quad + (u_{xx}\mathrm{I} - u_x\mathrm{D}_x + u\mathrm{D}_x^2)(3u^2) \\
&\quad + (u_{xxx}\mathrm{I} - u_{xx}\mathrm{D}_x + u_x\mathrm{D}_x^2 - u\mathrm{D}_x^3)(-u_x) \\
&= 18u^4 - 18uu_x^2 + 9u^2u_{xx} + u_{xx}^2 - 2u_xu_{xxx}
\end{aligned}
$$

Note: correct terms but incorrect coefficients!

Finally,

$$J = \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \, \frac{d\lambda}{\lambda}$$

$$= \int_0^1 \Big(18\lambda^3 u^4 - 18\lambda^2 u u_x^2 + 9\lambda^2 u^2 u_{xx} + \lambda u_{xx}^2$$

$$-2\lambda u_x u_{xxx} \Big) \, d\lambda$$

$$= \frac{9}{2} u^4 - 6 u u_x^2 + 3 u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx}$$

Final form of the flux:

$$\boxed{J = \frac{9}{2} u^4 - 6 u u_x^2 + 3 u^2 u_{xx} + \frac{1}{2} u_{xx}^2 - u_x u_{xxx}}$$

Application 2: Zakharov-Kuznetsov Equation

$$\boxed{u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0}$$

- Step 1: Compute the dilation invariance

  ZK equation is invariant under scaling symmetry

$$(t, x, y, u) \to (\frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})$$

  λ is an arbitrary parameter.

- Hence, the weights of the variables are

$$W(u) = 2, \;\; W(\mathrm{D}_t) = 3, \;\; W(\mathrm{D}_x) = 1, \;\; W(\mathrm{D}_y) = 1.$$

- A conservation law is invariant under the scaling symmetry of the PDE.

$$W(u) = 2, \;\; W(\mathrm{D}_t) = 3, \;\; W(\mathrm{D}_x) = 1, \;\; W(\mathrm{D}_y) = 1.$$

For example,

$$\mathrm{D}_t\Big(u^3 - \tfrac{3\beta}{\alpha}(u_x^2 + u_y^2)\Big) + \mathrm{D}_x\Big(3u^2(\tfrac{\alpha}{4}u^2 + \beta u_{xx}) - 6\beta u(u_x^2 + u_y^2)$$
$$+ \tfrac{3\beta^2}{\alpha}(u_{xx}^2 - u_{xy}^2) - \tfrac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}))\Big)$$
$$+ \mathrm{D}_y\Big(3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big) = 0$$

Rank $(\rho) = 6$, Rank $(J) = 8$.

Rank (conservation law) $= 9$.

Compute the density of selected **rank**, say, 6.

- Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

$$\{u^3, u_x^2, uu_{xx}, u_y^2, uu_{yy}, u_x u_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}$$

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

$$\boxed{\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y}$$

- Step 3: Compute the undetermined coefficients

Compute

$$\mathrm{D}_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t]$$

$$= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} \frac{\partial \rho}{\partial u_{kx\,\ell y}} \mathrm{D}_x^k \, \mathrm{D}_y^\ell \, u_t$$

$$= \Big(3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x + 2c_3 u_y \mathrm{D}_y + c_4 (u_y \mathrm{D}_x + u_x \mathrm{D}_y)\Big) u_t$$

Substitute $u_t = -\Big(\alpha u u_x + \beta (u_{xx} + u_{yy})_x\Big).$

$$
\begin{aligned}
E \;=\; -\mathrm{D}_t\rho \;=\; & 3c_1u^2(\alpha u u_x + \beta(u_{xx} + u_{xy})_x) \\
& + 2c_2u_x(\alpha u u_x + \beta(u_{xx} + u_{yy})_x)_x + 2c_3u_y(\alpha u u_x \\
& + \beta(u_{xx} + u_{yy})_x)_y + c_4(u_y(\alpha u u_x + \beta(u_{xx} + u_{yy})_x)_x \\
& + u_x(\alpha u u_x + \beta(u_{xx} + u_{yy})_x)_y)
\end{aligned}
$$

Apply the Euler operator (variational derivative)

$$
\mathcal{L}_{u(x,y)}E = \sum_{k=0}^{M_x}\sum_{\ell=0}^{M_y}(-\mathrm{D}_x)^k(-\mathrm{D}_y)^\ell \frac{\partial E}{\partial u_{kx\,\ell y}}
$$

$$
\begin{aligned}
&= -2\Big((3c_1\beta + c_3\alpha)u_xu_{yy} + 2(3c_1\beta + c_3\alpha)u_yu_{xy} \\
&\quad + 2c_4\alpha u_xu_{xy} + c_4\alpha u_yu_{xx} + 3(3c_1\beta + c_2\alpha)u_xu_{xx}\Big) \\
&\equiv 0
\end{aligned}
$$

Solve a parameterized linear system for the c_i:

$$3c_1\beta + c_3\alpha = 0, \;\; c_4\alpha = 0, \;\; 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \;\; c_2 = -\frac{3\beta}{\alpha}, \;\; c_3 = -\frac{3\beta}{\alpha}, \;\; c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

Final density of rank 6:

$$\boxed{\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)}$$

- Step 4: Compute the flux

Use the homotopy operator to invert Div:

$$\mathbf{J} = \mathrm{Div}^{-1} E = \left(\mathcal{H}^{(x)}_{u(x,y)} E, \; \mathcal{H}^{(y)}_{u(x,y)} E \right)$$

where

$$\mathcal{H}^{(x)}_{u(x,y)} E = \int_0^1 (I^{(x)}_u E)[\lambda u] \, \frac{d\lambda}{\lambda}$$

with

$$\mathcal{I}^{(x)}_u E = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix\,jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right.$$

$$\left. (-\mathrm{D}_x)^{k-i-1} \, (-\mathrm{D}_y)^{\ell-j} \right) \frac{\partial E}{\partial u_{kx\,\ell y}}$$

Similar formulas for $\mathcal{H}^{(y)}_{u(x,y)} E$ and $\mathcal{I}^{(y)}_u E$.

Let $A = \alpha u u_x + \beta(u_{xxx} + u_{xyy})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

Then,

$$\mathbf{J} = \left(\mathcal{H}^{(x)}_{u(x,y)} E,\, \mathcal{H}^{(y)}_{u(x,y)} E\right)$$

$$= \Big(\tfrac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u (3u_x^2 + u_y^2)$$

$$+ \frac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x (\tfrac{7}{2} u_{xyy} + 6u_{xxx})$$

$$- \frac{\beta^2}{\alpha} u_y (4u_{xxy} + \tfrac{3}{2} u_{yyy}) + \frac{\beta^2}{\alpha} (3u_{xx}^2 + \tfrac{5}{2} u_{xy}^2 + \tfrac{3}{4} u_{yy}^2)$$

$$+ \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_{xy} - 4\beta u u_x u_y$$

$$- \frac{3\beta^2}{4\alpha} u (u_{x3y} + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x (13u_{xxy} + 3u_{yyy})$$

$$- \frac{5\beta^2}{4\alpha} u_y (u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy} (u_{xx} + u_{yy})\Big)$$

However, $\mathrm{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (\mathrm{D}_y\theta,\ -\mathrm{D}_x\theta)$ is a curl term.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha}\Big(3u(u_{xxy} + u_{yyy}) + 10u_x u_{xy} + 5u_y(3u_{yy} + u_{xx})\Big)$$

Shorter flux:

$$\begin{aligned}
\tilde{\mathbf{J}} &= \mathbf{J} - \mathbf{K} \\
&= \Big(3u^2(\tfrac{\alpha}{4}u^2 + \beta u_{xx}) - 6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha}\Big(u_{xx}^2 - u_{yy}^2\Big) \\
&\qquad - \frac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})), \\
&\qquad 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big)
\end{aligned}$$

Software Demonstration

Software packages in Mathematica

Codes are available via the Internet:

URL: http://inside.mines.edu/$\sim$whereman/

Additional Examples

- Manakov-Santini system

$$
\begin{aligned}
u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y &= 0 \\
v_{tx} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} &= 0
\end{aligned}
$$

- Conservation laws for Manakov-Santini system:

$$
\begin{aligned}
&\mathrm{D}_t \Big(f u_x v_x \Big) + \mathrm{D}_x \Big(f (u u_x v_x - u_x v_x v_y - u_y v_y) \\
&\quad - f' y (u_t + u u_x - u_x v_y) \Big) + \mathrm{D}_y \Big(f (u_x v_y + u_y v_x + u_x v_x^2) \\
&\quad + f' (u - y u_y - y u_x v_x) \Big) = 0
\end{aligned}
$$

where $f = f(t)$ is arbitrary.

Conservation laws – continued:

$$
\begin{aligned}
&\mathrm{D}_t\Big(f(2u+v_x^2-yu_xv_x)\Big)+\mathrm{D}_x\Big(f\,(u^2+uv_x^2+u_yv\\
&-v_y^2-v_x^2v_y-y(uu_xv_x-u_xv_xv_y-u_yv_y))\\
&-f'y(v_t+uv_x-v_xv_y)+(f'-2fx)y^2(u_t+uu_x-u_xv_y)\Big)\\
&+\mathrm{D}_y\Big(f(v_x^3+2v_xv_y-u_xv-y(u_xv_x^2+u_xv_y+u_yv_x))\\
&+f'(v-y(2u+v_y+v_x^2))+(f'y^2-2fx)(u_xv_x+u_y)\Big)=0
\end{aligned}
$$

where $f=f(t)$ is arbitrary.

There are three additional conservation laws.

- (2+1)-dimensional Camassa-Holm equation

$$\boxed{(\alpha u_t + \kappa u_x - u_{txx} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{yy} = 0}$$

Interchange t with y

$$(\alpha u_y + \kappa u_x - u_{xxy} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{tt} = 0$$

Set $v = u_t$ to get

$$u_t = v$$

$$v_t = -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta u u_{xx} + 2u_{xx}^2 + 3u_x u_{xxx} + u u_{4x}$$

- Conservation laws for the Camassa-Holm equation

$$\mathrm{D}_t\Big(fu\Big) + \mathrm{D}_x\Big(\frac{1}{\alpha}f(\frac{3\beta}{2}u^2 + \kappa u - \frac{1}{2}u_x^2 - uu_{xx} - u_{tx})$$
$$+ (\frac{1}{2}f'y^2 - \frac{1}{\alpha}fx)(\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx}$$
$$- u_{txx})\Big) + \mathrm{D}_y\Big((\frac{1}{2}f'y^2 - \frac{1}{\alpha}fx)u_y - f'yu\Big) = 0$$

$$\mathrm{D}_t\Big(fyu\Big) + \mathrm{D}_x\Big(\frac{1}{\alpha}fy(\frac{3\beta}{2}u^2 + \kappa u - \frac{1}{2}u_x^2 - uu_{xx} - u_{tx})$$
$$+ y(\frac{1}{6}f'y^2 - \frac{1}{\alpha}fx)(\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx}$$
$$- u_{txx})\Big) + \mathrm{D}_y\Big(y(\frac{1}{6}f'y^2 - \frac{1}{\alpha}fx)u_y + (\frac{1}{\alpha}fx - \frac{1}{2}f'y^2)u\Big) = 0$$

where $f = f(t)$ is an arbitrary function.

- Khoklov-Zabolotskaya equation

describes e.g., sound waves in nonlinear media

$$\boxed{(u_t - uu_x)_x - u_{yy} - u_{zz} = 0}$$

Conservation law:

$$\mathrm{D}_t\Big(fu\Big) - \mathrm{D}_x\Big(\tfrac{1}{2}fu^2 + (fx+g)(u_t - uu_x)\Big)$$
$$+ \mathrm{D}_y\Big((fx+g)u_y - (f_y x + g_y)u\Big)$$
$$+ \mathrm{D}_z\Big((fx+g)u_z - (f_z x + g_z)u\Big) = 0$$

under the constraints $\Delta f = 0$ and $\Delta g = f_t$

where $f = f(t, y, z)$ and $g = g(t, y, z)$.

- Shallow water wave model (atmosphere)

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\boldsymbol{\Omega} \times \mathbf{u} + \nabla(\theta h) - \tfrac{1}{2} h \nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u} h) = 0$$

where $\mathbf{u}(x, y, t), \theta(x, y, t)$ and $h(x, y, t)$.

- In components:

$$u_t + u u_x + v u_y - 2\,\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x = 0$$

$$v_t + u v_x + v v_y + 2\,\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y = 0$$

$$\theta_t + u \theta_x + v \theta_y = 0$$

$$h_t + h u_x + u h_x + h v_y + v h_y = 0$$

- First few conservation laws of SWW model:

$$\rho_{(1)} = h \qquad \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(2)} = h\,\theta \qquad \mathbf{J}^{(2)} = h\,\theta \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(3)} = h\,\theta^2 \qquad \mathbf{J}^{(3)} = h\,\theta^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(4)} = h\left(u^2 + v^2 + h\theta\right) \qquad \mathbf{J}^{(4)} = h \begin{pmatrix} u\left(u^2 + v^2 + 2h\theta\right) \\ v\left(v^2 + u^2 + 2h\theta\right) \end{pmatrix}$$

$$\rho_{(5)} = \theta\left(2\Omega + v_x - u_y\right)$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

- More general conservation laws for SWW model:

$$\mathrm{D}_t\Big(f(\theta)h\Big) + \mathrm{D}_x\Big(f(\theta)hu\Big) + \mathrm{D}_y\Big(f(\theta)hv\Big) = 0$$

$$\begin{aligned}&\mathrm{D}_t\Big(g(\theta)(2\Omega + v_x - u_x)\Big)\\&+\mathrm{D}_x\Big(\tfrac{1}{2}g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y)\Big)\\&+\mathrm{D}_y\Big(\tfrac{1}{2}g(\theta)(4\Omega v - 2u_yv + 2vv_x + h\theta_x)\Big) = 0\end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$.

- Kadomtsev-Petviashvili (KP) equation

$$\boxed{(u_t + \alpha u u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0}$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

Equation be written as a conservation law

$$\mathrm{D}_t(u_x) + \mathrm{D}_x(\alpha u u_x + u_{xxx}) + \mathrm{D}_y(\sigma^2 u_y) = 0.$$

Exchange y and t and set $u_t = v$

$$\begin{aligned} u_t &= v \\ v_t &= -\frac{1}{\sigma^2}(u_{xy} + \alpha u_x^2 + \alpha u u_{xx} + u_{xxxx}) \end{aligned}$$

- Examples of conservation laws for KP equation (explicitly dependent on t, x, and y)

$$\mathrm{D}_t\Big(xu_x\Big) + \mathrm{D}_x\Big(3u^2 - u_{xx} - 6xuu_x + xu_{xxx}\Big) + \mathrm{D}_y\Big(\alpha x u_y\Big) = 0$$

$$\mathrm{D}_t\Big(yu_x\Big) + \mathrm{D}_x\Big(y(\alpha uu_x + u_{xxx})\Big) + \mathrm{D}_y\Big(\sigma^2(yu_y - u)\Big) = 0$$

$$\mathrm{D}_t\Big(\sqrt{t}u\Big) + \mathrm{D}_x\Big(\tfrac{\alpha}{2}\sqrt{t}u^2 + \sqrt{t}u_{xx} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{xxx}$$

$$+ \frac{\alpha\sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{xxx}\Big)$$

$$+\mathrm{D}_y\Big(x\sqrt{t}u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}}\Big) = 0$$

- More general conservation laws for KP equation:

$$\mathrm{D}_t\Big(fu\Big) + \mathrm{D}_x\Big(f(\tfrac{\alpha}{2}u^2 + u_{xx})$$
$$+(\tfrac{\sigma^2}{2}f'y^2 - fx)(u_t + \alpha u u_x + u_{3x})\Big)$$
$$+\mathrm{D}_y\Big((\tfrac{1}{2}f'y^2 - \sigma^2 fx)u_y - f'yu\Big) = 0$$

$$\mathrm{D}_t\Big(fyu\Big) + \mathrm{D}_x\Big(fy(\tfrac{\alpha}{2}u^2 + u_{xx})$$
$$+y(\tfrac{\sigma^2}{6}f'y^2 - fx)(u_t + \alpha u u_x + u_{3x})\Big)$$
$$+\mathrm{D}_y\Big(y(\tfrac{1}{6}f'y^2 - \sigma^2 fx)u_y + (\sigma^2 fx - \tfrac{1}{2}f'y^2)u\Big) = 0$$

where $f(t)$ is arbitrary function.

- Potential KP equation

  Replace u by u_x and integrate with respect to x.

$$\boxed{u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0}$$

- Examples of conservation laws
  (not explicitly dependent on x, y, t):

$$\mathrm{D}_t \Big(u_x \Big) + \mathrm{D}_x \left(\frac{\alpha}{2} u_x^2 + u_{xxx} \right) + \mathrm{D}_y \left(\sigma^2 u_y \right) = 0$$

$$\mathrm{D}_t \left(u_x^2 \right) + \mathrm{D}_x \left(\frac{2\alpha}{3} u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy} \right)$$

$$+\mathrm{D}_y \left(2\sigma^2 u_x u_y \right) = 0$$

Conservation laws for pKP equation – continued:

$$\mathrm{D}_t\Big(u_x u_y\Big) + \mathrm{D}_x\Big(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx}u_y - 2u_{xx}u_{xy}\Big)$$
$$+\mathrm{D}_y\Big(\sigma^2 u_y^2 - \tfrac{1}{3}u_x^3 - u_t u_x + u_{xx}^2\Big) = 0$$

$$\mathrm{D}_t\Big(2\alpha u u_x u_{xx} + 3u u_{4x} - 3\sigma^2 u_y^2\Big) + \mathrm{D}_x\Big(2\alpha u_t u_x^2 + 3u_t^2$$
$$- 2\alpha u u_x u_{tx} - 3u_{tx}u_{xx} + 3u_t u_{xxx} + 3u_x u_{t\,xx} - 3u u_{t\,xxx}\Big)$$
$$+\mathrm{D}_y\Big(6\sigma^2 u_t u_y\Big) = 0$$

Various generalizations exist.

- Generalized Zakharov-Kuznetsov equation

$$\boxed{u_t + \alpha u^n u_x + \beta (u_{xx} + u_{yy})_x = 0}$$

where n is rational, $n \neq 0$.

Conservation laws:

$$\mathrm{D}_t\Big(u\Big) + \mathrm{D}_x\Big(\tfrac{\alpha}{n+1}u^{n+1} + \beta u_{xx}\Big) + \mathrm{D}_y\Big(\beta u_{xy}\Big) = 0$$

$$\mathrm{D}_t\Big(u^2\Big) + \mathrm{D}_x\Big(\tfrac{2\alpha}{n+2}u^{n+2} - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\Big)$$
$$+ \mathrm{D}_y\Big(-2\beta u_x u_y\Big) = 0$$

- Third conservation law for gZK equation:

$$
\begin{aligned}
&\mathrm{D}_t\Big(u^{n+2} - \tfrac{(n+1)(n+2)\beta}{2\alpha}(u_x^2 + u_y^2)\Big)\\
&+ \mathrm{D}_x\Big(\tfrac{(n+2)\alpha}{2(n+1)}u^{2(n+1)} + (n+2)\beta u^{n+1}u_{xx}\\
&- (n+1)(n+2)\beta u^n(u_x^2 + u_y^2) + \tfrac{(n+1)(n+2)\beta^2}{2\alpha}(u_{xx}^2 - u_{yy}^2)\\
&- \tfrac{(n+1)(n+2)\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}))\Big)\\
&+ \mathrm{D}_y\Big((n+2)\beta u^{n+1}u_{xy} + \tfrac{(n+1)(n+2)\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big) = 0.
\end{aligned}
$$

Conclusions and Future Work

- The power of Euler and homotopy operators:
  - Testing exactness
  - Integration by parts: D_x^{-1} and Div^{-1}
- Integration of non-exact expressions

  Example: $f = u_x v + u v_x + u^2 u_{xx}$

  $\int f \, dx = uv + \int u^2 u_{xx} \, dx$
- Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)

- Broader class of PDEs (beyond evolution type)

  Example: short pulse equation (nonlinear optics)

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx}$$

  with non-polynomial conservation law

$$\mathrm{D}_t \left(\sqrt{1+6u_x^2}\right) - \mathrm{D}_x \left(3u^2\sqrt{1+6u_x^2}\right) = 0$$

- Continue the implementation in *Mathematica*
- Software: http://inside.mines.edu/∼whereman

Thank You

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3. W. Hereman, M. Colagrosso, R. Sayers, A. Ringler, B. Deconinck, M. Nivala, and M. S. Hickman, Continuous and Discrete Homotopy Operators and the Computation of Conservation Laws. In: Differential Equations with Symbolic Computation, Eds.: D. Wang and Z. Zheng, Birkhäuser Verlag, Basel (2005), Chapter 15, pp. 249-285.

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