A simplified Hirota method: Computation of solitary wave solutions and solitons through homogenization of degree

Willy Hereman

Department of Applied Mathematics and Statistics
Colorado School of Mines, Golden, Colorado

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Goals and Outline

- Homogenization of nonlinear PDEs
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- Application to non-solitonic PDEs
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- Concluding remarks

Homogenization of nonlinear PDEs

Example 1: The Burgers equation

The Cole-Hopf transformation

The Burgers equation

$$u_t + 2uu_x - u_{xx} = 0$$

can be **linearized** with a logarithmic derivative transformation due to Cole (1951) and Hopf (1950).

Integrate the Burgers equation with respect to x:

$$\partial_t \Big(\int^x u \, dx \Big) + u^2 - u_x = 0.$$

Substitute

$$u = c\,(\ln f)_x = c\left(\frac{f_x}{f}\right)$$

where c is a constant, to get

$$f(f_t - f_{xx}) + (c+1)f_x^2 = 0.$$

Setting $c = -1$ yields the heat equation $f_t - f_{xx} = 0.$

Then,

$$u(x,t) = -(\ln f)_x = -\frac{f_x}{f}$$

is the well-known Cole-Hopf transformation.

How to find the Cole-Hopf transformation?

As in the Painlevé test, substitute a Laurent series

$$u(x,t) = f^{\alpha}(x,t) \sum_{k=0}^{\infty} u_k(x,t) f^k(x,t)$$

$f(x,t)$ is manifold for the poles; α is negative integer.

Terms $f^{2\alpha-1}$ and $f^{\alpha-2}$ balance when $\alpha = -1$ and vanish for $u_0(x,t) = -f_x$. Truncating the Laurent series at the constant level in f

$$u(x,t) = -\frac{f_x}{f} + u_1(x,t) = -(\ln f)_x + u_1(x,t)$$

where $u_1(x,t)$ satisfies the Burgers equation.

Setting $u_1 = 0$ yields the Cole-Hopf transformation.

Find a kink solution of the Burgers equation

Substitute $f(x,t) = 1 + \mathrm{e}^{\theta} = 1 + \mathrm{e}^{k\,x - \omega\,t + \delta}$ into $f_t - f_{xx} = 0$. Then $\omega = -k^2$.

$$\begin{aligned}
u(x,t) &= -k\left(\frac{\mathrm{e}^{\theta}}{1+\mathrm{e}^{\theta}}\right) = -k\left(\frac{\mathrm{e}^{\theta}\mathrm{e}^{-\frac{\theta}{2}}}{(1+\mathrm{e}^{\theta})\mathrm{e}^{-\frac{\theta}{2}}}\right) \\
&= -k\left(\frac{\mathrm{e}^{\frac{\theta}{2}}}{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}}\right) = -\frac{1}{2}k\left(\frac{2\mathrm{e}^{\frac{\theta}{2}}}{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}}\right) \\
&= -\frac{1}{2}k\left(\frac{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}+\mathrm{e}^{\frac{\theta}{2}}-\mathrm{e}^{-\frac{\theta}{2}}}{\mathrm{e}^{\frac{\theta}{2}}+\mathrm{e}^{-\frac{\theta}{2}}}\right) = -\frac{1}{2}k\left(1+\tanh\frac{\theta}{2}\right)
\end{aligned}$$

with $\theta = kx + k^2 t + \delta$.

Equivalently,

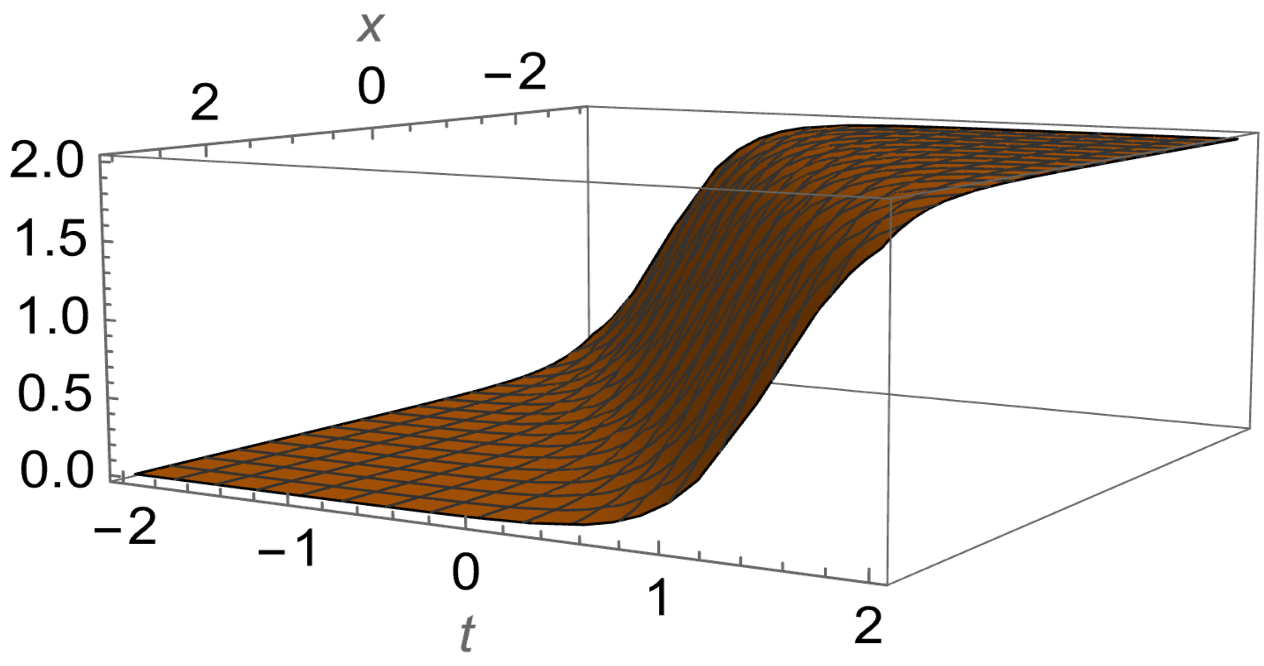
$$u(x,t) = \tfrac{1}{2}\tilde{k}\left(1 - \tanh\tfrac{\tilde{\theta}}{2}\right)$$

$$\text{with } \tilde{k} = -k,\ \tilde{\theta} = \tilde{k}x - \tilde{k}^2 t + \tilde{\delta}$$

$$u(x,t) = K\,(1 - \tanh\Theta)$$

$$\text{with } K = -\tfrac{k}{2},\ \Theta = Kx - 2K^2 t + \Delta.$$

Graph of kink solution for $K = 1$, $\Delta = 0$.



Example 2: Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Using a Laurent series, $f^{2\alpha-1}$ and $f^{\alpha-3}$ balance when $\alpha = -2$. The terms in f^{-5} and f^{-4} vanish when

$$u_0(x,t) = -f_x, \;\; u_1(x,t) = 2f_{xx}.$$

Hence

$$u(x,t) = -\frac{2{f_x}^2}{f^2} + \frac{2f_{xx}}{f} + u_2(x,t) = 2(\ln f)_{xx} + u_2(x,t)$$

where $u_2(x,t)$ solves the KdV equation (auto-Bäcklund transformation). Setting $u_2 = 0$ yields the Hirota transformation that “bilinearizes” the KdV equation.

Hirota's method

Example 2 cont.: KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Integrate with respect to x

$$\partial_t \left(\int^x u \, dx \right) + 3u^2 + u_{2x} = 0.$$

Substitute $u = c\,(\ln f)_{xx} = c\left(\frac{f f_{xx} - {f_x}^2}{f^2}\right)$ where c is a constant, to get

$$f^3(f_{xt} + f_{4x}) - f^2(f_x f_t - 3(c-1)f_{xx}^2 + 4f_x f_{3x})$$
$$+3(c-2)f_x^2(f_x^2 - 2f\,f_{xx}) = 0.$$

Set $c = 2$ to get a PDE that is **homogenous** of **second** degree in f

$$f(f_{xt} + f_{4x}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} = 0$$

which can be written in bilinear form

$$B(f \cdot f) \equiv \left(D_x D_t + D_x^4 \right) (f \cdot f) = 0$$

using Hirota's bilinear operators

$$\begin{aligned} D_x^n (f \cdot g) &= (\partial_x - \partial_{x'})^n f(x,t) g(x',t) \Big|_{x'=x} \\ &= \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \Big(\frac{\partial^j f}{\partial x^j} \Big) \Big(\frac{\partial^{n-j} g}{\partial x^{n-j}} \Big). \end{aligned}$$

Explicitly for $n = 4$

$$D_x^4(f \cdot g) = f_{4x}g - 4f_{3x}g_x + 6f_{xx}g_{xx} - 4f_xg_{3x} + fg_{4x}.$$

Leibniz rule for derivatives of products with every other sign flipped. Likewise

$$D_x^m D_t^n\,(f \cdot g) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')\Big|_{x'=x,t'=t}$$

$$= \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(-1)^{(n+m-i-j)} m!n!}{j!(m-j)!i!(n-i)!} \Big(\frac{\partial^{i+j} f}{\partial t^i \partial x^j}\Big)\Big(\frac{\partial^{n+m-i-j} g}{\partial t^{n-i} \partial x^{m-j}}\Big).$$

For example

$$D_x D_t(f \cdot g) = f_{xt}g - f_tg_x - f_xg_t + fg_{xt}.$$

Seek a solution of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t) = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots$$

ϵ is a bookkeeping parameter (not a small quantity).

$$O(\epsilon^0):\ B(1{\cdot}1) = 0$$

$$O(\epsilon^1):\ B(1{\cdot}f^{(1)} + f^{(1)}{\cdot}1) = 0$$

$$O(\epsilon^2):\ B(1{\cdot}f^{(2)} + f^{(1)}{\cdot}f^{(1)} + f^{(2)}{\cdot}1) = 0$$

$$O(\epsilon^3):\ B(1{\cdot}f^{(3)} + f^{(1)}{\cdot}f^{(2)} + f^{(2)}{\cdot}f^{(1)} + f^{(3)}{\cdot}1) = 0$$

$$O(\epsilon^4):\ B(1{\cdot}f^{(4)} + f^{(1)}{\cdot}f^{(3)} + f^{(2)}{\cdot}f^{(2)} + f^{(3)}{\cdot}f^{(1)} + f^{(4)}{\cdot}1) = 0$$

$$O(\epsilon^n):\ B\left(\sum_{j=0}^{n} f^{(j)}{\cdot}f^{(n-j)}\right) = 0 \quad \text{with } f^{(0)} = 1.$$

How to find exact solutions if the bilinear form is not known? Use a simplified Hirota method!

The quadratic equation

$$f(f_{xt} + f_{4x}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} = 0$$

is of the form $f\,\mathcal{L}f + \mathcal{N}(f, f) = 0$.

$\mathcal{L}$ is a linear differential operator, $\mathcal{N}$ is a quadratic differential operator. For the KdV equation

$$\begin{aligned} \mathcal{L}f &= f_{xt} + f_{4x} \\ \mathcal{N}(f, g) &= -f_x g_t + 3f_{xx}g_{xx} - 4f_x g_{3x} \end{aligned}$$

for auxiliary functions $f(x, t)$ and $g(x, t)$.

Substitute $f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t)$. Use Cauchy's product formula to group powers of ϵ.

Set the coefficients of powers of ϵ to zero:

$$O(\epsilon^1):\ \mathcal{L}f^{(1)} = 0$$

$$O(\epsilon^2):\ \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)})$$

$$\vdots \qquad\qquad \vdots$$

$$O(\epsilon^n):\ \mathcal{L}f^{(n)} = -\Big(\sum_{j=2}^{n-1} f^{(n-j)}\mathcal{L}f^{(j)} + \sum_{j=1}^{n-1} \mathcal{N}(f^{(n-j)}, f^{(j)})\Big),$$

$$n \geq 3.$$

For example, for $n = 3$

$$\mathcal{L}f^{(3)} = -\Big(f^{(1)}\mathcal{L}f^{(2)} + \mathcal{N}(f^{(2)}, f^{(1)}) + \mathcal{N}(f^{(1)}, f^{(2)})\Big).$$

The N-soliton solution of the KdV is generated from

$$f^{(1)} = \sum_{i=1}^{N} \mathrm{e}^{\theta_i} \equiv \sum_{i=1}^{N} \mathrm{e}^{k_i\,x-\omega_i\,t+\delta_i}.$$

Then $\mathcal{L}f^{(1)} = 0$ yields $\omega_i = k_i^3$.

Using $f^{(1)}$

$$\begin{aligned} -\mathcal{N}(f^{(1)}, f^{(1)}) &= -\sum_{i,j=1}^{N} 3k_i k_j^2 (k_i - k_j)\mathrm{e}^{\theta_i+\theta_j} \\ &= \sum_{1\le i<j\le N} 3k_i k_j (k_i - k_j)^2 \mathrm{e}^{\theta_i+\theta_j}. \end{aligned}$$

No terms in $\mathrm{e}^{2\theta_i}$! This determines the form of $f^{(2)}$

$$f^{(2)} = \sum_{1\le i<j\le N} a_{ij}\mathrm{e}^{\theta_i+\theta_j}$$

Next, compute

$$\mathcal{L}f^{(2)} = \sum_{1\le i<j\le N} 3k_ik_j(k_i+k_j)^2\, a_{ij}\, \mathrm{e}^{\theta_i+\theta_j}$$

and solve $\mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)})$ for

$$a_{ij} = \left(\frac{k_1-k_j}{k_i+k_j}\right)^2, \qquad 1\le i<j\le N.$$

Similarly, compute $f^{(3)}$. For example, for $N=3$

$$f^{(3)} = b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}$$

with

$$b_{123} = a_{12}\, a_{13}\, a_{23} = \left(\frac{(k_1-k_2)\,(k_1-k_3)\,(k_2-k_3)}{(k_1+k_2)\,(k_1+k_3)\,(k_2+k_3)}\right)^2.$$

For $N = 3$, one finds $f^{(n)} = 0$ for $n > 3$. Thus

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}\, e^{\theta_1+\theta_2} + a_{13}\, e^{\theta_1+\theta_3} + a_{23}\, e^{\theta_2+\theta_3}$$
$$+ b_{123}\, e^{\theta_1+\theta_2+\theta_3}$$

setting $\epsilon = 1$. No terms in $e^{2\theta_1}$, $e^{2\theta_2}$, $e^{2\theta_1+\theta_2}$, $e^{2\theta_2+\theta_1}$, ...

Finally, compute $u(x,t) = 2(\ln f)_{xx}$.

One-soliton solution of KdV equation

With $f = 1 + e^{\theta}$

$$u(x,t) = 2\left(\frac{f\, f_{xx} - {f_x}^2}{f^2}\right) = \frac{2k^2\, e^{\theta}}{\left(1+e^{\theta}\right)^2} = \frac{2k^2\, e^{\theta} e^{-\theta}}{\left(e^{-\frac{\theta}{2}}\left(1+e^{\theta}\right)\right)^2}$$

$$= \tfrac{1}{2}k^2 \operatorname{sech}^2\left[\tfrac{1}{2}(kx - k^3 t + \delta)\right] = 2K^2 \operatorname{sech}^2\left(Kx - 4K^3 t + \Delta\right)$$

Two-soliton solution of KdV equation

Using $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2}$,

$$u(x,t) = \frac{2\left(k_1^2\mathrm{e}^{\theta_1} + k_2^2\mathrm{e}^{\theta_2} + 2(k_1-k_2)^2\mathrm{e}^{\theta_1+\theta_2} + a_{12}(k_2^2\mathrm{e}^{\theta_1} + k_1^2\mathrm{e}^{\theta_2})\mathrm{e}^{\theta_1+\theta_2}\right)}{\left(1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}\right)^2}$$

Alternate forms:

$$u(x,t) = \frac{4\left(K_2^2 - K_1^2\right)\left((K_2^2 - K_1^2) + K_1^2\cosh(2\Theta_2) + K_2^2\cosh(2\Theta_1)\right)}{[(K_2 - K_1)\cosh(\Theta_2+\Theta_1) + (K_2+K_1)\cosh(\Theta_2-\Theta_1)]^2}$$

$$= 2\left(K_2^2 - K_1^2\right)\left(\frac{K_1^2\mathrm{sech}^2(\Theta_1) + K_2^2\mathrm{csch}^2(\Theta_2)}{[K_1\tanh(\Theta_1) - K_2\coth(\Theta_2)]^2}\right)$$

where $\Theta_j = K_j x - 4K_j^3 t + \Delta_j \quad (j = 1, 2)$.

N-soliton solution of KdV equation

$$f = \sum_{\mu=0,1} \mathrm{e}^{\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i\right]}$$

$\sum_{\mu=0,1}$ is over all combinations of
$\mu_1 = 0, 1, \;\; \mu_2 = 0, 1, \;\; \ldots, \;\; \mu_N = 0, 1.$

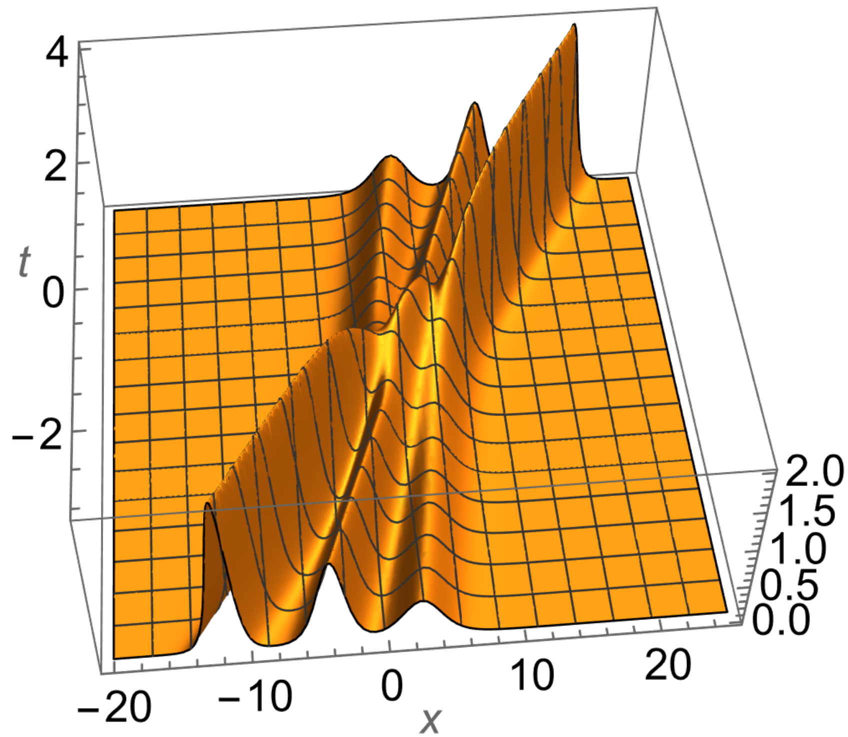
$\sum_{i<j}^{(N)}$ is such that $0 < i < j \leq N$ and $a_{ij} = \mathrm{e}^{A_{ij}}$.

Nice form of N-soliton solution (IST):

$$u(x,t) = 2\Big(\ln\det(I + M)\Big)_{xx}$$

where I is the $N \times N$ identity matrix and

$$M_{\ell m} = \frac{\mathrm{e}^{\Theta_\ell + \Theta_m}}{2(K_\ell + K_m)} \;\; \text{with } \Theta_\ell = K_\ell x - 4K_\ell^3 t + \Delta_\ell.$$



Bird's eye view of a 3-soliton collision for the KdV equation; $k_1 = 2,\ k_2 = \frac{3}{2},\ k_3 = 1,\ \delta_1 = \delta_2 = \delta_3 = 0.$

Application: Family of fifth-order KdV equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

Scaling $u = \frac{1}{\gamma}\tilde{u}$ yields

$$\tilde{u}_t + \frac{\alpha}{\gamma^2}\tilde{u}^2\tilde{u}_x + \frac{\beta}{\gamma}\tilde{u}_x\tilde{u}_{xx} + \tilde{u}\tilde{u}_{3x} + \tilde{u}_{5x} = 0.$$

Completely integrable cases:

$\frac{\alpha}{\gamma^2} = \frac{3}{10}, \;\; \frac{\beta}{\gamma} = 2 \;\; (\alpha = 30, \;\; \beta = 20, \gamma = 10)$ Lax

$\frac{\alpha}{\gamma^2} = \frac{1}{5}, \quad \frac{\beta}{\gamma} = 1 \;\; (\alpha = 5, \;\; \beta = 5, \gamma = 5)$ Sawada Kotera

or Caudrey–Dodd–Gibbon

$\frac{\alpha}{\gamma^2} = \frac{1}{5}, \quad \frac{\beta}{\gamma} = \frac{5}{2} \;\; (\alpha = 20, \;\; \beta = 25, \gamma = 10)$

Kaup–Kuperschmidt

Integrate

$$\partial_t\Big(\int^x u\,dx\Big) + \tfrac{1}{3}\alpha u^3 + \tfrac{1}{2}(\beta-\gamma)u_x^2 + \gamma u u_{xx} + u_{4x} = 0.$$

Substitute

$$u = c\,(\ln f)_{xx} = c\left(\frac{f\,f_{xx} - {f_x}^2}{f^2}\right)$$

where c is a constant, to get

$$\begin{aligned}
&6f^5(f_{xt}+f_{6x}) - 3f^4(2f_x f_t + \ldots + 12 f_x f_{5x})\\
&+2f^3\Big((\ldots)f_x^2 + \ldots + (\ldots)f_x^2 f_{4x}\Big)\\
&-3f^2 f_x^2\Big((\ldots)f_{xx}^2 + (\ldots)f_x f_{3x}\Big)\\
&+2f_x^4(360 - 6\beta c + \alpha c^2 - 12\gamma c)(3f\,f_{xx} - 2f_x^2) = 0.
\end{aligned}$$

Example 3: Lax equation

Using $\alpha = \frac{3}{10}\gamma^2$, $\beta = 2\gamma$, and $c = \frac{20}{\gamma}$ one gets a **cubic** (homogenous) equation

$$f^2(f_{xt} + f_{6x}) - f(f_x f_t - 5 f_{2x} f_{4x} + 6 f_x f_{5x})$$
$$+10(f_{2x}^3 - 2 f_x f_{2x} f_{3x} + f_x^2 f_{4x}) = 0.$$

Bilinear form consists of two coupled equations

$$\left(D_x D_s + D_x^4\right)(f \cdot f) = 0$$
$$\left(D_x D_t + D_x^6\right)(f \cdot f) - \tfrac{5}{3}\left(D_s^2 + D_s D_x^3\right)(f \cdot f) = 0$$

for only one function f but with an auxiliary time variable s. Ignore the bilinear form!

Write the cubic equation as

$$f^2 \mathcal{L} f + f \mathcal{N}_1(f, f) + \mathcal{N}_2(f, f, f) = 0$$

with

$$\begin{aligned} \mathcal{L} f &= f_{xt} + f_{6x} \\ \mathcal{N}_1(f, g) &= -(f_t g_x - 5 f_{2x} g_{4x} + 6 f_x g_{5x}) \\ \mathcal{N}_2(f, g, h) &= 10(f_{2x} g_{2x} h_{2x} - 2 f_x g_{2x} h_{3x} + f_x g_x h_{4x}). \end{aligned}$$

Substitute

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n \, f^{(n)}(x, t).$$

Perturbation scheme becomes

$$
\begin{aligned}
O(\epsilon^1):&\ \mathcal{L} f^{(1)} = 0\\
O(\epsilon^2):&\ \mathcal{L} f^{(2)} = -2 f^{(1)} \mathcal{L} f^{(1)} - \mathcal{N}_1(f^{(1)}, f^{(1)})\\
O(\epsilon^3):&\ \mathcal{L} f^{(3)} = -2 f^{(1)} \mathcal{L} f^{(2)} - 2 f^{(2)} \mathcal{L}(f^{(1)}) - {f^{(1)}}^2 \mathcal{L}(f^{(1)})\\
&\qquad - \mathcal{N}_1(f^{(1)}, f^{(2)}) - \mathcal{N}_1(f^{(2)}, f^{(1)})\\
&\qquad - f^{(1)} \mathcal{N}_1(f^{(1)}, f^{(1)}) - \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)})\\
&\vdots \qquad\qquad \vdots
\end{aligned}
$$

Start from $f^{(1)} = \sum_{i=1}^{N} \mathrm{e}^{\theta_i} = \sum_{i=1}^{N} \mathrm{e}^{k_i\, x - \omega_i\, t + \delta_i}$ and proceed as in KdV case.

One-soliton solution of Lax equation

$$\begin{aligned} u(x,t) &= \tfrac{5}{\gamma} k^2 \mathrm{sech}^2 \left[\tfrac{1}{2} (kx - k^5 t + \delta) \right] \\ &= \tfrac{20}{\gamma} K^2 \mathrm{sech}^2 \Big(Kx - 16K^5 t + \Delta \Big) \end{aligned}$$

where $k = 2K$ and $\delta = 2\Delta$.

Here, $\omega_i = k_i^5$, $a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2$ and $b_{123} = a_{12} a_{13} a_{23}$

(for 2- and 3-soliton solutions).

Example 4: Sawada-Kotera equation

Using $\alpha = \frac{1}{5}\gamma^2$, $\beta = \gamma$, and $c = \frac{30}{\gamma}$ one gets a **quadratic** equation

$$f(f_{xt} + f_{6x}) - f_x f_t - 10 f_{3x}^2 + 15 f_{xx} f_{4x} - 6 f_x f_{5x} = 0.$$

Bilinear representation

$$\left(D_x D_t + D_x^6\right)(f \cdot f) = 0$$

is similar to one for the KdV equation. Ignore it!

Define

$$\begin{aligned} \mathcal{L} f &= f_{xt} + f_{6x} \\ \mathcal{N}(f, g) &= -f_x g_t - 10 f_{3x} g_{3x} + 15 f_{xx} g_{4x} - 6 f_x g_{5x}. \end{aligned}$$

Proceed as in the KdV case.

One-soliton solution of Sawada-Kotera equation

$$
\begin{aligned}
u(x,t) &= \tfrac{15}{2\gamma}k^2\mathrm{sech}^2\left[\tfrac{1}{2}(kx-k^5t+\delta)\right] \\
&= \tfrac{30}{\gamma}K^2\mathrm{sech}^2\Big(Kx-16K^5t+\Delta\Big).
\end{aligned}
$$

where $k = 2K$ and $\delta = 2\Delta$.

Here, $\omega_i = k_i^5$. For the 3-soliton solution:

$$
\begin{aligned}
a_{ij} &= \frac{(k_i-k_j)^2\,(k_i^2-k_ik_j+k_j^2)}{(k_i+k_j)^2\,(k_i^2+k_ik_j+k_j^2)} = \frac{(k_i-k_j)^3\,(k_i^3+k_j^3)}{(k_i+k_j)^3\,(k_i^3-k_j^3)} \\
b_{123} &= a_{12}a_{13}a_{23}.
\end{aligned}
$$

Example 5: Kaup-Kupershmidt equation

Using $\alpha = \frac{1}{5}\gamma^2$, $\beta = \frac{5}{2}\gamma$, and $c = \frac{15}{\gamma}$ yields a **quartic** equation

$$4f^3(f_{xt} + f_{6x}) - f^2(4f_t f_x - 5{f_{3x}}^2 + 24 f_x f_{5x})$$
$$-30 f f_x (f_{2x} f_{3x} - 2 f_x f_{4x}) + 15 {f_x}^2 (3{f_{2x}}^2 - 4 f_x f_{3x}) = 0.$$

Bilinear form consists of two coupled equations

$$\left(D_x D_t + \frac{1}{16} D_x^6\right)(f\cdot f) + \frac{15}{4} D_x^2 (f\cdot g) = 0$$
$$D_x^4 (f\cdot f) - 4fg = 0$$

for two unknown functions f and g. Ignore bilinear form.

Continue with equation for f:

$$f^3 \mathcal{L} f \; + f^2 \mathcal{N}_1(f,f) + f \mathcal{N}_2(f,f,f) + \mathcal{N}_3(f,f,f,f) = 0$$

where

$$\begin{aligned}
\mathcal{L} f &= f_{xt} + f_{6x} \\
\mathcal{N}_1(f,g) &= -(4 f_t g_x - 5 f_{3x} g_{3x} + 24 f_x g_{5x}) \\
\mathcal{N}_2(f,g,h) &= -30 f_x (g_{2x} h_{3x} - 2 g_x h_{4x}) \\
\mathcal{N}_3(f,g,h,j) &= 15 f_x g_x (h_{2x} j_{2x} - 4 h_x j_{3x}).
\end{aligned}$$

First three equations of the 'perturbation' scheme:

$$O(\epsilon^1): \ \mathcal{L}f^{(1)} = 0$$

$$O(\epsilon^2): \ \mathcal{L}f^{(2)} = -\mathcal{N}_1(f^{(1)}, f^{(1)})$$

$$\begin{aligned} O(\epsilon^3): \ \mathcal{L}f^{(3)} = & -3f^{(1)}\mathcal{L}f^{(2)} - 2f^{(1)}\mathcal{N}_1(f^{(1)}, f^{(1)}) \\ & -\mathcal{N}_1(f^{(2)}, f^{(1)}) - \mathcal{N}_1(f^{(1)}, f^{(2)}) \\ & -\mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}). \end{aligned}$$

The one-soliton solution of KK equation

With $f^{(1)} = \mathrm{e}^{\theta} = \mathrm{e}^{kx-\omega t+\delta}$, $\mathcal{L}f^{(1)} = 0$ yields $\omega = k^5$.

Compute

$$-\mathcal{N}(f^{(1)}, f^{(1)}) = 15k^6\mathrm{e}^{2\theta}.$$

Thus $f^{(2)}$ is of the form

$$f^{(2)} = a\mathrm{e}^{2\theta}.$$

Since

$$\mathcal{L}f^{(2)} = 240ak^6\mathrm{e}^{2\theta}$$

one gets $a = \frac{1}{16}$.

Next verify that $f^{(n)} = 0$ for $n \geq 3$.

Setting $\epsilon = 1$, for the one-soliton solution

$$f = 1 + \mathrm{e}^{\theta} + \frac{1}{16}\mathrm{e}^{2\theta}.$$

Then $u = \frac{15}{\gamma}(\ln f)_{xx}$ yields

$$u = \frac{240}{\gamma}k^2 \left(\frac{\mathrm{e}^{\theta}(16 + \mathrm{e}^{\theta} + \mathrm{e}^{2\theta})}{(16 + 16\mathrm{e}^{\theta} + \mathrm{e}^{2\theta})^2} \right)$$

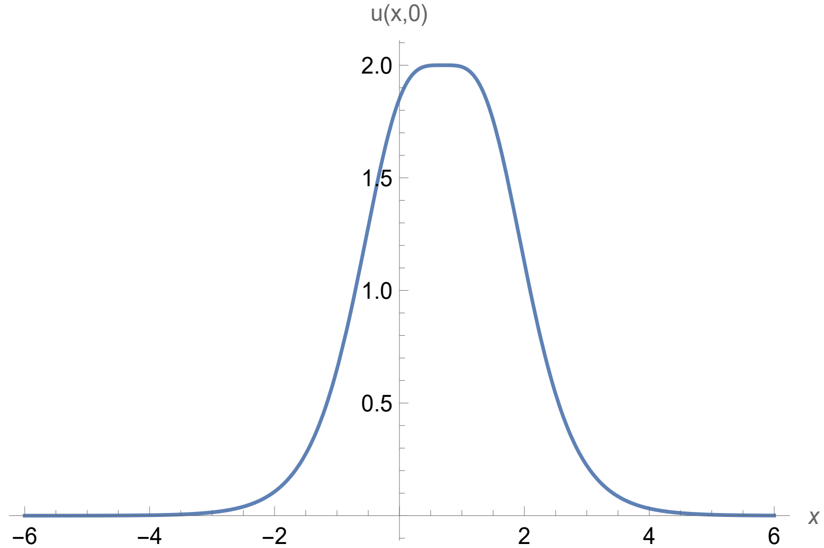
which solves $u_t + \frac{1}{5}\gamma^2 u^2 u_x + \frac{5}{2}\gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$

One-soliton solution can also be written as

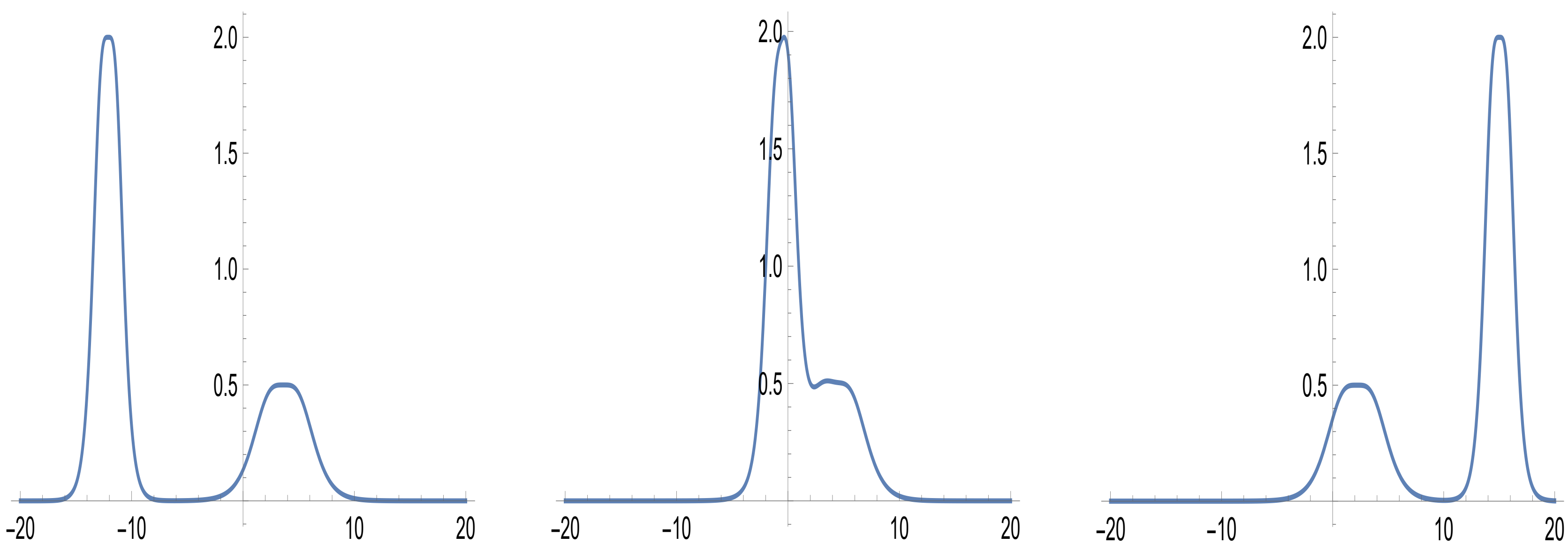
$$u = \frac{240}{\gamma}k^2 \left(\frac{\left(1 - \tanh^2(\frac{\theta}{2})\right)\left(21 - 30\tanh(\frac{\theta}{2}) + 13\tanh^2(\frac{\theta}{2})\right)}{\left(33 - 30\tanh(\frac{\theta}{2}) + \tanh^2(\frac{\theta}{2})\right)^2} \right)$$

where $\theta = kx - k^5 t + \delta$.

Graph of one-soliton solution ($\gamma = 10, k = 2, \delta = 0$).



Collision of two-solitons ($k_1 = 2,\ k_2 = 1, \delta_1 = \delta_2 = 0$).



The two-soliton solution of KK equation

Start from

$$f^{(1)} = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$$

where $\theta_i = k_i x - \omega_i t + \delta_i$ with $\omega_i = k_i^5 \ (i = 1, 2)$.

Compute

$$\begin{aligned} -\mathcal{N}_1(f^{(1)}, f^{(1)}) &= 15k_1^6 \mathrm{e}^{2\theta_1} + 15k_2^6 \mathrm{e}^{2\theta_2} \\ &\quad +10k_1k_2(2k_1^4 - k_1^2k_2^2 + 2k_2^4)\mathrm{e}^{\theta_1+\theta_2}. \end{aligned}$$

Thus $f^{(2)}$ must be of the form

$$f^{(2)} = a\mathrm{e}^{2\theta_1} + b\mathrm{e}^{2\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2}.$$

Proceed with

$$\mathcal{L}f^{(2)} = 240ak_1^6 \mathrm{e}^{2\theta_1} + 240bk_2^6 \mathrm{e}^{2\theta_2} + 20a_{12}k_1k_2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)\mathrm{e}^{\theta_1+\theta_2}.$$

Hence, $a = b = \frac{1}{16}$, and

$$a_{12} = \frac{2k_1^4 - k_1^2k_2^2 + 2k_2^4}{2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}.$$

Therefore

$$f^{(2)} = \frac{1}{16}\mathrm{e}^{2\theta_1} + \frac{1}{16}\mathrm{e}^{2\theta_2} + \frac{(2k_1^4 - k_1^2k_2^2 + 2k_2^4)\,\mathrm{e}^{\theta_1+\theta_2}}{2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}.$$

At the next level of recursion

$$f^{(3)} = b_{12}\left(\mathrm{e}^{\theta_1+2\theta_2} + \mathrm{e}^{2\theta_1+\theta_2}\right)$$

with

$$b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{16(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)}.$$

Next,

$$f^{(4)} = b_{12}^2 \mathrm{e}^{2\theta_1+\theta_2} = \frac{(k_1 - k_2)^4(k_1^2 - k_1k_2 + k_2^2)^2 \, \mathrm{e}^{2(\theta_1+\theta_2)}}{256(k_1 + k_2)^4(k_1^2 + k_1k_2 + k_2^2)^2}.$$

After verification that all $f^{(n)}$ are zero, for $n \geq 5$

$$\begin{aligned} f &= 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \frac{1}{16}\mathrm{e}^{2\theta_1} + \frac{1}{16}\mathrm{e}^{2\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &\quad + b_{12}\left(\mathrm{e}^{2\theta_1+\theta_2} + \mathrm{e}^{\theta_1+2\theta_2}\right) + b_{12}^2\mathrm{e}^{2(\theta_1+\theta_2)} \end{aligned}$$

and $u = \frac{15}{\gamma}(\ln f)_{xx}$.

The three-soliton solution of KK equation

First six equations in the perturbation scheme must be solved. Start with $f^{(1)} = \sum_{i=i}^{3} \mathrm{e}^{\theta_i}$. Then

$$f^{(2)} = \frac{1}{16} \sum_{i=i}^{3} \mathrm{e}^{2\theta_i} + \sum_{1 \le i < j \le 3} a_{ij} \mathrm{e}^{\theta_i + \theta_j}$$

with

$$a_{ij} = \frac{2k_i^4 - k_i^2 k_j^2 + 2k_j^4}{2(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, \qquad 1 \le i < j \le 3.$$

Compute $f^{(3)}$ through $f^{(6)}$; at that term the expansion of f truncates. Verify that $f^{(n)} = 0$ for $n \ge 7$.

Summary of the results:

$$f^{(3)} = \sum_{1\le i<j\le 3} b_{ij}\left(\mathrm{e}^{2\theta_i+\theta_j} + \mathrm{e}^{\theta_i+2\theta_j}\right) + c_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}$$

where

$$b_{ij} = \frac{(k_i - k_j)^2(k_i^2 - k_i k_j + k_j^2)}{16(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2)}, \qquad 1 \le i < j \le 3$$

and

$$
\begin{aligned}
c_{123} \;=\; & \frac{1}{D}[(2k_1^4 - k_1^2k_2^2 + 2k_2^4)(k_3^8 + k_1^4k_2^4) \\
& +(2k_1^4 - k_1^2k_3^2 + 2k_3^4)(k_2^8 + k_1^4k_3^4) \\
& +(2k_2^4 - k_2^2k_3^2 + 2k_3^4)(k_1^8 + k_2^4k_3^4)] \\
& -\frac{1}{2D}[(k_1^2 + k_2^2)(k_1^4 + k_2^4)(k_3^6 + k_1^2k_2^2k_3^2) \\
& +(k_1^2 + k_3^2)(k_1^4 + k_3^4)(k_2^6 + k_1^2k_2^2k_3^2) \\
& +(k_2^2 + k_3^2)(k_2^4 + k_3^4)(k_1^6 + k_1^2k_2^2k_3^2) \\
& +12k_1^4k_2^4k_3^4]
\end{aligned}
$$

with

$$
D = 4 \prod_{1 \le i < j \le 3} (k_i + k_j)^2 (k_i^2 + k_ik_j + k_j^2).
$$

Solve step by step to get

$$
\begin{aligned}
f^{(4)} &= \sum_{1\le i<j\le 3} b_{ij}^2 \mathrm{e}^{2(\theta_i+\theta_j)} + 16\Big(a_{23}b_{12}b_{13}\mathrm{e}^{2\theta_1+\theta_2+\theta_3} \\
&\quad + a_{13}b_{12}b_{23}\mathrm{e}^{\theta_1+2\theta_2+\theta_3} + a_{12}b_{13}b_{23}\mathrm{e}^{\theta_1+\theta_2+2\theta_3}\Big). \\
f^{(5)} &= 16^2\, b_{12}b_{13}b_{23}\Big(b_{12}\mathrm{e}^{2\theta_1+2\theta_2+\theta_3} + b_{13}\mathrm{e}^{2\theta_1+\theta_2+2\theta_3} \\
&\quad + b_{23}\mathrm{e}^{\theta_1+2\theta_2+2\theta_3}\Big). \\
f^{(6)} &= 16\,(16 b_{12}b_{13}b_{23})^2 \mathrm{e}^{2(\theta_1+\theta_2+\theta_3)}.
\end{aligned}
$$

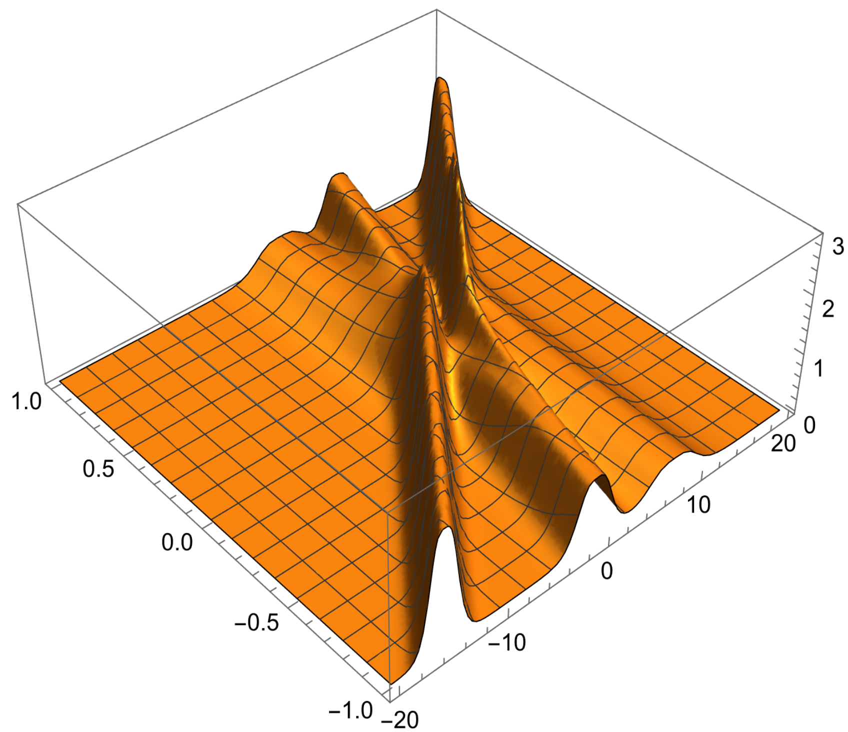
Finally

$$f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} + f^{(6)}.$$

Then $u = \frac{15}{\gamma}(\ln f)_{xx}$ solves

$$u_t + \frac{1}{5}\gamma^2 u^2 u_x + \frac{5}{2}\gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

Mathematica demonstration



Bird's eye view of a 3-soliton collision for the KK equation; $k_1 = 2, k_2 = \frac{3}{2}, k_3 = 1, \delta_1 = \delta_2 = \delta_3 = 0$.

Example 6: Fisher equation with convection

$$u_t + \alpha u u_x - u_{xx} - u(1-u) = 0.$$

Use $u(x,t) = -\frac{2}{\alpha}(\ln f(x,t))_x = -\frac{2}{\alpha}\left(\frac{f_x}{f}\right)$ to get an homogenous equation of lowest degree (quadratic)

$$f(f_{xxx} + f_x - f_{xt}) + f_x(f_t - f_{xx} + \tfrac{2}{\alpha} f_x) = f\,\mathcal{L}f + \mathcal{N}(f,f) = 0.$$

Seek $f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n \, f^{(n)}(x,t)$.

$\mathcal{L} f^{(1)} = \mathcal{L}(\sum_{i=1}^{N} \mathrm{e}^{\theta_i})$ determines $\omega_i = -(1 + k_i^2)$.

Consequently

$$\mathcal{L}f^{(2)} = -\sum_{i=1}^{N} k_i(1 + \tfrac{2}{\alpha}k_i)\mathrm{e}^{2\theta_i} - \sum_{1\le i<j\le N} \tfrac{4}{k}k_i k_j(k_i + k_j)\mathrm{e}^{\theta_i+\theta_j}.$$

Including terms $\mathrm{e}^{2\theta_i}$ in $f^{(2)}$ does not help (perturbation scheme does not terminate). Hence, $k_i = -\frac{\alpha}{2}$ and $N = 1$. Then

$$f(x,t) = 1 + \mathrm{e}^{\theta} = 1 + \mathrm{e}^{-\frac{\alpha}{2}x + \frac{1}{4}(4+\alpha^2)t + \delta}.$$

With $u = -\frac{2}{\alpha}\left(\frac{f_x}{f}\right)$

$$u(x,t) = \frac{\mathrm{e}^{\theta}}{1 + \mathrm{e}^{\theta}} = \tfrac{1}{2}\left(1 - \tanh[\tfrac{1}{2}(\tfrac{\alpha}{2}x - \tfrac{1}{4}(4 + \alpha^2)t + \delta)]\right).$$

Example 7: Fisher equation

$$u_t - u_{xx} - u(1-u) = 0.$$

Truncated Laurent series reveals that

$$u(x,t) = -6(\ln f)_{xx} + \tfrac{6}{5}(\ln f)_t.$$

The equation for f is **quadratic**:

$$\begin{aligned}
& f(f_{4x} + f_{xx} - \tfrac{6}{5} f_{xxt} + \tfrac{1}{5} f_{tt} - \tfrac{1}{5} f_t) \\
& -4 f_x f_{3x} + 3 f_{xx}^2 - f_x^2 - \tfrac{6}{5} f_t f_{xx} + \tfrac{12}{5} f_x f_{xt} + \tfrac{1}{25} f_t^2 \\
& = f\,\mathcal{L} f + \mathcal{N}(f,f) = 0.
\end{aligned}$$

$\mathcal{L} e^{\theta} = 0$ yields $\omega = -5k^2$ or $\omega = -(1+k^2)$. Next, $\mathcal{N}(e^{\theta}, e^{\theta}) = 0$ determines $k = \pm \frac{1}{\sqrt{6}}$. Thus, $\omega = -\frac{5}{6}$.

With $f = 1 + \mathrm{e}^{\theta}$

$$u(x,t) = \frac{\mathrm{e}^{2\theta}}{(1+\mathrm{e}^{\theta})^2} = \frac{1}{(1+e^{-\theta})^2} = \tfrac{1}{4}\Big(1 + \tanh\left(\tfrac{\theta}{2}\right)\Big)^2.$$

Explicitly for $k = -\frac{1}{\sqrt{6}}$

$$u(x,t) = \tfrac{1}{4}\left(1 - \tanh\left(\tfrac{1}{2}\left[\tfrac{1}{\sqrt{6}}x - \tfrac{5}{6}t + \delta\right]\right)\right)^2.$$

Example 8: FitzHugh-Nagumo equation with convection term

$$u_t + \alpha u u_x - u_{xx} + u(1-u)(a-u) = 0.$$

Use $u = \sqrt{m}\,(\ln f)_x = \sqrt{m}\left(\frac{f_x}{f}\right)$ where $\alpha = \frac{m-2}{\sqrt{m}}$.

Then

$$f(f_{xxx} - a f_x - f_{xt}) + f_x\left(f_t - (m+2) f_{xx} + \sqrt{m}(1+a) f_x\right)$$
$$= f\,\mathcal{L} f + \mathcal{N}(f, f) = 0.$$

Seek $f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t)$.

$\mathcal{L} f^{(1)} = \mathcal{L}(\sum_{i=1}^{N} \mathrm{e}^{\theta_i})$ determines $\omega_i = a - k_i^2$.

Finally, $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$ and

$$u(x,t) = \frac{\mathrm{e}^{\theta_1} + a\,\mathrm{e}^{\theta_2}}{1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}}$$

where

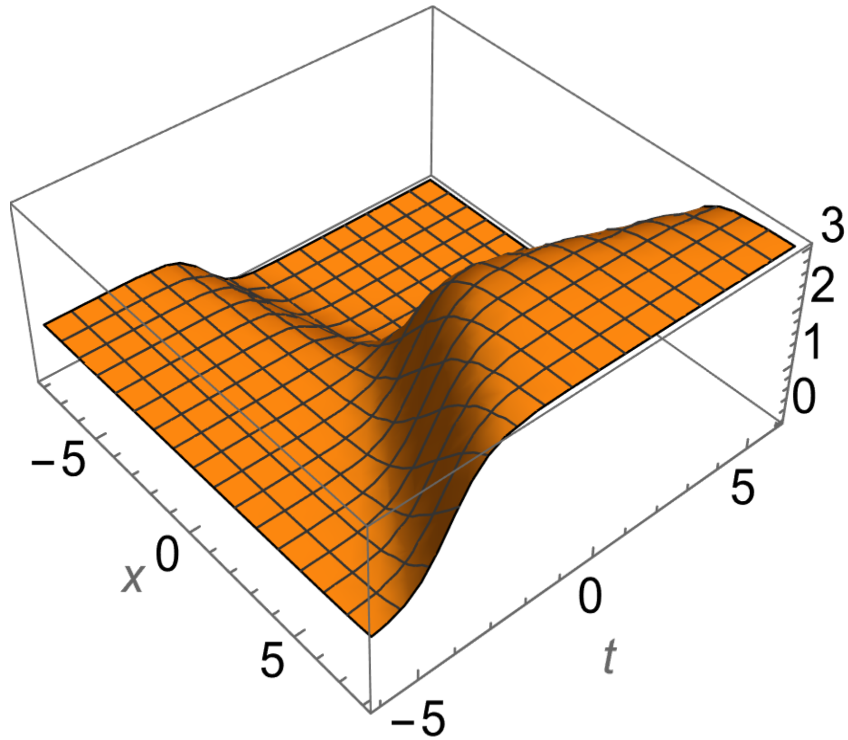
$\theta_1 = \frac{1}{\sqrt{m}}x + \left(\frac{1-am}{m}\right)t + \delta_1$ and $\theta_2 = \frac{a}{\sqrt{m}}x + \left(\frac{a(a-m)}{m}\right)t + \delta_2$.

The solution describes two coalescent wave fronts.

Since $\alpha = \frac{m-2}{\sqrt{m}}$ the values of m are

$$m = \tfrac{1}{2}\left(4 + \alpha^2 \pm \alpha\sqrt{8 + \alpha^2}\right).$$

Graph of $u(x,t)$ for $a = 3, \alpha = 1\,(m = 4), \delta_1 = \delta_2 = 0$.



For $m = 2$ one gets the solution of FitzHugh-Nagumo equation without convection ($\alpha = 0$).

Example 9: Combined KdV-Burgers equation

$$u_t + 6uu_x + u_{xxx} - 5\beta u_{xx} = 0.$$

Truncated Laurent series reveals that

$$u(x,t) = 2(\ln f)_{xx} - 2\beta(\ln f)_x.$$

Substitute into the (integrated) KdV-Burgers equation

$$\partial_t \Big(\int^x u\,dx \Big) + 3u^2 + u_{xx} - 5\beta u_x = 0$$

to get

$$\begin{aligned} & f(f_{xt} - \beta f_t + 5\beta^2 f_{2x} - 6\beta f_{3x} + f_{4x}) \\ & -f_x f_t + \beta^2 f_x^2 + 6\beta f_x f_{2x} + 3f_{2x}^2 - 4f_x f_{3x} = 0. \end{aligned}$$

As before $f\,\mathcal{L}f + \mathcal{N}(f,f) = 0$. Then, $\mathcal{L}\mathrm{e}^{kx-\omega t+\delta} = 0$ yields $(\beta - k)(\omega - k^3 + 5\beta k^2) = 0$.

Case 1: $\omega = k^2(k - 5\beta)$ and $\beta \neq k$.

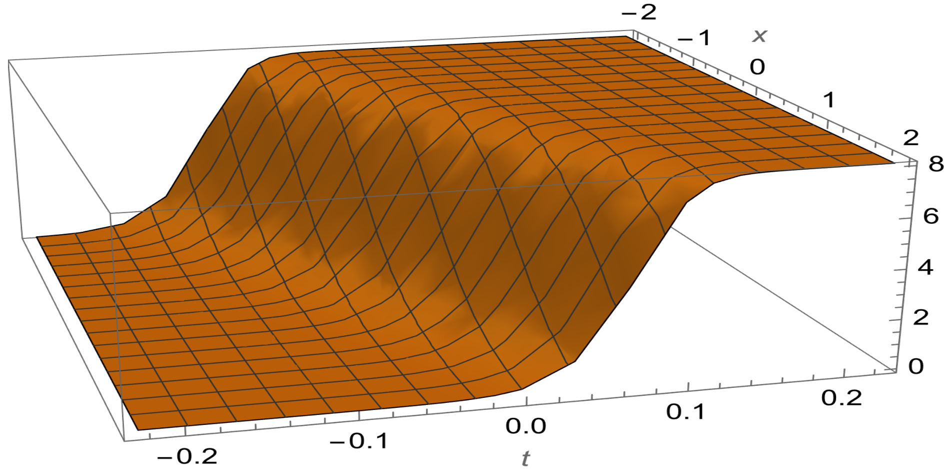
Then $\mathcal{N}(\mathrm{e}^\theta, \mathrm{e}^\theta) = 0$ determines $k = -\beta$. So, $\omega = -6\beta^3$.

First solution:

$$u(x,t) = 2\beta^2 \left(\frac{\mathrm{e}^\theta (2 + \mathrm{e}^\theta)}{(1 + \mathrm{e}^\theta)^2} \right) = -\tfrac{1}{2}\beta^2 \Big(3 - \tanh\tfrac{\theta}{2}\Big)\Big(1 + \tanh\tfrac{\theta}{2}\Big)$$

with $\theta = -\beta x + 6\beta^3 t + \delta$.

Graph of the solution for $\beta = 2$, $\delta = 0$.



Case 2: $\beta = k$.

Then $\mathcal{N}(\mathrm{e}^{\theta}, \mathrm{e}^{\theta}) = 0$ determines $\omega = -6\beta^3$.

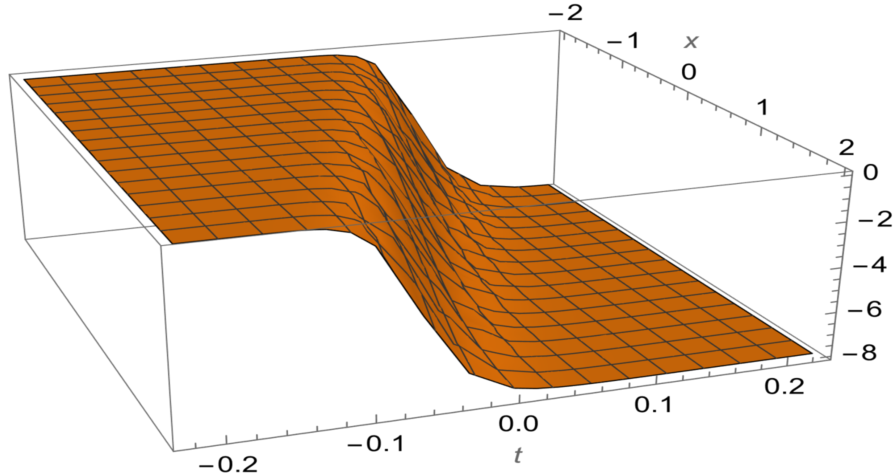
Second solution:

$$u(x,t) = -2\beta^2 \left(\frac{\mathrm{e}^{2\tilde{\theta}}}{(1+\mathrm{e}^{\tilde{\theta}})^2} \right) = -\tfrac{1}{2}\beta^2 \Big(1 + \tanh \tfrac{\tilde{\theta}}{2} \Big)^2$$

with $\tilde{\theta} = \beta x + 6\beta^3 t + \tilde{\delta}$.

Case 2: $\beta = k$.

Graph of the solution for $\beta = 2$, $\tilde{\delta} = 0$.



Concluding remarks

- Shed some light on how Hirota's method works.
- Hirota transformation and homogenization (of degree) are crucial.
- You can still proceed without knowing the bilinear forms.
- Bilinear form is useful, e.g., to proof existence of solitons solutions, etc.
- Hirota's method can also be used to find rational solutions.
- Hirota's method applies to discrete equations.
- Simplified approach applies to nonlinear PDEs that are not solitonic.

Thank You

Extra examples are given on the next slides

Extra example: The mKdV equation

$$u_t + 24u^2u_x + u_{xxx} = 0.$$

Laurent series (two branches combined):

$$u = \tfrac{1}{2}i\left(\frac{F_x}{F} - \frac{G_x}{G}\right) = \tfrac{1}{2}i\left(\ln\left(\frac{F}{G}\right)\right)_x$$

Let $F = f + ig,\ G = f - ig$

$$u = \tfrac{1}{2}i\left(\ln\left(\frac{f+ig}{f-ig}\right)\right)_x = \left(\arctan\left(\tfrac{f}{g}\right)\right)_x = \frac{f_x g - f g_x}{f^2+g^2}.$$

This is Hirota's transformation for the mKdV equation!

Integrate

$$u_t + 24u^2 u_x + u_{xxx} = 0$$

with respect to x:

$$\partial_t \Big(\int^x u \, dx \Big) + 8u^3 + u_{xx} = 0.$$

Applying Hirota's transformation

$$u = \left(\arctan\left(\tfrac{f}{g}\right) \right)_x = \frac{f_x g - f g_x}{f^2 + g^2}$$

yields

$$\begin{aligned} & f^3(g_t + g_{3x}) - g^3(f_t + f_{3x}) - f^2(f_t g + 3f_x g_{xx} + 3f_{xx} g_x + f_{3x} g) \\ & + g^2(f g_t + 3f_{xx} g_x + 3f_x g_{xx} + f g_{3x}) + 6f g_x(f_x^2 + g_x^2) \\ & - 6f_x g(f_x^2 + g_x^2) + 6fg(f_x f_{xx} - g_x g_{2x}) = 0 \end{aligned}$$

which can be regrouped as

$$
\begin{aligned}
&(f^2+g^2)(f_t g - f g_t + f_{3x} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{3x}) \\
&-6(f_x g - f g_x)(f f_{xx} - f_x^2 + g g_{xx} - g_x^2) = 0,
\end{aligned}
$$

and recast into bilinear form:

$$
\begin{aligned}
&(D_t + D_x^3)(f \cdot g) = 0 \\
&D_x^2(f \cdot f + g \cdot g) = 0.
\end{aligned}
$$

Seek

$$
\begin{aligned}
f &= f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots \\
g &= g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + \dots .
\end{aligned}
$$

Then $f^{(0)} = g^{(1)} = 0$ (or, equivalently, $g^{(0)} = f^{(1)} = 0$).

One-soliton solution of mKdV equation

With $f = \mathrm{e}^{\theta}$ and $g = 1$

$$u \;=\; \frac{f_x}{1+f^2} = \frac{k\,\mathrm{e}^{\theta}}{1+\mathrm{e}^{2\theta}} = \tfrac{1}{2}k\,\mathrm{sech}\,\theta$$

$$= \tfrac{1}{2}k\,\mathrm{sech}\,(kx - k^3t + \delta) = K\mathrm{sech}\left(2K(x - 4K^2t + \Delta)\right)$$

with $k = 2K,\ \delta = 2K\Delta$.

Two-soliton solution of mKdV equation

$$\begin{aligned} f &= e^{\theta_1} + e^{\theta_2} \\ g &= 1 - a_{12} e^{\theta_1+\theta_2} \end{aligned}$$

with $\theta_i = k_i x - k_i^3 t + \delta_i, \;\; a_{12} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$.

Eventually,

$$u = \frac{k_1 e^{\theta_1} + k_2 e^{\theta_2} + a_{12}\,(k_1 e^{\theta_2} + k_2 e^{\theta_1}) e^{\theta_1+\theta_2}}{1 + e^{2\theta_1} + e^{2\theta_2} + \dfrac{8k_1k_2}{(k_1+k_2)^2}\, e^{\theta_1+\theta_2} + {a_{12}}^2\, e^{2\theta_1+2\theta_2}}.$$

Three-soliton solution of mKdV equation

$$\begin{aligned} f &= \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} - b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \\ g &= 1 - a_{12}\,\mathrm{e}^{\theta_1+\theta_2} - a_{13}\,\mathrm{e}^{\theta_1+\theta_3} - a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \end{aligned}$$

with $\theta_i = k_i x - k_i^3 t + \delta_i,\ a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2,\ b_{123} = a_{12}a_{13}a_{23}$.

N-soliton solution of mKdV equation

$$u(x,t) = \frac{1}{2i}\left(\ln \frac{\det(I + iM)}{\det(I - iM)}\right)_x$$

where I is the $N \times N$ identity matrix and

$$M_{\ell m} = \frac{\mathrm{e}^{\Theta_\ell + \Theta_m}}{2(K_\ell + K_m)} \ \text{ with } \Theta_\ell = K_\ell x - 4K_\ell^3 t + \Delta_\ell.$$

Other Equations and Their Bilinear Forms

Type I

$$\text{KdV equation}: \quad u_t + 6uu_x + u_{xxx} = 0$$

$$u = 2(\ln f)_{xx}$$

$$(D_x D_t + D_x^4)(f \cdot f) = 0$$

Type II

mKdV equation : $\quad u_t + 6u^2 u_x + u_{xxx} = 0$

$$u = 2\Big(\arctan\left(\frac{f}{g}\right)\Big)_x$$

$$(D_t + D_x^3)(f\cdot g) = 0$$

$$D_x^2(f\cdot f + g\cdot g) = 0.$$

Alternative 1 : $\quad u = \frac{G}{F}$

$$(D_t + D_x^3)(G\cdot F) = 0$$

$$D_x^2(F\cdot F) - 2G^2 = 0$$

Alternative 2 : $\quad v = i\Big(\ln\left(\frac{f^*}{f}\right)\Big)_x$

$$(D_t + D_x^3)(f^*\cdot f) = 0$$

$$D_x^2(f^*\cdot f) = 0$$

Type III

$$\text{sine}-\text{Gordon equation}: \quad u_{xt} = \sin u$$

$$u = 2i \ln\left(\frac{f^*}{f}\right), \quad \sin u = \frac{1}{2i}\left(\left(\frac{f}{f^*}\right)^2 - \left(\frac{f^*}{f}\right)^2\right)$$

$$D_x D_t (f \cdot f) = -\frac{1}{2}(f^{*2} - f^2)$$

$$\text{Alternative 1}: \quad u = 4 \arctan\left(\frac{G}{F}\right)$$

$$(D_x D_t - I)(F \cdot G) = 0$$

$$D_x D_t (F \cdot F - G \cdot G) = 0$$

$$\text{Alternative 2}: \quad u = 4 \arctan\left(\frac{\tilde{F} - \tilde{G}}{\tilde{F} + \tilde{G}}\right)$$

$$(D_x D_t - I)(\tilde{F} \cdot \tilde{F} - \tilde{G} \cdot \tilde{G}) = 0$$

$$D_x D_t (\tilde{F} \cdot \tilde{G}) = 0$$

Type IV

$$\text{NLS equation}: \quad iu_t + u_{xx} + |u|^2 u = 0$$

$$u = \frac{G}{F}, \quad \text{with F real}, \quad |u|^2 = 2(\ln F)_{xx}$$

$$(iD_t + D_x^2)(G \cdot F) = 0$$

$$D_x^2(F \cdot F) = GG^*$$

Type V

Benjamin–Ono equation : $u_t + 2uu_x + \mathcal{H}u_{xx} = 0$

Hilbert transform $\mathcal{H}(w)(t) = \frac{1}{\pi}\,\text{p.v.} \int_{-\infty}^{\infty} \frac{w(t')}{t-t'}\,dt'$

$$u = i\Big(\ln\Big(\frac{f^*}{f}\Big)\Big)_x$$

$$(iD_t + D_x^2)(f\cdot f^*) = 0,$$

Coupled systems

$$\text{Hirota} - \text{Satsuma system}$$

$$u_t + 3u_x v + u_{xxx} = 0$$

$$v_t - a(6vv_x + v_{xxx}) - 2buu_x = 0$$

$$u = \frac{G}{F}, \quad v = 2(\ln F)_{xx}$$

$$(D_t + D_x^3)(F \cdot G) = 0$$

$$(D_x D_t - aD_x^4)(F \cdot F) = bG^2$$

N-soliton solution only exists if $a = \frac{1}{2}$.

Polynomial versions of sine-Gordon equation

$$u_{xt} = \sin u$$

$$f = \mathrm{e}^{iu}$$

$$f^3 - f(f_{xt} + 1) + f_x f_t = 0$$

Alternative :

$$\phi = u_x, \quad \psi = \cos u \, - 1$$

$$\phi_{xt} - \phi - \phi\psi = 0$$

$$\phi_t^2 + 2\psi + \psi^2 = 0$$