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WAVELETS: Theory and Applications
An Introduction

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Introduction to WAVELETS: Theory and Applications

LECTURE NOTES
HOMEWORK ASSIGNMENT & EXAM
BOOKS ON WAVELETS

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Wed.: 10-10:50, Fri.: 10-10:50, 11-11:50

Topics—Outline

- Introduction: History, motivation, applications
- Key ideas and definitions
- Haar and Daubechies wavelets
- Scaling and wavelet functions
- Multi-resolution analysis
- Wavelet analysis: decomposition and reconstruction
- Fast Fourier Transform (FFT) versus Fast Wavelet Transform (FWT)
- Vanishing moments, smoothness, approximation
- Low and high pass filters
- Quadrature Mirror Filters (QMF)
- Construction of Daubechies' wavelets
- Construction of scaling and wavelet functions
- Selected applications
- Software demonstration

- INTRODUCTION (André Weideman)
- BACKGROUND AND EXPERIENCE WITH WAVELETS
  * 1993- Work with Gregory Beylkin at CU - Boulder.
  * 1993 - '94 Sabbatical year devoted to wavelets and applications.
  * 1993 - Short course in Ghent, Belgium (my alma mater).
  * 1994
    - Work on coiflets (with Monzon and Beylkin),
    - work on Dubuc-Deslauriers' subdivision scheme and wavelets,
    - work on Battle-Lemarié spline based wavelets.
  * Course on wavelets at CSM-Golden, CO (1995).
  * Short course on wavelets in Antwerp. Fall 2000 (Also, May 1995).
  * 1998 Paper on coiflets.
  * Currently wavelet issues related to applications (facial recognition, fingerprints, etc.).
  * Belgian connection with Ingrid Daubechies and Wim Sweldens.
- PLAN AT STELLENBOSCH
  * Gentle introduction to wavelets (concepts).
  * American style: ask questions.
  * Media
    o Lecture notes in library.
    o Copies of slides in library.
    o Reference books in library.
    o Review papers in office.
    o Wavelet Explorer (*Mathematica).*
- WHO HAS EXPERIENCE WITH WAVELETS?
  * o Theory.
    o Applications? Which?
  * Who has heard about wavelets and where, what context?
  * Relation to field of study.

  * Background in mathematics.
  * Reasons to take course?

- GRADE FOR WAVELET SHORT COURSE
  * One assignment 4 or 5 problems.
  * One examination question (open book).

- INFO ABOUT WAVELETS
  o long list of books (copy in library).
  o thousands of papers.
  o two dozen good review papers.
  o Wavelet Digest (Wavelets for Kids) http://www.wavelet.org/wavelet.
  o Electronic resources.
  o Wavelet Software.
  o Info on WEB (see course Weideman).

DEMO OF WAVELET EXPLORER

To get to Wavelet Explorer.

- Start *Mathematica*.
- Click on Help.
- Click on Add-ons.
- Click on Wavelet Explorer.

To get intro to Wavelet Explorer
From wavelet Explorer   Pick Fundamentals of Wavelets

To use it in your own notebook in *Mathematica*

In [1]   Needs [“Wavelets‘ Wavelets‘”].
In [2]   ? *aub*.
    DaubechiesFilter[n] (*brief description*)
In [3]   ??DaubechiesFilter[n] (*for full description with all options*)
In [4]   DaubechiesFilter[3].
    For filtertaps of Daubechies averaging filter with 3 vanishing moments.

GENERAL IDEAS ABOUT WAVELETS

* Arena (setting) Mathematics behind data, signal, image analysis or processing (Engr.)and numerical analysis (Mathematics/Applied Mathematics).

* Wavelets is at interface of engineering and mathematics (compare with mathematical physics, mathematical biology, mathematical finance, ...)

* Counterpart to Fourier analysis

FFT $\longleftrightarrow$ FWT (or discrete DWF)
Fast Fourier transform *Fast wavelet transform*

Great Discovery of Jean-Baptiste Fourier (1768-1830).

A periodic signal (sound, function) can be decomposed in harmonics (sines or cosines, or complex exponentials).

* Shortcomings of Fourier analysis (*weaknesses*)

- sines and cosines wiggle (extend) infinitely far,
- does not handle gaps or localized signals well, or faulty data (too many coefficients needed to make cancellations to zero),
- a transient sharp "blip" needs many Fourier components. Huge number of non-zero coefficients,
- does not handle fast variations (highly oscillatory signal) well,
- trouble at discontinuities (overshoot, undershoot, Gibbs phenomena).

* Strong points of Fourier analysis

- functions are analytic and simple,
- simple (sines and cosines) orthogonal functions,
- fast FFT is $n \log n$ algorithm - due to trick invented in 1965 by J. Tuckey and J. Cooley (same trick applies for wavelets),
- few coefficients in series are needed to represent a smooth periodic function,
- very applicable recursive algorithms,
- caused a revolution in scientific computing,
- continuous and discrete transforms are available,

→ Try to keep what's good and try to improve on bad properties ⟹ Wavelets.

* Key requirements

- o a few functions (no longer sines/cosines) should give good approximation,
- o orthogonal functions,
- o we want localization in both time (via translations) and (space) frequency (wavelength) via dilations and contractions,
- o we want fast, recursive algorithm,
- o zoom in - zoom out property,
- o something highly applicable,
- o mathematically on solid foundation (requires new mathematical concepts, ideas, ...construction),
- o continuous and discrete transforms.

* Cost:

⟹ no longer analytical functions,

⟹ not as simple as Fourier series or Fourier transform.

WHAT IS A WAVELET?

"a little wiggle, ripple"

French: ondelette

How many types of wavelets are there? - About 2 dozen practical types.
(Daubechies' wavelets, Meyer wavelets, biorthogonal wavelets, Battle-Lemarié wavelets, ...)

HOW DOES ONE GET TO WAVELET BASIS?

In Fourier analysis:
Take sinusoids (or complex exponentials) as basis functions and then study the properties of the Fourier series (or Fourier integral)
In wavelet analysis:
One poses the desired properties and then derives the resulting basis functions.

[For instance, multiresolution is a key property, orthogonality also, and perhaps symmetry, vanishing moments property].

Key idea:

You want to construct a basis for $\mathbb{L}^2(\mathbb{R})$ (= collection/space of all square integrable functions)

$$f \in \mathbb{L}^2(\mathbb{R}) \text{ if } \int_{-\infty}^{+\infty} f(x)^2 dx < \infty$$

CONCEPTUAL ANALOGIES (OTHER PERSPECTIVES)

1. **The game of averages and differences and vanishing moments**

Student #1 has X_1 amount of money in bank account
Student #2 has X_2 amount of money in bank account

You can either give X_1, X_2 itself or give the

average

$$S_1 = \frac{X_1 + X_2}{2} \Longleftrightarrow X_1 = S_1 + D_1$$

difference

$$D_1 = \frac{X_1 - X_2}{2} \Longleftrightarrow X_2 = S_2 - D_1$$

Advantage if

$$X_1 = X_2 \Longrightarrow D_1 = 0$$

.

If $X_1 = X_2$ follow the same constant trend

then simple "differences" creates a zero student number

"he same amount in account" $\longrightarrow$ difference = zero, $\longrightarrow$ average = how much each has.

Question: Is it possible to generalise this simple idea? YES.

Look at it from a "signal data" processing point of view

$$\underbrace{X_1 \qquad X_2} \longrightarrow \underbrace{X_3 \qquad X_4} \qquad X_5 \ldots$$

move over in steps of two.

$$\text{for averages} \qquad \bullet \left(\frac{1}{2}\right) = h_0 \quad \bullet \left(\frac{1}{2}\right) = h_1$$

$$\text{for differences} \qquad \bullet \left(\frac{1}{2}\right) = g_0 \quad \bullet \left(\frac{1}{2}\right) = g_1$$

$$\begin{aligned} S_1 &= \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2 &&= h_0X_1 + h_1X_2 \\ D_1 &= \tfrac{1}{2}X_1 - \tfrac{1}{2}X_2 &&= g_0X_1 + g_1X_2 \end{aligned}$$

$$\begin{array}{lrcl} \text{Choices} & \left(\frac{1}{2}, \frac{1}{2}\right) & = & h_0, h_1 \\ & \left(\frac{1}{2}, -\frac{1}{2}\right) & = & (g_0, g_1) \\ \text{or} & (1,1) & = & (h_0, h_1) \\ & (1,-1) & = & (g_0, g_1) \\ \text{or} & \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) & = & (h_0, h_1) \\ & \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) & = & (g_0, g_1) \qquad \textit{Haar case} \\ \text{Similarly} & S_2 & = & h_0X_3 + h_1X_4 \\ & D_2 & = & g_0X_3 + g_1X_4, \ \text{etc.} \end{array}$$

Leaping ahead

If $X_1, X_2, X_3, X_4, \ldots$ lie on line can we find h_0, h_1 and g_0, g_1 so that D_1, D_2 etc. are zero? No.

We need four numbers

h_0, h_1, h_2, h_3	for averaging	(low pass filter)
g_0, g_1, g_2, g_3	for differencing so that	(high pass filter)

$$D_1 = g_0X_1 + g_1X_2 + g_2X_3 + g_4X_4 = 0$$

$$\text{if} \quad X_1, X_2, X_3, X_4, \ldots \ \text{are on } \underline{\text{line}}$$

$$\text{Also,} \quad D_2 = 0 \ \text{ etc.}$$

If X_i are on parabola we will need

$$\begin{array}{ll} (h_0, \ldots, h_5) & \text{for averaging} \\ (g_0, \ldots, g_5) & \text{for differencing, so} \end{array}$$

$$\begin{array}{lll} \text{that} & g_0 X_1 + \ldots + g_5 X_6 = D_1 = 0 \\ \text{if} & X_i \quad \text{follow quadratic trend} \end{array}$$

$$\begin{array}{ccc} a + & bX + & cX^2 \\ \downarrow & \downarrow & \downarrow \end{array}$$

.constant linear and quadratic combined

We will need six numbers.

Etc. can be done for x^n also.

If X_i follow on $a + bX + cX^2 + \ldots + fX^n$
then you need $2n$ "magical" numbers with special properties so that

$$\begin{array}{lll} D_1 & = & 0\ , \\ D_2 & = & 0\ , \text{ etc.} \end{array}$$

Examples of data following trends

* Pictures: cone and ball on tiled floor. Show and explain image compression.
* Cartoon: how are the Disney cartoons made? Game for kids: picture book with slightly different (shifted figures) on frozen background.
* Video: TV broadcasting. Transmit only the differences.
  $\longrightarrow$ Life is full of redundancy!

2. **Vanishing moments**

If you could construct a wavelet function $\psi(x)$ which corresponds to "differencing" then you could require that $\psi(x)$ annihilates constant, linear, quadratic, cubic, ... trends ... up to degree say x^{M-1}

$$\text{so} \qquad \int x^{\ell} \psi(x)\, dx = 0 \quad \text{for} \quad \underbrace{\ell = 0, 1, 2, \ldots M-1}_{M \text{ moments}}$$

Since a function $f(x)$ can be written in McLaurin series

$$f(x) = \underbrace{f(0)}_{constant} + \underbrace{f'(0)x}_{linear} + \overbrace{\underbrace{\frac{f''(0)}{2}x^2}}^{quadratic} + \ldots + \underbrace{\frac{f^{(M-1)}}{(M-1)!}x^{M-1}}_{degree\ M-1} + \ldots$$

you would have

$$\int f(x)\psi(x)dx = 0 + 0 + 0 \ldots + 0 + \underbrace{\int \frac{x^M}{M!}\psi(x)dx + \ldots}_{highly\ oscillating\ part}$$

⟶ Wavelets would be sensitive to fast oscillating pieces of $f(x)$ and annihilate slower varying pieces of $f(x)$.

3. **Fingerprint story**

Popularized wavelets
(FBI competition for fingerprint compression)

(Old systems for identification was based on facial features. Bertillion 1884).

- 37 million fingerprint cards for criminals (some in duplicate),
- 250 million fingerprint cards in total (job applications, immigration, background checks for employment ..., weapon purchases, ...),
- 60 km of “thin” cardboard,
- each card has 10 rolled imprints of fingers, 2 unrolled thumb prints, two full hands,
- cataloguing is several million cards behind,
- request for identity check may take between 100 - 140 days,
- enormous office building full of these cards.

⟶ Need for automation and electronic storage (via scanning), saves space. Allows for on-line verification (in shortest possible time, while suspect is being held).
Should be accessible from every police car (digitizing tablet, modem, ...).

- FBI has approximately 350 employees to handle the verification requests, cataloguing, removing duplicates, etc. ...

- Automatic system: IAFIS (Automatic Fingerprint Identification System).
  * cards are digitized and stored on 2 computers in a nuclear attack proof building (cellar 20 m deep in Clarksburg, West Virginia),
  * old card collection is moved to Federicksburg,
  * scanning all cards took 5 years,
  * cost: $640 million,
  * every police car has a scanner and they can fully automatically check the enormous database.
- FBI get 45,000 requests per day for verification.
- FBI adds 5,000 fingerprints to database per day.
- Average checking time (for routine stuff, like for application for weapons permit) takes 30 seconds.
- Computer database is 40 terabytes.
  1 byte = 8 bids (zero or one)
  40×10^{12} bytes (= 40,000 PC with harddrive of 10 gigabytes). *Even compressed!*

How does it get that large?

Each square inch fingerprint image is broken into 500×500 grid of small boxes, called pixels.

Each pixel is given a gray-scale from 0 to 255

$$\downarrow \quad \downarrow$$

white black

$\longrightarrow$ darkness of one pixel requires 8 bits = 1 byte.

Start multiplying: 1 whole card $\cong$ 10 MB.

(Mega byte = 10^6 bytes

Giga byte = 10^9 bytes

Tera byte = 10^{12} bytes)

$\Longrightarrow$ 2500 Tera byte = 2500 10^{12}

$$
\begin{aligned}
250 \text{ million cards } &\cong 2500 10^6 \underbrace{10^6}_{\text{MB}} \\
&\cong 2500 \times 10^2 \times \quad \underbrace{10\text{GB}}_{\text{harddrive}} \\
&\cong \underline{250,000}\text{PC's.} \qquad \text{(quarter million PC's.)}
\end{aligned}
$$

For $\approx$ 37 million criminals alone

$\Longrightarrow$ 37,000 PC's.   1,000 cards fill up one PC!

Impossible without serious data compression for storage and transmission (via JPEG (Joint Photographic Experts Group) or something else.

$\longrightarrow$ FBI competition won by Ron Coifman (Yale) and Bradley and Brislawn, at Los Alamos National Laboratory (biorthogonal wavelets).

DATA COMPRESSION:

Sending 1 card without compression via 56K modem would take $\approx$ 20 minutes

Standard compression ratios 3/1 or 2/1
Wavelet compression ratios 15/1 or 20/1   (5% kept)

$\longrightarrow$ with 20/1 compression

transmission time = 1 minute.

Space savings also factor 15 or 20
$\longrightarrow$ 1/20 of space

(For cirminals' card storage $\longrightarrow$ less than 2,000 PC's)

Wavelet algorithms closer to home:

* wavelet chip in modems (Aware Corp.)
* JPEG 2000 standard for internet pictures (you see the details come in).

APPLICATIONS OF WAVELETS

Signal and Data processing:

- analysis (eg. detection of edges, detection of faults, abrupt changes, defects like "cracks" in materials, . . .),
- compression (reduction of storage space, fast transmission, as in fingerprint application),
- smoothing (attenuation of noise, denoising),
- synthesis (reconstruction after compression or/and smoothening, or other types of modification).

Types of signals

1D : sound, speech, time-series (e.g. history of a stock in financial markets),

2D : pictures, maps, . . . ,

3D : spatial diffusion (e.g. of heat of body, dopamine injected in brain, . . .).

SAMPLE APPLICATIONS

$\longrightarrow$ Reconstruction of Johannes Brahms' early recording (1889) of First Hungarian Dance (story in Scientific American 1993).

Music speech

$\longrightarrow$ Medical applications: analysis of

– MRI scan,
– electro cardiogram,
– ultrasonic images,
– mammograms (tumor detection),
– various other modern scanning, visualization methods in medicine.

Medical imaging

$\longrightarrow$ Military applications:
formation of images based on partial radar data (distinguish ambulance from tank).

Military imaging

$\longrightarrow$ High speed data transmission:

- single pictures (come gradually in focus),

- video (explain principle).

Digital communication

$\longrightarrow$ Denoising technology:
(project for ADA Technologies: detection of metals like vanadium, mercury, lead in smoke, laser spectroscopy $\longrightarrow$ least squares method and wavelets).

Data analysis

$\longrightarrow$ Mathematical, engineering applications:
fast PDE solvers, $Ax = b$ solvers, numerical analysis

FFT vs FWT

$\longrightarrow$ Geophysical applications
seismic methods for detection of rock formations, pockets of gas, detection of oil fields, ...

CSM

Massive data

$\longrightarrow$ Finance:
analysis of time-series (prediction of behaviour of stocks, other financial intruments ...) .

Money matters

$\longrightarrow$ Metereology:
analysis of weather data, climate, temperature in oceans,....

Massive data

SHORT HISTORY

"Wavelets come from a synthesis of ideas".

- First wavelet: Haar 1910 (box functions).
- Engineering experiments in 1960's $\longrightarrow$ 1980's.
- Experiments with windowed Fourier Transforms.
- 1980: Jean Morlet (geophysicist)
  - wavelets for analysis of seismic data.
- 1980 Alex Grossmann: theoretical study of wavelets and wavelet transform.
- 1985: Yves Meyer (harmonic analyst)

- connections with other fields in maths,
- connection with singular integral operators.

- 1986 - 1988: Ingrid Daubechies (VUB colleague)
  - frames (generalized basis for Hilbert space),
  - 1988 - major breakthrough,
  - "Daubechies" family of orthonormal wavelets with compact support,
  - connection with coherent states.
  - Inspired by work of Mallat and Meyer on multi-resolution analysis and applications in decomposition and reconstruction of images (computer graphics).
  - "Ten lectures on Wavelets".
- Beyond 1988: exponential growth in theoretical developments.
  (5,000 papers in Science Citation Index).
  Applications in physics (coherent states), quantum field theory, electrical engineering, computer science, statistics, etc., etc., ...

"SYNTHESIS OF IDEAS"

SCALING AND WAVELET FUNCTIONS

* Wavelet is little ripple, a little wave.

$\psi(x)$, not symmetric, such that
$\int \psi(x)dx = 0$, and such that(*admissibility condition)*
$\psi(x)$ will have special properties.

* Start with "mother" wavelet, concentrated, say, in interval [0,1].

* Generate other wavelet by moving the mother wavelet left or right in unit steps, and by dilating or compressing the mother wavelet by repeated factors of 2.

Compare with images (zoom into a picture)
dilate (stretch) $\xrightarrow{zoom\ out}$ lower resolution details of picture,
compress (squeeze) $\xrightarrow{zoom\ in}$ higher resolution details of picture.

Look at TV-screen:
large screen (blurred image) expanded image
small screen (sharper image) compressed image.

Definition (Mathematical)
Let $\mathbb{L}^2\mathbb{R}$ be the Hilbert space of square integrable functions on real line.
A wavelet is a funtion $\psi \in \mathbb{L}^2(\mathbb{R})$ such that

$$2^{j/2}\psi(2^j x - k) \;\; j,\, k \in \mathbb{Z}$$

is an orthonormal basis of $\mathbb{L}^2(\mathbb{R})$

There are two families in the game:
$\varphi(x)$ scaling function $\longrightarrow \varphi_{j,k}(x)$
$\downarrow$
father where $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) = 2^{j/2}\varphi(2^j(x - k/2^j))$
$\uparrow$ $\uparrow$
2 parameter family *kids (boys)*
$\psi(x)$ wavelet function $\longrightarrow \psi_{j,k}(x) \longrightarrow$ *kids(girls)*
$\downarrow$
mother where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) = 2^{j/2}\psi(2^j(x - k/2^j))$

↑

2 parameter family

Haar wavelet   Daubechies wavelet

(DAUB4) 2 vanishing moments

Picture mother

Picture father

Pictures of scaling and wavelet functions

Key features (desired properties).

* The basis should be orthonormal "vectors" (here functions) should be mutually orthogonal, and properly normalized

  $\longrightarrow$ fast algorithms to compute coefficients.

* The basis functions (wavelets) should have vanishing moments
  $\longrightarrow$ sparsity (lots of zeros) everywhere.

* The algorithms should be recursive. True due to the multi-resolution properties of the basis (always devide or multiply by factor 2).

* There should be no need for the functions themselves

  so, the family $\{2^{j/2}\psi(2^j x - k)\}_{j,k}$ is not needed.

  Instead one uses "filter coefficients or taps" for averaging, differencing (you want to maintain a constant norm, independent of j).

* Normalization

  If $\qquad \psi_{j,k}(x) = c\psi(2^j x - k)$

  Require $\qquad \|\psi_{j,k}(x)\|^2 = \|\psi_{0,0}(x)\|^2 = \|\psi\|^2$

  $\int \psi_{j,k}(x)^2 dx = c^2 \int \psi^2(2^j x - k)dx$, set $z = 2^j x - k$

  $= c^2 2^{-j} \int \psi^2(z)dz = c^2 2^{-j}\|\psi\|^2 = 1.\|\psi\|^2$

  so we need $\quad c = 2^{j/2}$.

Note: Daubechies wavelets

If ψ is a Daubechies wavelet with M vanishing moments that generates an orthonormal basis of $\mathbb{L}^2(\mathbb{R})$ then

ψ is supported on $[0, 2M - 1]$ or $[-M + 1, M]$ if translated to left over $M - 1$

φ is supported on $[0, 2M-1]$.

Vanishing moments

2 vanishing moments means $\int \psi(x)\, dx = 0, \quad \int x\, \psi(x)\, dx = 0$

4 vanishing moments $\int x^\ell\, \psi(x)\, dx = 0$ for $\ell = 0, 1, 2, 3$

M vanishing moments $\int x^\ell\, \psi(x)\, dx = 0$ $\ell = 0, 1, 2, \ldots, M-1.$

Wavelets with vanishing moments will annihilate (ignore, or be blind to) certain trends; linear (*including const*), quadratic, cubic, quartic, quintic, etc. and only be sensitive (pick out) higher degree oscillations.

Expanding and contracting wavelets.
Compare with music.
ψ quarter note at middle C (*where your start*)
$\frac{1}{V2}\psi(\frac{x}{2})$ half note a lower octave (*stretch*)
$2\psi(4x)$ 16^{th} note at 2 octaves higher (*contract*)

THE REFINEMENT OR TWO-SCALE DIFFERENCE EQUATION

Haar case

$$\varphi(x) = \text{characteristic function on } (0,1)$$

$$\varphi(x) = \begin{cases} 1 & , \quad 0 \le x < 1 \\ 0 & , \quad \text{elsewhere } (x < 0 \text{ or } x \ge 1). \end{cases}$$

Observe

$$\int_0^1 \varphi(x) dx = 1 = \int_0^1 \varphi^2(x) dx.$$

Also, for scaling function

$$\varphi(x) = \varphi(2x) + \varphi(2x-1).$$

Refinement equation for Haar case.

Other names: dilation equation; two-scale difference equation.

Verify

$$\begin{array}{rl} 0 \le x < 1/2 & 1 = 1 + 0 \\ 1/2 \le x \le 1 & 1 = 0 + 1 \\ x < 0 \;\text{ or }\; x \ge 1 & 0 = 0 + 0. \end{array}$$

Also, for wavelet function

$$\psi(x) = \varphi(2x) - \varphi(2x-1).$$

<u>Note:</u> wavelet function can be defined in terms of scaling function.

<u>IN GENERAL (DAUBECHIES WAVELET AND OTHERS)</u>

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k \varphi(2x-k)$$

$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k \varphi(2x-k)$$

Equivalently

$$\varphi(x/2) = \sqrt{2}\sum_{k=0}^{L-1} h_k \varphi(x-k), \text{ similar for } \psi(x/2)$$

$$L = 2 \quad \text{(length of filter)}$$

<u>For Haar case</u>

$$\begin{aligned} \varphi(x) &= \varphi(2x) + \varphi(2x-1) \\ &= \sqrt{2}\left(\tfrac{1}{\sqrt{2}}\varphi(2x) + \tfrac{1}{\sqrt{2}}\varphi(2x-1)\right) \\ \psi(x) &= \varphi(2x) - \varphi(2x-1) \\ &= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\varphi(2x) - \tfrac{1}{\sqrt{2}}\varphi(2x-1)\right) \\ \longrightarrow &\ (h_0, h_1) = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = H \\ \longrightarrow &\ (g_0, g_1) = \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) = G \end{aligned}$$

Filters for averaging and differencing.

In general

$$\begin{aligned} H &= (h_0, h_1, h_2, \ldots, h_{L-1}) && \text{low pass filter} \\ G &= (g_0, g_1, g_2, \ldots, g_{L-1}) && \text{high pass filter.} \end{aligned}$$

Always come in pairs

Note: As vectors $\overrightarrow{H} \perp \overrightarrow{G}$

i.e. $\varphi \perp \psi$ as functions

$H \longrightarrow G$ reverse the order and switch every other sign

$$H = (h_0, h_1, \ldots, h_{L-1}) \longrightarrow G = (h_{L-1}, -h_{L-2}, \ldots, -h_0).$$

Reverse (of refinement equation)

$$\varphi(2x) = \frac{1}{2}(\varphi(x) + \psi(x))$$

$$\varphi(2x-1) = \frac{1}{2}(\varphi(x) - \psi(x))$$

Later, general:

$$\varphi(2x - \ell) = \sum_k (a_{\ell-2k}\varphi(x-k) + b_{\ell-k}\psi(x-k))$$

Remarks

1. In general there is no (explicit) analytical form for $\varphi(x)$ or $\psi(x)$
2. $\varphi(x)$ is evaluated via the refinement equation at dyadic points $x = \frac{k}{2^j}$ $j, k \in \mathbb{Z}$

   $\longrightarrow$ (*Assignment plus old notes. Eigenvalue problem*).

   Example:

$$\begin{aligned} \text{Set } x = 1/2 \quad \varphi(1/2) &= \overbrace{\varphi(1)}^{0} + \overbrace{\varphi(0)}^{1} = 1 \\ \text{Also} \quad \varphi(-1/2) &= \varphi(-1) + \varphi(-2) \\ &= 0 + 0 = 0 \end{aligned}$$

So, if φ is known at integers then φ is known at the halves

Recursive!

$$\varphi(1/4) = \varphi(1/2) + \varphi(-1/2) = 1$$

General $\varphi(2^{j-1}n) = \varphi(2^j n) + \varphi(2^j n - 1)$

3. To be able to use $\{\varphi(x-k)\}$ $k \in \mathbb{Z}$ to approximate even simple function we assume
   that $\varphi(x)$ and its translates form a partition of unity.

$$\sum_{k \in \mathbf{Z}} \varphi(x-k) = 1, \quad \forall x \in \mathbb{R}$$

   Proof in Fourier domain

$$\hat{\varphi}(0) = 1 \Longrightarrow \sum_k \varphi(x-k) = 1.$$

   More general:

$$f(t) = \sum_n f_n(t) \text{ where } f_n(t) = \varphi(t-n) f(t) \longrightarrow \text{localized function.}$$

   For Haar case:

$$\varphi(x) = \varphi(2x) + \varphi(2x-1)$$

4. Refinement equations were known in context of splines.

   Examples

   * Piecewise linear spline (hat function)

$$N_2 = \begin{cases} x & , \ 0 \le x < 1 \\ 2-x & , \ 1 \le x < 2 \\ 0 & , \ \text{elsewhere} \end{cases}$$

   Satisfies

$$N_2(x) = \frac{1}{2} N_2(2x) + N_2(2x-1) + \frac{1}{2} N_2(2x-2)$$

   * Piecewise quadratic spline

$$N_3(x) = \begin{cases} 1/2x^2 & , \ 0 \le x \le 1 \\ 3/4 - (x-3/2)^2 & , \ 1 \le x < 2 \\ 1/2(x-3)^2 & , \ 2 \le x \le 3 \\ 0 & , \ \text{elsewhere} \end{cases}$$

   Satisfies

$$N_3(x) = \frac{1}{4} N_3(2x) + \frac{3}{4} N_3(2x-1) + \frac{3}{4} N_3(2x-2) + \frac{1}{4} N_3(2x-3)$$

5. In general

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k\varphi(2x-k) \overset{\int \varphi(x)dx=1}{\Longrightarrow} \sum_{k=0}^{L-1} h_k = \sqrt{2} \qquad \textit{linear}$$

$$\psi(x) = \sqrt{2}\sum g_k\varphi(2x-k) \overset{\int \psi(x)dx=0}{\Longrightarrow} \sum_{k=0}^{L-1} g_k = 0 \qquad \textit{linear}$$

Proof - Integrate both sides

$$1 = \int \varphi(x)dx = \sqrt{2}\sum_{k=0}^{L-1} h_k \int \varphi\overbrace{(2x-k)}^{z}dx$$

$$= \frac{\sqrt{2}}{2}\sum_{k=0}^{L-1} h_k \underbrace{\int \varphi(z)dz}_{1}$$

$$= \frac{1}{\sqrt{2}}\sum_{k=0}^{L-1} h_k$$

$$\longrightarrow \sum_{k=0}^{L-1} h_k = \sqrt{2}$$

Verify for Haar case $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\int \psi(x)dx = 0 \Longrightarrow \sum_k g_k = 0$

Similar proof for $\sum_k g_k = 0$

Verify for Haar case $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

6. Orthogonality properties

$$\varphi(x) \perp \varphi(x-k), \qquad \forall k \in \mathbb{Z}_0 \quad (k \neq 0)$$

Inner product in $\mathbb{L}^2(\mathbb{R})$

$$< f, g > = \int_{-\infty}^{+\infty} \overline{f(t)}\, g(t)\, dt$$

$$\text{So } < \varphi, \varphi_n > = < \varphi, \varphi(x-n) > = \delta_{no}$$

One can show that

$$\varphi(x) \perp \varphi(x-n), \qquad \text{if } n \neq 0$$

and

$$\varphi^2(x) \text{ normalized } (n=0) \int \varphi^2(x)dx = 1$$

leads to *(No proof here - proof in extended notes)*

(i)

$$\sum_{k=0}^{L-1} h_k^2 = 1 \qquad \longleftarrow \underline{\textit{quadratic}} \qquad (n=0) \textit{ case above}$$

(ii)

$$\sum_{\ell=0}^{L-1-2m} h_{2m+\ell}\, h_\ell = 0 \qquad\qquad m = 1, 2, \ldots, \frac{L}{2} - 1 \quad (n \neq 0\,)\textit{above}$$

Show in detail via shifts of two

$h_0\ h_1\ h_2\ h_3\ \ldots$

$h_0\ h_1\ \ldots$

Verify for Haar case

$$\text{(i)} \qquad (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$L = 2 \qquad m = 0 \qquad$ - no condition of type (ii)

$\varphi \perp \psi \qquad (h_0, h_1) \perp (g_0, g_1) = (h_1, -h_0)$

so, $\qquad h_0 h_1 - h_1 h_0 = 0.$

For

$$L \;=\; 8 \;\begin{cases} h_0h_2 + h_1h_3 + h_2h_4 + h_3h_5 + h_4h_6 + h_5h_7 = 0 \\ h_0h_4 + h_1h_5 + h_2h_6 + h_3h_7 = 0 \\ h_0h_6 + h_1h_7 = 0 \end{cases}$$

7. Admissibility Condition

$$\int \psi(x)dx = 0$$

$$\psi(x) = h_0\varphi(2x) - h_1\varphi(2x-1)$$

$$\int \psi(x)dx = 0 = \frac{h_1}{2}\overbrace{\int \varphi(z)dz}^{1} - \frac{h_0}{2}\overbrace{\int \varphi(z)dx}^{1}$$

$$\Longrightarrow h_0 = h_1$$

In general,

$$\sum_{k=0}^{L-1} h_{2k} = \sum_{k=0}^{L-1} h_{2k+1} = \frac{1}{\sqrt{2}}, \quad \text{and also } \sum_{k=0}^{L-1} (-1)^k h_k = 0 \qquad (H \perp G)$$

REPRESENTING FUNCTIONS IN A WAVELET BASIS

Multi-resolution

For Haar case (1910)

$\{\psi_{j,k}(x)\}_{k,j\in\mathbf{Z}}$ is a fully orthonormal basis for $\mathbb{L}^2(\mathbb{R})$.

So for $f(x) \in \mathbb{L}^2(\mathbb{R})$

$$\begin{aligned} f(x) &= \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_{jk}\psi_{j,k}(x) \\ &= \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a_{jk}2^{j/2}\psi(2^j x - k) \end{aligned}$$

$\uparrow$ *wavelets*

In practice, Haar also used

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k\varphi(x-k) + \sum_{k=-\infty}^{+\infty} \sum_{j=0}^{+\infty} d_{jk}\psi_{j,k}(x)$$

$\downarrow$ *translates at scale* $j = 0$

$\downarrow$ *adding weighted wavelets at different scales, starting from* $j = 0$ *and allowing translates also.*

Effect of j

$x \longleftrightarrow 2^j x$ space $\longleftrightarrow$ scale

$t \longleftrightarrow 2^j t$ time $\longleftrightarrow$ frequency (via dilation)

Effect of k localization in space (or time)
(via translation)

RESOLUTION - MULTI-RESOLUTION

A few examples

1. Our number system has multi-resolution.
   Compare with computation of $\frac{\text{circumference circle}}{2\text{ radius}} = \pi$

   $\pi = 3.1415926535$

   j'aime a faire connaitre le nombre utile aux sages.

   $3 = 3.\frac{1}{10^0}$ $\longrightarrow$ *what scaling function shows at a given resolution*

   $3.1 = 3.\frac{1}{10^0} + \underbrace{1.\frac{1}{10}}$ $\longrightarrow$ *better resolution*

   wavelet contribution - encodes the detail

   $$3.14 = 3.\frac{1}{10^0} + \underbrace{1.\frac{1}{10}} + \underbrace{4.\frac{1}{10^2}} + \dots$$

   Other direction: stretching (adding more and more detail), the scaling function too much we end up seeing nothing at all. Represent 3.14159 in $10 = 10^1$ or $100 = 10^2$ scales.

2. Picture of Dali's painting.
   "Gala naked watching the sea".

   (Salvador Dali, 1976, Dali Museum, Figueres, Spain).

   Zoom in: Look at detail (close-up) $\longrightarrow$ Gala (Dali's wife).
   Zoom out: Look at coarse picture (from afar) $\longrightarrow$ Abraham Lincoln.

   Dali was aware of images at different scales of resolution.

Basic notions (First for Haar case)
Inspiration for multi-resolution properties.

1. Consider the scaling function φ and the associated 2 parameter scaling family

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) \qquad j,k \in \mathbb{Z}$$

For fixed j : $\{\varphi_{j,k}(x)\}_{k\in \mathbf{Z}}$ spans a subspace V_j of $\mathbb{L}^2(\mathbb{R})$.

More precise: We define the subspace $\longrightarrow$ closure.

$$V_j \;=\; \text{span}_{k\in \mathbf{Z}} \left\{ \varphi_{j,k}(x) \right\} \qquad\qquad j \text{ fixed}$$

and $\varphi_{j,k}(x)$ is a basis for V_j, the space of piecewise constant functions (with dyastic points $\frac{k}{2^j}$).

Example:

$$\text{If} \qquad f(x) \in V_0 \qquad \text{then} \qquad f(x) = \sum_{k\in \mathbf{Z}} a_k \overbrace{\varphi_{0,k}(x)}^{\varphi(x-k)}$$

$V_0 =$ space of piecewise constant functions over integers.

$V_1 =$ space of ... over halves = larger space than V_0,

because the functions $\varphi_{1,k}(x) = \sqrt{2}\varphi(2x-k)$

$= \sqrt{2}\varphi(2(x-k/2))$

are narrower and translated in smaller steps.

can represent finer detail

So $\qquad V_0 \subset V_1$

In general $\qquad \ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$

$V_j \subset V_{j+1} \longrightarrow$ Question: what "lies" between them?

2. Consider the wavelet function ψ and the associated 2 parameter family

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \qquad j,k \in \mathbb{Z}.$$

For fixed j : $\qquad \{\psi_{j,k}(x)\}_{k\in \mathbf{Z}} \qquad$ spans W_j

$\{\psi_{j,k}\}_{k\in\mathbf{Z}}$ is basis for W_j, again with piecewise functions on dyadic grid $\{k/2^j\}$ is a fully orthonormal basis.

3. $\{\psi_{j,k}\}_{(j,k\in\mathbf{Z}}$

   basis for $\mathbb{L}^2\mathbb{R}$. This basis is complete.

   It is an infinite basis in terms of wavelet functions.

4. Practical alternative

   Take the set

$$\begin{aligned}\{\varphi_{j_0,k}(x) &= j_0/2\varphi(2^{j_0}x-k),\\ \psi_{j,k} &= 2^{j/2}\psi(2^jx-k)\}_{j\geq 0,\,k\in\mathbf{Z}}.\end{aligned}$$

   which also forms a basis for $\mathbb{L}^2(\mathbb{R})$.

   In practice, one quite often takes $j_0=0$

   and therefore takes

$$\begin{aligned}&\{\varphi(x),\varphi(x\pm 1),\varphi(x\pm 2),\ldots,\ldots\\ &\psi(x),\psi(x\pm 1),\psi(x\pm 2),\ldots\\ &\sqrt{2}\psi(2x\pm 1),\sqrt{2}\psi(2x\pm 2),\ldots\\ &\quad\vdots\\ &2^{j/2}\psi(2^jx-k)\}\end{aligned}$$

   as the basis

   Note: The set $\{\varphi_{j_0,k}(x)\}_{k\in\mathbf{Z}}$ spans the same subspace as $\{\psi_{j,k}(x)\}_{j\leq j_0,\,k\in\mathbf{Z}}$.

5. Visualization of Haar's multi-resolution

   V_0 coarse

   V_1 finer $\qquad V_0\subset V_1\subset V_2$

   V_2 still finer

   $\vdots$

   V_j by having j sufficiently large the subspace V_j is quite large.

   Using the wavelets $\psi_{j,k}(x)$ takes care of the differences.

   The precise relationship between V_j, W_j and $\mathbb{L}^2(\mathbb{R})$ is given via the Multiresolution Analysis (MRA).

6. Beyond the Haar case

   The same construction is possible, the same concepts and ideas hold for many types of wavelet families (Daubechies wavelets in particular).

   Advantage: We will be able to use more regular (smoother) functions φ and ψ at the cost of a more elaborate construction and loss of analytic or closed forms, longer filters, etc.

Multi-resolution analysis (Meyer and Mallat - 1986)

Multi-resolution analysis of $\mathbb{L}^2(\mathbb{R})$ is an increasing sequence of closed (nested) subspaces

$$\{V_j\}_{j\in\mathbf{Z}} \text{ of } \mathbb{L}^2(\mathbb{R}),$$
$$\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots$$

Such that

(i) $\bigcap_{k\in\mathbf{Z}} V_j = \{0\}$ or $\lim_{j\to-\infty} V_j = \{0\}$ (no overlap, separate spaces)

(ii) $\bigcup_{k\in\mathbf{Z}} V_j = \mathbb{L}^2(\mathbb{R})$ or $\lim_{j\to\infty} V_j = \mathbb{L}^2(\mathbb{R})$ $\left(\bigcup_j V_j \text{ is dense in } \mathbb{L}^2(\mathbb{R})\right)$

(iii) For any $f \in \mathbb{L}^2(\mathbb{R})$, for any $j \in \mathbb{Z}$

$$f(x) \in V_j \text{ iff } f(2x) \in V_{j+1} \; \left(\text{Also } f(x) \in V_0 \Longleftrightarrow f(2^j x) \in V_j\right)$$

(iv) For any $f \in \mathbb{L}^2(\mathbb{R})$, for any $k \in \mathbb{Z}$

$$f(x) \in V_0 \quad \text{iff} \quad f(x-k) \in V_0$$

(v) $\exists \varphi \in V_0$ such that

$$\{\varphi(x-k)\}_{k\in\mathbf{Z}} \text{ is an orthonormal basis of } V_0$$

.

Notes:

(1) φ is the scaling function.

(2) ψ will come as a by-product.

(3) Riesz basis: A countable set $\{f_n\}$ of a Hilbert space $\mathbb{L}^2(\mathbb{R})$ is a Riesz basis if $\forall f \in \mathbb{L}^2(\mathbb{R})$ it can be uniquely written as

$$f = \sum_{n \in \mathbb{N}} c_n f_n \text{ and } \exists A, B > 0 \qquad \text{such that}$$

$$A\|f\|^2 \leq \sum_n |c_n|^2 \leq B\|f\|^2$$

Now define W_j as the orthogonal complement of V_j in V_{j+1} ,
i.e. $\quad V_j \oplus Wj = V_{j+1}, \;\; j \in \mathbb{Z}$.

Then $\mathbb{L}^2(\mathbb{R})$ is represented as the direct sum

$$\mathbb{L}^2(\mathbb{R}) = \bigoplus_{j \in \mathbf{Z}} W_j$$

<u>In practice</u>

1. There is a coarsest scale "n" and

$$V_{-n} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$$

$$\longrightarrow \quad \mathbb{L}^2(\mathbb{R}) = V_{-n} \oplus W_{j, j \geq -n} \qquad \textit{low resolution}$$

2. There is a finite number of scales, select $j = 0$ to be the coarsest scale

$$\begin{array}{ccc} V_0 \subset & V_1 \subset \ldots \subset V_n, & \qquad V_0 \subset \mathbb{L}^2(\mathbb{R}) \\ \downarrow & \downarrow & \\ \textit{coarsest} & \textit{finest} & \end{array}$$

3. In numerical implementations the subspace V_0 is <u>finite</u> dimensional.

Be aware of other choices in the literature where $\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x - k)$
Then

$$V_j = V_{j+1} \oplus W_{j+1} \qquad\qquad \underline{\underline{j}} \longrightarrow \underline{\underline{-j}}$$

and

$$\supset V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset \ldots$$

EXPLANATION OF MULTI-RESOLUTION ANALYSIS

Haar basis on [0,1]

V_0 is spanned by $\varphi(x)$

W_0 is spanned by $\psi(x)$

$V_0 \perp W_0$ and $V_1 = V_0 \oplus W_0$

V_1 = space of all piecewise constant functions on half intervals.

Basis for V_1 consists of $\sqrt{2}\varphi(2x)$ and $\sqrt{2}\varphi(2x-1) \longrightarrow \sqrt{2}\varphi(2x-k)$, $k = 0, 1$.

$$\sqrt{2}\varphi(2x) = \frac{\sqrt{2}}{2}(\varphi(x) + \psi(x))$$

$$\sqrt{2}\varphi(2x-1) = \frac{\sqrt{2}}{2}(\underset{\substack{\uparrow \\ \in V_0}}{\varphi(x)} - \underset{\substack{\uparrow \\ \in W_0}}{\psi(x)})$$

Since $V_0 \subset V_1 \longrightarrow \varphi(x)$ can be written in terms of

$$\sqrt{2}\varphi(2x) \text{ and } \sqrt{2}\varphi(2x-1)$$

Indeed

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\varphi(2x) + \sqrt{2}\varphi(2x-1)\right)$$

Sine

$$\begin{aligned} W_0 \subset V_1 \longrightarrow \psi(x) &= \varphi(2x) - \varphi(2x-1) \\ &= \frac{1}{\sqrt{2}}\left(\sqrt{2}\varphi(2x) - \sqrt{2}\varphi(2x-1)\right) \end{aligned}$$

Basis for W_1 consists of $\sqrt{2}\psi(2x)$ and $\sqrt{2}\psi(2x-1) \longrightarrow \sqrt{2}\psi(2x-k)$, $k = 0, 1$.

V_2 is spanned by $2\varphi(2^2x-k) = 2\varphi(4x-k), \quad k = 0, 1, 2, 3$
$\longrightarrow$ space of piecewise functions on quarter interval.

W_2 is spanned by $2\psi(4x-k), \quad k = 0, 1, 2, 3$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

So, a function $f \in V_2$ can be written in terms of

$$\varphi(x), \psi(x), 2\psi(2x) \text{ and } \sqrt{2}\psi(2x-1)$$

But V_2 is also spanned by

$$2\varphi(4x), 2\varphi(4x-1), 2\varphi(4x-2) \text{ and } 2\varphi(4x-3).$$

In general,

V_j is spanned by $2^{j/2}\varphi(2^j x - k)$ for fixed j and $k = 0, 1, \ldots, 2^j - 1$

W_j is spanned by $2^{j/2}\psi(2^j x - k)$, and $k = 0, 1, \ldots, 2^j - 1$

$$\begin{aligned} V_{j+1} &= V_j \oplus W_j \\ &= V_{j-1} \oplus W_{j-1} \oplus W_j \\ &= \cdots \\ &= \cdots \\ V_{j+1} &= V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_j \end{aligned}$$

and the translates of wavelets on the right are also translates of scaling functions on the left.

Next step, move from [0,1] to $\mathbb{R}$ by allowing all translations (no restrictions on index k).

A basis for $\mathbb{L}^2\mathbb{R}$ consists of $\varphi(x-k)_{k \in \mathbf{Z}}$ together with

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k) \text{ with } j \geq 0,\ k \in \mathbb{Z}$$

Another basis for $\mathbb{L}^2\mathbb{R}$ consists of

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \quad \forall j \in \mathbb{Z},\ \forall k \in \mathbb{Z}.$$

Orthogonality Issues

1. For fixed j $\quad V_j \perp W_j \quad \int \varphi_{j,k}\psi_{j,\ell}dx = 0\ \forall k, \ell$
2. Since $W_j \subset V_{j+1}$ and $W_{j+1} \perp V_{j+1}$ $\quad \int \psi_{j,k}(x)\psi_{j+1,\ell}dx = 0$

$$\Longrightarrow W_j \perp W_{j+1}$$

Then $W_{j+1} \subset V_{j+2}$ and $W_{j+2} \perp V_{j+2} \implies W_{j+1} \perp W_{j+2}$

All W_j are mutually orthogonal.

3. Since $V_j \subset V_{j+1}$ and $W_{j+1} \perp V_{j+1}$ $\quad \int \psi_{p,k}\psi_{q,\ell}dx = 0$ if $p \neq q$

$$\Longrightarrow V_j \perp W_{j+1} \qquad \int \varphi_{j,k}\psi_{j+1,\ell}dx = 0.$$

4. It is <u>not</u> true that $V_j \perp V_{j+1}$, <u>but</u> $\varphi \perp$ its translates.

Representing a function in a wavelet basis

There are 5 different approaches (points of view).

Each one leads to a <u>key</u> interpretation, to key properties and to an algorithm to perform wavelet decomposition.

Approaches:

1. inner product method,
2. matrix method (change of basis),
3. convolution method,
4. FWT method,
5. projection matrices method.

<u>Example to illustrate the methods</u>

Take $f(x) = 9\varphi(4x) + 1.\varphi(4x-1) + 2\varphi(4x-2) + 0.\varphi(4x-3)$.

Rewrite in basis $2^{j/2}\varphi(2^jx - k)$ on V_2

here $j = 2$, $k = 0, 1, 2, 3$

$$\begin{aligned} f(x) &= \frac{9}{2}\underbrace{2\varphi(4x-0)} + \frac{1}{2}\underbrace{2\varphi(4x-1)} + 1\,\underbrace{2\varphi(4x-2)} + 0\,\underbrace{\varphi(4x-3)} \\ &= \frac{9}{2}\varphi_{2,0}(x) + \frac{1}{2}\varphi_{2,1}(x) + 1\varphi_{2,2}(x) + 0\varphi_{2,3}(x) \end{aligned}$$

basis vectors for V_2

We want to write

$$f(x) = a\varphi(x) + b\psi(x) + c\sqrt{2}\psi(2x) + d\sqrt{2}\psi(2x-1)$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \searrow \quad \swarrow \\ \in V_2 & \in V_0 & \in W_0 & \in W_1 \end{array}$$

There are 5 different methods to compute a, b, c and d.

METHOD #1:Inner product method (not efficient)

$$\begin{aligned}
a &= < f(x), \varphi(x) >= \frac{9}{2} < \varphi_{2,0}, \varphi > + \frac{1}{2} < \varphi_{2,1'}\varphi > +1 < \varphi_{2,2'}\varphi > +0 \\
&= \frac{9}{2} < 2\varphi(4x), \varphi(x) > + \frac{1}{2} < 2\varphi(4x-1), \varphi(x) > + < 2\varphi(4x-2), \varphi(x) > \\
&= \frac{9}{2} 2.\ \frac{1}{4} + \frac{1}{2} .2.\ \frac{1}{4} + 2.\frac{1}{4} + 0 \\
&= \frac{9}{4} + \frac{1}{4} + \frac{2}{4} = \underline{\underline{3.}}
\end{aligned}$$

Similarly

$$\begin{aligned}
b &= < f(x), \psi(x) >= \cdots & = 2 \\
c &= < f(x), \sqrt{2}\psi(2x) >= \cdots & = 4/\sqrt{2} \\
d &= < f(x), \sqrt{2}\psi(2x-1) >= \cdots & = 1/\sqrt{2}
\end{aligned}$$

$$\longrightarrow \ (x) = 3.\varphi(x) + 2\psi(x) + \frac{4}{\sqrt{2}}.\sqrt{2}\psi(2x) + \frac{1}{\sqrt{2}}.\sqrt{2}\psi(x-1).$$

So,

$$f(x) = 3\varphi(x) + 2\psi(x) + \frac{4}{\sqrt{2}}\sqrt{2}\psi 2x + \frac{1}{\sqrt{2}}\sqrt{2}\psi(2x-1)$$

$$\begin{array}{cccc}
f^{(2)}(x) = & f^{(0)} + & g^{(0)} + & g^{(1)} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\in V_2 & \in V_0 & \in W_0 & \in W_1 \\
 & \text{1 coeff} & \text{1 coeff} & \text{2 coeffs}
\end{array}$$

In general $g^{(j)}$ is made up with functions on mesh $\frac{1}{2^j}$.

Full decomposition

$$\begin{array}{rl}
f^{(n)}(x) = & f^{(0)} + g^{(0)} + g^{(1)} + g^{(2)} + \cdots + g^{(n-1)} \\
\#\ \text{coefficients} & \underbrace{1 + 1 + 2 + 4 + 8 + \cdots + 2^{n-1}}_{2^n \ \text{coefficients}}
\end{array}$$

After J steps of decomposition

$$\begin{array}{rl}
f^{(n)}(x) = & f^{(n-J)}(x) + g^{(n-J)}(x) + \cdots + g^{(n-1)}(x) \\
\#\ \text{coefficients} & 2^{n-J} + \cdots 2^{n-2} + 2^{n-1} = 2^n \ \text{coefficients}
\end{array}$$

METHOD #2: **Matrix Method**

$$\begin{array}{lcl} \text{Old basis} \quad B & = & \{2\varphi(4x), 2\varphi(4x-1), 2\varphi(4x-2), 2\varphi(4x-3)\} \\ \text{New basis} \quad B' & = & \{\varphi(x), \psi(x), \sqrt{2}\psi(2x), \sqrt{2}\psi(2x-1)\} \end{array}$$

$$\begin{bmatrix} 9/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Longleftrightarrow \vec{f} = R\ \vec{b}$$

↑ old coordinates

↑ $R =$ reconstruction matrix

If matrix with as columns the coordinates of the new basis vectors in terms of the old basis.

Column #1 :

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

because

$$\varphi(x) = \underline{\underline{\frac{1}{2}}}2\varphi(4x) + \underline{\underline{\frac{1}{2}}}2\varphi(4x-1) + \underline{\underline{\frac{1}{2}}}2\varphi(4x-2) + \underline{\underline{\frac{1}{2}}}2\varphi(4x-3)$$

Similarly, for columns 2, 3 and 4

$$\psi(x) = \underline{\underline{\frac{1}{2}}}2\varphi(4x) + \underline{\underline{\frac{1}{2}}}2\varphi(4x-1) - \underline{\underline{\frac{1}{2}}}2\varphi(4x-2) - \underline{\underline{\frac{1}{2}}}2\varphi(4x-3)$$

$$\sqrt{2}\psi(2x) = \frac{1}{\sqrt{2}}2\varphi(4x) - \frac{1}{\sqrt{2}}2\varphi(4x-1) + 0.2\varphi(4x-2) + 0.2\varphi(4x-3)$$

$$\sqrt{2}\psi(2x-1) = 0.2\varphi(4x) - 0.2\varphi(4x-1) + \sqrt{2}2\varphi(4x-2) - \frac{1}{\sqrt{2}}2\varphi(4x-3)$$

$\vec{b} = D\ \vec{f}$ $\qquad$ $D =$ decomposition matrix

We need $\qquad \vec{b} = R^{-1}\ \vec{f} = R^{\top}\ \vec{f}$

because the columns of R are orthonormal vectors.

$$\longrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 9/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

So,
$$\begin{aligned} f^{(2)}(x) = f(x) &= 3\varphi(x) + 2\psi(x) + \tfrac{4}{\sqrt{2}}\sqrt{2}\psi(2x) + \tfrac{1}{\sqrt{2}}\sqrt{2}\psi(2x-1) \\ &= 3\varphi(x) + 2\psi(x) + 4\psi(2x) + 1\psi(2x-1) \end{aligned}$$

Picture

$$\rightsquigarrow 9\psi(4x) + \varphi(4x-1) + 2\varphi(4x-2) + 0\varphi(4x-3)$$

METHOD #3: **Convolution Method**

For Haar case $H = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\ G = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$\begin{array}{ccccccc} \vec{s}^{(n)} & \xrightarrow{H} & \vec{s}^{(n-1)} & \xrightarrow{H} & \vec{s}^{(n-2)} & \cdots & \longrightarrow \ \vec{s}^{(0)} \\ G\downarrow & & G\downarrow & & G\downarrow & \downarrow G & \\ \vec{d}^{(n-1)} & & \vec{d}^{(n-2)} & & \vec{d}^{(n-3)} & \vec{d}^{(0)} & \end{array}$$

$\underbrace{\qquad\qquad}_{\text{set aside}}$ stop at level zero.

$$\begin{bmatrix} \vec{s}^{(0)} \\ \cdots\cdots \\ \vec{d}^{(0)} \\ \cdots\cdots \\ \vec{d}^{(1)} \\ \cdots\cdots \\ \vec{d}^{(2)} \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ \vec{d}^{(n-1)} \end{bmatrix}.$$

$$\begin{array}{ccccc} \vec{s}^{(2)} = 9/2\ \ 1/2\ \ 1\ \ 0 & \xrightarrow{H} & \vec{s}^{(1)} = 5/\sqrt{2}\ \ 1/\sqrt{2} & \xrightarrow{H} & \vec{s}^{(0)} = 3 \\ \downarrow G & & \downarrow G & & \\ \vec{d}^{(1)} = 4/\sqrt{2}\ \ 1/\sqrt{2} & & d^{(0)} = 2 & & \end{array}$$

$$\begin{bmatrix} \vec{s}^{(0)} \\ \cdots \\ \vec{d}^{(0)} \\ \cdots \\ \vec{d}^{(1)} \end{bmatrix}$$

Assemble the result:

$$\begin{bmatrix} 9/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \overset{D}{\Longrightarrow} \begin{bmatrix} 3 \\ \cdots\cdots \\ 2 \\ \cdots\cdots \\ 4/\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

<u>In general</u>

$$\vec{s}^{\,n} = \left.\begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}\right\} \quad 2^n \text{ data} \quad \longrightarrow \quad \left.\begin{matrix} \begin{bmatrix} \vec{s}^{\,(0)} \\ \cdots\cdot \\ \vec{d}^{\,(0)} \\ \cdots\cdot \\ \vec{d}^{\,(1)} \\ \cdots\cdot \\ \vec{d}^{\,(2)} \\ \cdots\cdot \\ \vdots \\ \vdots \\ \cdots\cdot \\ \vec{d}^{\,n-1} \end{bmatrix} & \begin{matrix} 1 \\ \\ 1 \\ \\ 2 \\ \\ 4 \\ \\ 8 \\ \\ \vdots \\ \\ 2^{n-1} \end{matrix} \end{matrix}\right\} \quad 2^n \text{ data}$$

<u>Convolution</u>

$$\begin{matrix} & 9/2 \quad 1/2 & 1 \quad 0 \\ & \searrow \; \swarrow & \searrow \; \swarrow \\ \text{Average:} & 5/\sqrt{2} & 1/\sqrt{2} \\ \text{Difference:} & 4/\sqrt{2} & 1/\sqrt{2} \end{matrix} \quad \longrightarrow \quad \begin{matrix} & 5/\sqrt{2} \quad 1/\sqrt{2} \\ & \searrow \; \swarrow \\ \text{Average:} & 3 \\ \text{Difference:} & 2 \end{matrix}$$

METHOD #4: **Fast Wavelet Transform**

Decompose the matrix D into a product of 3 sparser matrices.

Decompose at level 1: $V_2 = V_1 \oplus W_1$

$$\begin{array}{ll} h_0 \;\; h_1 \longrightarrow \\ g_0 \;\; g_1 \longrightarrow \end{array} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 9/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 4/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{array}{l} - \text{ low frequency} \\ - \text{ high frequency} \\ - \text{ low frequency} \\ - \text{ high frequency} \end{array}$$

$$\text{or} \qquad M_1 \vec{f} = \vec{b_1}$$

butterfly (structured) matrix

Bring high frequencies to bottom via a permutation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} \\ 4/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 1/\sqrt{2} \\ 4/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{array}{l} - \text{ low frequencies} \\ - \text{ low frequencies} \\ - \text{ high frequencies} \\ - \text{ high frequencies} \end{array}$$

$$\text{or} \qquad M_2 \vec{b_1} = \vec{b_2}$$

very sparse

Decompose at level 2: $V_1 = V_0 \oplus W_0$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} \\ 1/\sqrt{2} \\ 4/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\text{or} \qquad M_3 \vec{b_2} = \vec{b}$$

So

$$\left.\begin{array}{rcl} M_1 \vec{f} & = & \vec{b_1} \\ M_2 \vec{b_1} & \leqq & \vec{b_2} \\ M_3 \vec{b_2} & \leqq & \vec{b} \end{array}\right\} \begin{array}{rcl} M_3 M_2 M_1 \vec{f} & = & \vec{b} \\ = D \vec{f} & = & \vec{b} \end{array}$$

$$D = M_3 M_2 M_1$$

Count number of operations

For

$$D\,\vec{f}=\vec{b} \qquad \longrightarrow\sim N^2 \quad \text{operations (not counting additions)}$$

D is a dense $N \times N$ matrix 12 multiplications

For

$$\left.\begin{array}{lcll} M_1\,\vec{f} & = & \vec{b_1} & \text{8 multiplications} \\ M_2\,\vec{b_1} & = & \vec{b_2} & \text{no cost, only shuffling} \\ M_3\,\vec{b_2} & = & \vec{b} & \text{4 multiplications} \end{array}\right\} \sim 12 \text{ multiplications}$$

In general $N\log_2 N$ algorithm instead of N^2algorithm.

Compare with FFT Tukey, Cooley (1965)
(*See slides)*

* Same idea: write dense matrix as product of sparser "butterfly" and permutation matrices.

* Computation count $N\log_2 N$ versus N^2
If $N = 1024$ $N^2 = 1,048576$

$N\log_2 N = 10240.$

- Show M_1 for Daubechies case (DAUB4)

METHOD #5 Projection matrices Method

See e.g. Daubechies' Ten Lectures on Wavelets.

- Comparison with FFT.

QUADRATURE MIRROR FILTERS
(A look in the Fourier domain)

The Fourier transform.

There are 3 equivalent choices, depending on where one puts the "2π".

Each choice exists of two versions, depending on whether the minus is put in the exponent of the direct <u>or</u> the inverse transform.

So, we have

1. $$(\mathcal{F}f)(\omega) = \hat{f}(\omega) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

   $\rightarrow \omega$ is in radians per second (1 radian $= \frac{1}{2\pi}$ circle)

2. $$\hat{f}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} e^{-2\pi i\omega t} dt$$

   $\rightarrow \omega$ in Hertz

3. $$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

   $\rightarrow$ in radians.

For these notes:

<u>Direct Transformation:</u> $\hat{f}(\omega) = (\mathcal{F}f)(\omega) = \int_{-\infty}^{+\infty} e^{-2\pi i\omega t} f(t)\, dt$

<u>Inverse Transformation:</u> $f(t) = (\mathcal{F}^{-1}\hat{f})(t) = \int_{-\infty}^{+\infty} e^{2\pi i\omega t} f(\omega)\, d\omega$

1. Apply the Fourier Transformation to the refinement equation

(in real space)

$$\begin{aligned} \varphi(x) &= \sqrt{2} \sum_{k=0}^{L-1} h_k \varphi(2x-k) \\ \hat{\varphi}(\omega) &= \int_{-\infty}^{+\infty} e^{-2\pi i\omega x} \varphi(x) dx \\ &= \sqrt{2} \sum_{k=0}^{L-1} h_k \int_{-\infty}^{+\infty} e^{-2\pi i\omega x} \varphi(2x-k) dx \end{aligned}$$

Set $\quad u = 2x - k \quad \longrightarrow \quad x = \dfrac{u+k}{2} \quad \longrightarrow \quad dx = \dfrac{du}{2}$

$$\begin{aligned} &= \frac{\sqrt{2}}{2} \sum_{k=0}^{L-1} h_k \int_{-\infty}^{+\infty} e^{-2\pi i\omega[\frac{u+k}{2}]} \varphi(u) du \\ &= \frac{1}{\sqrt{2}} \sum_{k=0}^{L-1} h_k \int_{-\infty}^{+\infty} e^{-2\pi i \frac{\omega}{2} k} e^{-2\pi i \frac{\omega}{2} u} \varphi(u) du \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{L-1} h_k e^{-2\pi i \frac{\omega}{2} k} \int_{-\infty}^{+\infty} e^{-2\pi i (\frac{\omega}{2}) u} \varphi(u) du$$

$$= \frac{1}{\sqrt{2}} \sum_{k-0}^{L-1} h_k e^{-2\pi i \frac{\omega}{2} k} \hat{\varphi}\left(\frac{\omega}{2}\right)$$

Define

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{k=0}^{L-1} h_k e^{-2\pi i k \omega}$$

Then

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right)$$

or equivalently,

$$\hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega) \quad \text{refinement equation in Fourier domain!}$$

Notes: $m_0(\omega)$ is a trigonometric polynomial

$$m_0(\omega + 1) = m_0(\omega)$$

$$\text{If we know } m_0(\omega) \quad \Longrightarrow \quad \text{we know } h_k.$$

2. Keep refining

Since

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right)$$

then $\omega \to \omega/2$ $\quad \hat{\varphi}\left(\frac{\omega}{2}\right) = m_0\left(\frac{\omega}{4}\right) \hat{\varphi}\left(\frac{\omega}{4}\right)$

then $\omega \to \omega/2$ $\quad \hat{\varphi}\left(\frac{\omega}{4}\right) = m_0\left(\frac{\omega}{8}\right) \hat{\varphi}\left(\frac{\omega}{8}\right)$

We get

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right) m_0\left(\frac{\omega}{4}\right) m_0\left(\frac{\omega}{8}\right) \cdots m_0\left(\frac{\omega}{32}\right) \hat{\varphi}\left(\frac{\omega}{32}\right)$$

$$\vdots$$

Keep going $\omega \longrightarrow \omega/2$

$$\hat{\varphi}(\omega) = \prod_{j=1}^{N} m_0\left(\frac{\omega}{2^j}\right) \hat{\varphi}\left(\frac{\omega}{2^N}\right)$$

Take limit for $N \longrightarrow \infty$

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0\left(\frac{\omega}{2^j}\right)\hat{\varphi}(0)$$

But

$$\hat{\varphi}(0) = \int e^{-2\pi i 0 t}\varphi(t)dt = \int \varphi(t)dt = 1$$

So,

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0\left(\frac{\omega}{2^j}\right)$$

Infinite refinement of the two-scale difference relation in the Fourier domain.

Similarly,

$$\hat{\psi}(\omega) = m_1(\omega/2)\hat{\varphi}(\omega/2)$$

$$\text{or} \qquad \hat{\psi}(2\omega) = m_1(\omega)\hat{\varphi}(\omega)$$

$$\text{with} \qquad m_1(\omega) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\sum_{k=0}^{L-1} g_k e^{-2\pi i k\omega}$$

Also, by continuing the refinement process

$$\begin{aligned}
\hat{\psi}(\omega) &= m_1(\omega/2)\hat{\varphi}(\omega/2) \\
&= m_1(\omega/2)m_0(\omega/4)\hat{\varphi}(\omega/4) \\
&= m_1(\omega/2)m_0(\omega/4)m_0(\omega/8)\hat{\varphi}(\omega/8) \\
&\qquad\qquad \vdots
\end{aligned}$$

$$\Longrightarrow \hat{\psi}(\omega) = m_1\left(\frac{\omega}{2}\right)\prod_{j=2}^{\infty} m_0\left(\frac{\omega}{2^j}\right)$$

Since $G \perp H$ so $g_k = (-1)^k h_{L-1-k}$

one has $m_1(\omega) = e^{-2\pi i(1-L)\omega}\overline{m_0\left(\omega + \frac{1}{2}\right)}$

3. Change of notation

Set $z = e^{-2\pi i\omega}$

Then $\quad m_0(\omega) \quad \longrightarrow \quad H(z) = \sqrt{2}m_0(\omega(z))$

$$H(z) = \sum_{k=0}^{L-1} h_k z^k$$

And

$$m_1(\omega) \to G(z) = \sqrt{2}m_1(\omega(z))$$

$$G(z) = \sum_{k=0}^{L-1} g_k z^k$$

Also, $m_1(\omega)$ is related to $\overline{m_0(\omega+1/2)} \quad \Longrightarrow \quad G(z) = -z^{L-1}H(-\frac{1}{z})$

DERIVATION OF THE QMF CONDITION

(Quadrature mirror filters)

Tools needed

Parseval's Identity

$$< f, g >= \int_{-\infty}^{+\infty} \overline{f(x)}g(x)dx = \int_{-\infty}^{+\infty} \overline{\hat{f}(\omega)}g(\omega)d\omega =< \hat{f}, \hat{g} >$$

Consequence (Plancherel's formula)

$$\begin{aligned} \parallel f \parallel^2 &= \int \overline{f} f dx = \int |f|^2 dx \\ &= \int \overline{\hat{f}}(\omega) f(\omega) d\omega = \int |\hat{f}(\omega)|^2 d\omega = \parallel \hat{f} \parallel^2 \end{aligned}$$

($\mathcal{F}$ is an isometric isomorphism)

First, we prove that the following statements are equivalent

(i)

$$< \varphi(x-k), \varphi(x-\ell) >= \int_{-\infty}^{+\infty} \varphi(x-k)\varphi(x-\ell)dx = \delta_{kl}$$

i.e. $\varphi \perp \varphi$ translates.

(ii)

$$\hat{\varphi}(\omega) \text{ satisfies } \int_{-\infty}^{+\infty} e^{-2\pi ik\omega}|\hat{\varphi}(\omega)|^2 d\omega = \delta_{k0}$$

(iii)

$$\sum_{k=-\infty}^{+\infty} |\hat{\varphi}(\omega+k)|^2 = 1$$

<u>Proof:</u> Define the function $K(\omega) = \sum_{k=-\infty}^{+\infty} |\hat{\varphi}(\omega+k)|^2$

Then

(1) $K(\omega)$ is 1 periodic

$$K(\omega+1) = \sum_{k\in\mathbf{Z}} |\ \hat{\varphi}(\omega+k+1)\ |^2 = \sum_{\ell\in\mathbf{Z}} |\ \hat{\varphi}(\omega+\ell)\ |^2 = K(\omega)$$

(2)

$$\int_0^1 K(\omega)d\omega = \int_0^1 \sum_k |\ \hat{\varphi}(\omega+k)\ |^2\ d\omega$$

$$= \sum_k \int_0^1 |\ \hat{\varphi}(\omega+k)\ |^2\ d\omega$$

Set $z = \omega + k$

$$= \sum_k \int_k^{k+1} |\ \hat{\varphi}(z)\ |^2\ dz = \int_{-\infty}^{+\infty} |\ \hat{\varphi}(z)\ |^2\ dz$$

$$\overset{\text{Plancherel}}{=} \int_{-\infty}^{+\infty} |\ \varphi(z)\ |^2\ dz = 1.$$

So, $K(\omega)$ is 1 periodic and integrable (integrates to 1)

So, $K(\omega)$ has a Fourier series

$$K(\omega) = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi in\omega}$$

Where

$$\begin{aligned} c_n &= \int_0^1 e^{-2\pi in\omega} K(\omega)d\omega \\ &= \int_0^1 e^{-2\pi in\omega} \sum_{k=-\infty}^{+\infty} |\ \hat{\varphi}(\omega+k)\ |^2\ d\omega \end{aligned}$$

Set $\omega + k = z$

$$= \sum_{k=-\infty}^{+\infty} \int_k^{k+1} e^{-2\pi i n(z-k)} \mid \hat{\varphi(z)} \mid^2 dz$$
$$= \sum_{k=-\infty}^{+\infty} \int_k^{k+1} e^{-2\pi i n z} \mid \hat{\varphi}(z) \mid^2 dz$$
$$= \int_{-\infty}^{+\infty} e^{-2\pi i n z} \overline{\hat{\varphi}(z)} \hat{\varphi}(z) dz$$
$$= \int_{-\infty}^{+\infty} \hat{\varphi}(z-n) \hat{\varphi}(z) dz$$

$$\overset{\text{Parseval}}{=} \int_{-\infty}^{+\infty} \varphi(x-n)\varphi(x)dx = \delta_{no}$$

So, $\quad c_n = \delta_{n0}$

Hence, $\quad K(\omega) = \sum_{n=-\infty}^{+\infty} \delta_{no} e^{2\pi i n \omega} = 1$

$$\Longrightarrow \quad \sum_{k=-\infty}^{+\infty} \mid \hat{\varphi}(\omega+k)^2 \mid = 1.$$

Consequences

(1) If $\quad \hat{\varphi}(0) = 1$

Then from $\quad \sum_{k=-\infty}^{+\infty} \mid \hat{\varphi}(\omega+k) \mid^2 = \mid \Longrightarrow \hat{\varphi}(k) = \delta_{k0}$

One can show that $\quad \hat{\varphi}(k) = \delta_{k0} \Longleftrightarrow \sum_{k=-\infty}^{+\infty} \mid \hat{\varphi}(\omega+k) \mid^2 = 1.$

(2)

$$\sum_{k=-\infty}^{+\infty} \varphi(x-k) = 1 \qquad \text{Partition of unity}$$

Proof: Similar to the above by setting

$$P(x) = \sum_k \varphi(x-k)$$

showing that $\quad P(x+1) = P(x) \quad$ and $P(x)$ integrable

$$\text{Compute} \quad c_n = \int_0^1 e^{-2\pi i n x} \sum_k \varphi(x-k) dx$$
$$= \cdots = \delta_{n0} \quad , \quad \text{etc.}$$

(3)

$$\hat{\varphi}(k) = \delta_{k0} \quad \Longleftrightarrow \quad \sum_{k \in \mathbf{Z}} \varphi(x-k) = 1 \qquad \forall x$$

Compare with Poisson summation formula

$$\sum_{n=-\infty}^{+\infty} \hat{f}(n) = \sum_{k=-\infty}^{+\infty} f(k)$$

Now we can show that

$$\mid m_0(\omega) \mid^2 + \mid m_0(\omega + 1/2) \mid^2 = 1 \qquad \text{QMF condition}$$

or, equivalently,

$$H(z)H(1/z) + H(-z)H(-1/z) = 2$$

Proof

use $\quad \hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega) \quad (1)$

Denote

$$\begin{aligned} K(\omega) &= \sum_k \mid \hat{\varphi}(\omega + k) \mid^2 = 1 \\ K(2\omega) &= \sum_k \mid \hat{\varphi}(2\omega + k) \mid^2 = 1 \\ &\overset{(1)}{=} \sum_k \mid \hat{\varphi}(\omega + \frac{k}{2}) \mid^2 \mid m_0(\omega + \frac{k}{2}) \mid^2 \end{aligned}$$

Split in even and odd terms

$$\begin{aligned} &= \sum_{\ell\ (k=2\ell)} \mid \hat{\varphi}(\omega + \ell) \mid^2 \mid m_0(\omega + \ell) \mid^2 \\ &+ \sum_{\ell\ (k=2\ell+1)} \mid \hat{\varphi}(\omega + \ell + \frac{1}{2}) \mid^2 \mid m_0(\omega + \ell + \frac{1}{2}) \mid^2 \end{aligned}$$

$m_0(\omega)$ is 1-periodic

$$\begin{aligned} &= \sum_\ell \mid m_0(\omega) \mid^2 \mid \hat{\varphi}(\omega + \ell) \mid^2 + \sum_\ell \mid m_0(\omega + \frac{1}{2}) \mid^2 \mid \hat{\varphi}(\omega + \ell + \frac{1}{2}) \mid^2 \\ &= \mid m_0(\omega) \mid^2 \underbrace{\sum_\ell \mid \hat{\varphi}(\omega + \ell) \mid^2}_{1} + \mid m_0\left(\omega + \frac{1}{2}\right) \mid^2 \underbrace{\sum_\ell \mid \hat{\varphi}\left(\omega + \ell + \frac{1}{2}\right) \mid^2}_{1} \\ &= \mid m_0(\omega) \mid^2 + \mid m_0(\omega + \frac{1}{2}) \mid^2 = 1 \quad \text{QMF} \end{aligned}$$

Rewrite in terms of $H(z)$

$$H(z)H(1/z) + H(-z)H(-1/z) = 2$$

<u>Proof</u>

$$m_0(\omega) = \frac{1}{\sqrt{2}}H(z) \qquad \text{since} \quad z = e^{-2\pi i\omega}$$

$$\begin{aligned}\overline{m_0(\omega)} = \frac{1}{\sqrt{2}}\sum h_k\overline{e^{-2\pi ik\omega}} &= \frac{1}{\sqrt{2}}\sum_k h_k e^{2\pi i\omega k} \\ &= \tfrac{1}{\sqrt{2}}H(1/z)\end{aligned}$$

$$\begin{aligned}m_0(\omega + 1/2) &= \frac{1}{\sqrt{2}}\sum_k h_k e^{-2\pi ik(\omega+1/2)} \\ &= \frac{1}{\sqrt{2}}\sum_k h_k(-1)^k e^{-2\pi ik\omega} \\ &= \frac{1}{\sqrt{2}}\sum_k h_k(-z)^k = \frac{1}{\sqrt{2}}H(-z) \\ \overline{m_0(\omega + 1/2)} &= \tfrac{1}{\sqrt{2}}H(-1/z)\end{aligned}$$

<u>So</u>

$$m_0(\omega)\overline{m_0(\omega)} + m_0(\omega + 1/2)\overline{m_0(\omega + 1/2)} = 1$$

$$\longrightarrow \frac{H(z)}{\sqrt{2}}\frac{H(1/z)}{\sqrt{2}} + \frac{H(-z)}{\sqrt{2}}\frac{H(-1/z)}{\sqrt{2}} = 1$$

<u>or</u>

$$H(z)H(1/z) + H(-z)H(-1/z) = 2$$

Expresses that $\varphi \perp \varphi$ translates.

Similarly,

$$\varphi \perp \psi \qquad G(z)H(1/z) + G(-z)H(-1/z) = 0$$

$$\psi \perp \psi \qquad G(z)G(1/z) + G(-z)G(-1/z) = 2$$

For reversal $\qquad H \longleftrightarrow G$

$$G(z) = -z^{L-1}H(-1/z)$$

Vanishing moments for $\psi(x)$

Theorem. The following statements are equivalent.

(i) The wavelet function ψ has M vanishing moments i.e. $\int x^m \psi(x) dx = 0 \quad , \quad m = 0, 1, 2, \cdots, M-1$

(ii) The filter coefficients h_k satisfy $\sum_{k=0}^{L-1} (-1)^k k^m h_k = 0,$ $m = 0, 1, 2, \cdots, M-1$

(iii) $H(z)$ has $z = -1$ as a zero with multiplicity M

$$\begin{array}{lrcl} \text{i.e.} & H(-1) = H'(-1) & = & \cdots = H^{(M-1)}(-1) = 0 \\ \text{or} & H^{(m)}(-1) & = & 0 \quad , \quad m = 0, 1, 2, \cdots, M-1 \\ \text{Hence,} & H(z) & = & \sqrt{2}(\frac{1+z}{2})^M Q(z) \end{array}$$

(iv) The matrix which determines the values of $\varphi(x)$ at integers has eigenvalues $1, \frac{1}{2}, (\frac{1}{2})^2, \cdots, (\frac{1}{2})^{M-1}$

(v) The Fourier transform of ψ , i.e. $\hat{\psi}(\omega)$, has a M-fold zero at $\omega = 0$

(vi) The Fourier transform of φ, i.e. $\hat{\varphi}(\omega)$, has a M-fold zero at $\omega = k$, except at 0 where $\hat{\varphi}(0) = 1$.

Proof (some samples, to get the flavour)

$$\begin{aligned} \hat{\psi}(\omega) &= \int_{-\infty}^{+\infty} \psi(x) e^{-2\pi i \omega x} dx \\ \hat{\psi}(0) &= \int_{-\infty}^{+\infty} \psi(x) dx = 0 \longrightarrow \hat{\psi}(0) = 0 \\ \psi'(0) &= \frac{d\hat{\psi}}{d\omega}|_{\omega=0} = \int_{-\infty}^{+\infty} \psi(x)(-2\pi i x) e^{-2\pi i \omega x} dx \end{aligned}$$

$$\text{So,} \quad \int x\psi(x) dx = 0 \quad \longleftrightarrow \quad \hat{\psi}'(0) = 0, \quad \text{etc.}$$

$$\hat{\psi}(\omega) = m_1(\omega/2)\hat{\varphi}(\omega/2) \quad \longrightarrow \quad \hat{\psi}(0) = 0 \quad \Longrightarrow \quad m_1(0) = 0 \longrightarrow H(-1) = 0, \text{etc.}$$

and also $\hat{\varphi}(0) = 1$.

Example: DAUB6. (3 vanishing moments)

$$\int x^m \psi(x) dx = 0 \qquad m = 0, 1, 2$$

So, $$H(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^3 Q(z) \qquad (1)$$

On the other hand $$H(z) = \sum_{k=0}^{5} h_k z^k = h_0 + h_1 z + \cdots + h_5 z^5 \qquad (2)$$

$\longrightarrow Q(z)$ is degree 2

So, $Q(z) = a_0 + a_1 z + a_2 z^2$

$$H(1) = \sqrt{2} Q(1) = \sum_{k=0}^{5} h_k = \sqrt{2} \quad (\text{ due to normalization})$$

So, $Q(1) = 1 \quad \Longrightarrow a_0 + a_1 + a_2 = 1$

Put $$H(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^3 Q(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^3 (a_0 + a_1 z + a_2 z^2) \qquad (3)$$

into QMF

and take coeff of z^0, z or $\frac{1}{z}$, z^2 or $\frac{1}{z^2}$

$$\Longrightarrow \begin{cases} 10(a_0^2 + a_1^2 + a_2^2) + 15(a_0 a_1 + a_1 a_2) + 6 a_0 a_2 &= 16 \\ 3(a_0^2 + a_1^2 + a_2^2) + 8(a_0 a_1 + a_1 a_2) + 10 a_0 a_2 &= 0 \\ a_0 a_1 + a_1 a_2 + 6 a_0 a_2 = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} a_0^2 + a_1^2 + a_2^2 &= 19/4 \\ a_0 a_1 + a_1 a_2 &= -9/4 \\ a_0 a_2 = 3/8 \end{cases}$$

Solve

$$\begin{aligned} a_0 &= \tfrac{1}{4}\left(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}\right) \\ a_1 &= \tfrac{1}{4}\left(2 - 2\sqrt{10}\right) = \tfrac{1}{2}\left(1 - \sqrt{10}\right) \\ a_2 &= \tfrac{1}{4}\left(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}\right) \end{aligned}$$

Substitute in (3), compare with (2)

$$\Longrightarrow \begin{cases} h_0 &= \frac{1}{16\sqrt{2}}\left(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\right) \\ \vdots & \vdots \\ h_5 &= \frac{1}{16\sqrt{2}}\left(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right) \end{cases}$$

DAUBECHIES WAVELETS: SOLVING THE QMF

We will solve

$$| m_0(\omega) |^2 + | m_0(\omega + 1/2) |^2 = 1 \qquad \text{QMF}$$

Set

$$| m_0(\omega) |^2 = M_0(\omega)$$

then

$$| m_0(\omega + 1/2) |^2 = M_0(\omega + 1/2)$$

$$\text{QMF} \quad : \quad M_0(\omega) + M_0(\omega + 1/2) = 1$$

For wavelets with M vanishing moments we have

$$m_0(\omega) = \left(\frac{1 + e^{-2\pi i\omega}}{2}\right)^M \mathcal{L}(\omega)$$

$\rightarrow$ trigonometric polynomial in $e^{-2\pi i\omega}$

$$M_0(\omega) = \left[\left(\frac{1 + e^{-2\pi i\omega}}{2}\right)\left(\frac{1 + e^{2\pi i\omega}}{2}\right)\right]^M \mathbb{L}(\omega)$$

where $\begin{aligned} \mathbb{L}(\omega) &= \overline{\mathcal{L}(\omega)}\mathcal{L}(\omega) \\ &= |\mathcal{L}(\omega)|^2 \end{aligned}$

$$= \cos^{2M}(\pi\omega)\mathbb{L}(\omega)$$

$\rightarrow$ is trigonometric polynomial in $e^{-2\pi i\omega}$
$\rightarrow$ can be written in $\sin^2 \pi\omega$
$\rightarrow$ say $P(\sin^2 \pi\omega)$

So we will assume the form

$$\begin{aligned} M_0(\omega) &= \cos^{2M}(\pi\omega)P(\sin^2 \pi\omega) \\ \text{Then} \qquad M_0(\omega + 1/2) &= \sin^{2M}(\pi\omega)P(\cos^2 \pi\omega) \end{aligned}$$

$$\begin{array}{lll} \text{Set} & y = \sin^2 \pi\omega & M_0(\omega) = (1-y)^M P(y) \\ \text{Then} & 1-y = \cos^2 \pi\omega & M_0(\omega + 1/2) = y^M P(1-y) \end{array}$$

So we need to find $P(y)$ such that

$$(1-y)^M P(y) + y^M P(1-y) = 1$$

Note: Doing this all for $z = e^{-2\pi i\omega}$ would give

$$\underbrace{\frac{(z+1)^{2M}}{(4z)^M}}_{(1-y)^M} \overbrace{Q(z)Q(1/z)}^{P(y)} + \underbrace{(-1)^M \frac{(z-1)^{2M}}{(4z)^M}}_{y^M} \underbrace{Q(-z)Q(-1/z)}_{P(1-y)} = 1$$

Step 1: Find $P(y)$

Most general solution

$$P(y) = \sum_{k=0}^{M-1} \binom{M+k+1}{k} y^k + y^M R\left(\frac{1}{2} - y\right)$$

where $R(y)$ is an odd polynomial chosen so that $P(y) \geq 0$ for $0 \leq y \leq 1$

Let's consider the case $\underline{R=0}$ $\Longleftrightarrow$ minimum length $\mathbb{L}$ where $L = 2M$

(If $R \neq 0$ then $M > 2L$, more freedom).

Argument: $P(y)$ polynomial in y

$$P(y) \sim (1-y)^{-M} \quad \underset{\text{Taylor}}{=\!=} \quad \sum_{k=0}^{\infty} \binom{M+k+1}{k} y^k$$

$\longrightarrow$ must be truncated at degree $M-1$

$P(1-y) \sim y^{-M}$

Candidate

$$P(y) = \sum_{k=0}^{M-1} \binom{M+k+1}{k} y^k$$

$$\text{or} \quad Q(z)Q(1/z) = \sum_{k=0}^{M-1} (-1)^k (1-z)^{2k} (4z)^{-k} \frac{(M+k-1)!}{k!(M-1)!}$$

Example: DAUB6 case

LHS

$$\left.\begin{array}{rcl} Q(z) & = & a_0 + a_1 z + a_2 z^2 \\ Q\left(\frac{1}{z}\right) & = & a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} \end{array}\right\}$$

$$Q(z)Q\left(\frac{1}{z}\right) = a_0^2 + a_1^2 + a_2^2 + (a_0 a_1 + a_1 a_2)\left(z + \frac{1}{z}\right) + a_0 a_2 \left(z^2 + \frac{1}{z^2}\right)$$

RHS

$$\sum_{k=0}^{2} (-1)^k \frac{(1-z)^{2k}}{(4z)^k} \binom{2+k}{k} = \binom{2}{0} - \binom{3}{1}\frac{(1-z)^2}{4z} + \binom{4}{2}\frac{(1-z)^4}{(4z)^2}$$

$$= \frac{19}{4} - \frac{9}{4}\left(z + \frac{1}{z}\right) + \frac{3}{8}\left(z^2 + \frac{1}{z^2}\right)$$

$$\Longrightarrow \begin{cases} a_0^2 + a_1^2 + a_2^2 = 19/4 \\ a_0 a_1 + a_1 a_2 = -9/4 \\ a_0 a_2 = 3/8 \end{cases}$$

$$a_0 + a_1 + a_2 = 1 \qquad \text{since} \quad Q(1) = 1$$

Solve

$$\begin{cases} a_0 & = & \frac{1}{4}\left(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}\right) \\ a_1 & = & \frac{1}{2}\left(1 - \sqrt{10}\right) \\ a_2 & = & \frac{1}{4}\left(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}\right) \end{cases}$$

$$\longrightarrow \begin{array}{rcl} H(z) & = & \sqrt{2}\left(\frac{1+z}{2}\right)^3 \left(a_0 + a_1 z + a_2 z^2\right) \\ & \equiv & h_0 + h_1 z + \cdots + h_5 z^5 \end{array}$$

$$\longrightarrow \begin{cases} h_0 & = & \frac{1}{16\sqrt{2}}\left(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}\right) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ h_5 & = & \frac{1}{16\sqrt{2}}\left(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}\right) \end{cases}$$

GENERAL POLYNOMIAL SOLUTION OF QMF

$$(1-y)^M P(y) + y^M P(1-y) = 1$$

Use Theorem of Bezout (gives existence and uniqueness of polynomial solutions).

Theorem: If $p_1(x)$ and $p_2(x)$ are two polynomials of degree n_1 and n_2 with no common zero, then there exists two unique polynomials $q_1(x)$ and $q_2(x)$ of degree $\leq n_2 - 1$ and $\leq n_1 - 1$ so that

$$p_1(x)q_1(x) + p_2(x)q_2(x) = 1$$

Apply Bezout's theorem

$$\begin{array}{rcll} p_1(y) & = & (1-y)^M & \text{degree} M \\ p_2(y) & = & y^M & \text{degree} M \end{array}$$

$$\exists \; q_1(y) \text{ and } q_2(y) \text{ so that}$$

$$(1-y)^M q_1(y) + y^M q_2(y) = 1$$

Now substitute $1-y$ for y

$$\longrightarrow y^M q_1(1-y) + (1-y)^M q_2(1-y) = 1$$

Since $\quad q_1(y)$ and $q_2(y)$ are unique $\quad \Longrightarrow q_1(y) = q_2(1-y)$

So, $\quad (1-y)^M q_1(y) + y^M q_1(1-y) = 1$

and $\quad q_1(y) \quad$ is of degree $\quad \leq M-1$

So $\quad P(y) = P_M(y)$ (the polynomial we sought)

Exists and is unique

$$\longrightarrow P_M(y) = \sum_{k=0}^{M-1} \binom{M+k-1}{k} y^k$$

Step 2 How do we get $Q(z)$ or $m_0(\omega)$ now that we have $M_0(\omega) = \mid m_0(\omega) \mid^2$

or $P(y) = Q(z)Q(1/z)$

$\Longrightarrow$ **Use spectral Factorization Theorem of Riesz**

Let $A(\xi)$ be a positive trigonometric polynomial invariant under the symmetry $\xi \longrightarrow -\xi$

then $A(\xi)$ is of the form

$$A(\xi) = \sum_{m=0}^{M} a_m \cos(m\xi) \quad , \quad a_m \in \mathbb{R}$$

and there then exists a trigonometric polynomial $B(\xi)$ of degree M, i.e.

$$B(\xi) = \sum_{m=0}^{M} b_m e^{im\xi} \quad , \quad b_m \in \mathbb{R}$$

such that $\mid B(\xi) \mid^2 = A(\xi)$

$B(\xi)$ is "root".

In practice: Factor $A(\xi)$ and investigate the complex (or real) root structure.

Example: DAUB 6

$$A(\xi) = Q(z)Q(1/z) = \frac{\frac{3}{8} - \frac{9}{4}z + \frac{19}{4}z^2 - \frac{9}{4}z^3 + \frac{3}{8}z^4}{z^2}$$

4 roots

$$\left.\begin{matrix} z_1 \\ \overline{z_1} \end{matrix}\right\} \text{2 inside unit circle} \qquad \begin{aligned} |z_1| &= 0.325406 \\ \mathrm{Re}(z_1) &= 0.28725 \end{aligned}$$

$$\left.\begin{matrix} z_1^{-1} \\ \overline{z}_1^{-1} \end{matrix}\right\} \text{2 outside unit circle}$$

$$\begin{aligned} Q(z) &= cte\left(z^2 - 2z\,\mathrm{Re}(z_1) + \mid z_1 \mid^2\right) \\ Q(1) &= 1 \longrightarrow Q(z) = 1.8818688(z^2 - 0.57450z + 0.1058894) \\ \text{Then } H(z) &= \sqrt{2}\left(\tfrac{1+z}{2}\right)^3 Q(z) = h_0 + h_1 z + \ldots + h_5 z^5 \\ &\longrightarrow h_1,\ h_2,\ \cdots,\ h_5 \quad \text{values} \end{aligned}$$

Doing so one gets the numerical values of the DAUB6 filter taps.
For the roots outside the unit circle in the Gauss plane one gets

$$
\begin{aligned}
h_0 &= \tfrac{1}{16\sqrt{2}}\left(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\right) &\cong& \; 0.3326705529500825\cdots \\
h_1 &= \tfrac{1}{16\sqrt{2}}\left(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}\right) &\cong& \; 0.8068915093110924\cdots \\
h_2 &= \tfrac{1}{8\sqrt{2}}\left(5-\sqrt{10}+\sqrt{5+2\sqrt{10}}\right) &\cong& \; 0.4598775021184914\cdots \\
h_3 &= \tfrac{1}{8\sqrt{2}}\left(5-\sqrt{10}-\sqrt{5+2\sqrt{10}}\right) &\cong& \; -0.1350110200102545\cdots \\
h_4 &= \tfrac{1}{16\sqrt{2}}\left(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}\right) &\cong& \; -0.0854412738820267\cdots \\
h_5 &= \tfrac{1}{16\sqrt{2}}\left(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right) &\cong& \; 0.03522629188570\cdots
\end{aligned}
$$

Using the roots inside the unit circle one gets the filter in reversed order.

Construction of scaling and wavelet filters

Eigenvalue-Eigenvector Problem

Example: DAUB6 $(L = 6)$

$$\varphi(x) = \sqrt{2}\sum_{k-0}^{l-1} h_k \varphi(2x-k)$$

Assume that φ is bounded to an interval [a, b].

Then

$$[a,b] \overset{\text{matching}}{\longleftrightarrow} \left[\frac{a}{2}\,,\,\frac{b}{2}\right]\;,\;\left[\frac{a+1}{2}\,,\,\frac{b+1}{2}\right],\ldots,\left[\frac{a+L+1}{2}\,,\,\frac{b+L-1}{2}\right]$$

So

$$a \equiv \tfrac{a}{2} \longrightarrow \underline{a=0}$$

$$
b \equiv \tfrac{b+L-1}{2} \longrightarrow \quad
\begin{aligned}
\not{2}b &= \not{b}+L-1 \\
b &= L-1
\end{aligned}
$$

So $\varphi(x)$ lives on interval $[0, L-1]$
Same for $\varphi(x)$

$$\text{Supp } \varphi = \text{Supp } \psi = [0, L-1]$$

$$
\begin{aligned}
\text{For} \quad L &= 6 \qquad\qquad \text{Set} \quad c_k = \sqrt{2}h_k \\
\varphi(0) &= c_0\varphi(0) + 0 \qquad\qquad (\text{Since} \quad \varphi(-1) = 0, \text{etc.}) \\
\varphi(1) &= c_0\varphi(2) + c_1\varphi(1) + c_2\varphi(0) + 0 \\
\varphi(2) &= c_0\varphi(4) + c_1\varphi(3) + c_2\varphi(2) + c_3\varphi(1) + c_4\varphi(0) \\
\varphi(3) &= c_0\varphi(6) + c_1\varphi(5) + c_2\varphi(4) + c_3\varphi(3) + c_4\varphi(2) + c_5\varphi(1) \\
\varphi(4) &= c_3\varphi(5) + c_4\varphi(4) + c_5\varphi(3) \\
\varphi(5) &= c_5\varphi(5)
\end{aligned}
$$

Since all $\quad c_i \neq 0,\ i = 0, \cdots, 5 \quad \Longrightarrow \quad \varphi(0) = \varphi(5) = 0$

$$
\begin{cases}
\varphi(0) = 0 \\
\varphi(1) = c_0\varphi(2) + c_1\varphi(1) \\
\varphi(2) = c_0\varphi(4) + c_1\varphi(3) + c_2\varphi(2) + c_3\varphi(1) \\
\varphi(3) = c_2\varphi(4) + c_3\varphi(3) + c_4\varphi(2) + c_5\varphi(1) \\
\varphi(4) = c_4\varphi(4) + c_5\varphi(3) \\
\varphi(5) = 0
\end{cases}
$$

Write non-trivial equations in matrix form.

$$
\begin{bmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \end{bmatrix} = \begin{bmatrix} c_1 & c_0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 \\ c_5 & c_4 & c_3 & c_2 \\ 0 & 0 & c_5 & c_4 \end{bmatrix} \begin{bmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \end{bmatrix}
$$

$$
\vec{\varphi} = A\,\vec{\varphi}
$$

Eigenvalue - Eigenvector Problem

$$
\lambda\,\vec{v} = A\,\vec{v} \quad \text{where} \quad \lambda = 1, \quad \vec{v} \text{ unknown.}
$$

Due to partition of unity $\sum v_i = 1$, so,

$$
\varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) = 1 \quad (*)
$$

Eigenvalues of matrix A are $\frac{1}{4}, \frac{1}{2}, 1, \frac{1-\sqrt{10}}{8}$

If you use

$$
\begin{aligned}
c_0 &= \tfrac{1}{16}\left(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\right)\\
c_1 &= \tfrac{1}{16}\left(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}\right)\\
c_2 &= \tfrac{1}{8}\left(5-\sqrt{10}+\sqrt{5+2\sqrt{10}}\right)\\
c_3 &= \tfrac{1}{8}\left(5-\sqrt{10}-\sqrt{5+2\sqrt{10}}\right)
\end{aligned}
$$

Using $(*)$ you get for the eigenvector components

$$
\begin{aligned}
\varphi(1) &= 1.28634\\
\varphi(2) &= -0.385837\\
\varphi(3) &= 0.0952675\\
\varphi(4) &= 0.00423435
\end{aligned}
\longrightarrow
$$

allows one to compute φ , ψ at dyadic points ...

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TW Honns: Numeriese Algoritmes Huiswerk # 4
(Vir 3/4/01) 2001

INSTRUCTIONS:

(i) General instructions (a) and (b) from the previous assignments apply.

(ii) The computational part of the assignment should be done with software of your choice.

Recommended: MATLAB, Mathematica, Fortran or C.

Task 1: Wavelet Decomposition & Reconstruction with Haar Wavelets

Consider a function $f(x)$, that is piecewise constant on quarter-intervals:

$$f(x) = \begin{cases} 2, & 0 \leq x < \frac{1}{4} \\ 0, & \frac{1}{4} \leq x < \frac{1}{2} \\ 6, & \frac{1}{2} \leq x < \frac{3}{4} \\ 4, & \frac{3}{4} \leq x < 1 \end{cases} \tag{1}$$

(i) Express $f(x)$ in terms of the scaling function $\phi_{2,0}(x) = 2\phi(4x)$ and its translates $\phi_{2,1}(x) = 2\phi(4x-1), \phi_{2,2}(x) = 2\phi(4x-2), \phi_{2,3}(x) = 2\phi(4x-3)$.

Let $\vec{f}_{\mathcal{B}}$ denote the coordinate vector of $f(x)$ with respect to the (old) basis $\mathcal{B}$, made up with the above 4 functions belonging to the scaling family $\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$. What is the explicit form of $\vec{f}_{\mathcal{B}}$?

(ii) Express $f(x)$ in terms of the wavelet (new) basis $\mathcal{B}' = \{\phi_{0,0}(x) = \phi(x), \psi_{0,0}(x) = \psi(x), \psi_{1,0}(x) = \sqrt{2}\psi(2x), \psi_{1,1}(x) = \sqrt{2}\psi(2x-1)\}$.

Let $\vec{f}_{\mathcal{B}'}$ denote the coordinate vector of $f(x)$ with respect to the (new) basis $\mathcal{B}'$, made up with the above 4 functions. What is the explicit form of $\vec{f}_{\mathcal{B}'}$?

(iii) Find the explicit form of the transformation matrix $\mathbf{D}$ such that $\vec{f}_{\mathcal{B}'} = \mathbf{D}\vec{f}_{\mathcal{B}}$. The matrix $\mathbf{D}$ is called the *decomposition* matrix.

(iv) Find the explicit form of the transformation matrix $\mathbf{R}$ such that $\vec{f}_{\mathcal{B}} = \mathbf{R}\vec{f}_{\mathcal{B}'}$. The matrix $\mathbf{R}$ is called the *reconstruction* matrix.

(v) Show that the matrices $\mathbf{D}$ and $\mathbf{R}$ are both orthogonal matrices.

(vi) Show that $\mathbf{D}$ can be written as the product of three matrices, i.e. $\mathbf{D} = \mathbf{D_2P_1D_1}$, where $\mathbf{D_1}$ is the matrix for averaging and differencing at level 1, $\mathbf{D_2}$

is the matrix for averaging and differencing at level 2, and $\mathbf{P_1}$ is a permutation matrix which brings averages together (at the top) and differences together (at the bottom).

(vii) Verify your analytic work on the computer. Define the matrices $\mathbf{D_1}, \mathbf{D_2}$ and $\mathbf{P_1}$ and compute their product to obtain $\mathbf{D}$. Compute the matrix $\mathbf{R}$ explicitly. Verify the formulas in (iii) and (iv).

(viii) Compute $\vec{f}_{\mathcal{B}'}$ via repeated convolutions with the low-pass filter $H = (h_0, h_1) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the high-pass filter $G = (g_0, g_1) = (h_1, -h_0) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
How may times can you apply each filter if you start with the data defined by $f(x)$?

(ix) If there are 4 data entries (the four values of $f(x)$) then all the matrices are size 4×4.
Generalize the above problem to sizes 8×8 and 16×16.
For instance, for the 8×8 case, $\mathcal{B} = \{2\sqrt{2}\phi(8x - k),\ k = 0, 1, 2, ..., 7\}$ and $\mathcal{B}' = \{\phi(x), \psi(x), \sqrt{2}\psi(2x), \sqrt{2}\psi(2x-1), 2\psi(4x), 2\psi(4x-1), 2\psi(4x-2), 2\psi(4x-3)\}$.
Set up the matrices $\mathbf{D}$ and $\mathbf{R}$ for each case.
(x) Show that for the 8×8 case, the matrix $\mathbf{D}$ can be written as $\mathbf{D} = \mathbf{D_3P_2D_2P_1D_1}$. The explicit forms of all the decomposition and permutation matrices should be in your computer program.

(xi) Show that for the 16×16 case, the matrix $\mathbf{D}$ can be written as $\mathbf{D} = \mathbf{D_4P_3D_3P_2D_2P_1D_1}$. The explicit forms of all the decomposition and permutation matrices should be in your computer program.

(xii) Apply the Haar decomposition and reconstruction algorithm to the following data sets

$$\vec{f}_{\mathcal{B}} = 2\sqrt{2}[1, 3, 0, 5, 7, 4, 2, 10]^T \tag{2}$$

and

$$\vec{f}_{\mathcal{B}} = [4, 20, 0, 8, 4, 12, 12, 16, 24, 28, 4, 8, 0, 12, 36, 24]^T \tag{3}$$

Determine $\vec{f}_{\mathcal{B}'}$ for both cases. Have the computer do the computations!

Task 2: Construction of Daubechies Filter(s) (DAUB4) with 2 vanishing moments

This task involves the computation of the Daubechies filter(s) (DAUB4), H and G, each with 4 filter taps, corresponding to wavelets with two vanishing moments

($M = 2$).

The goal is to find the real numbers (preferably 16 digits beyond the decimal point) h_0, h_1, h_2 and h_3 such that

(i) $G(g_0, g_1, g_2, g_3)$ is a high-pass filter which annihilates both constant and linear trends (or signals). Thus, we require two vanishing moments for ψ : i.e.

$$\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} x\psi(x)dx = 0. \tag{4}$$

(ii) $H(h_0, h_1, h_2, h_3)$ is normalized in the usual way:

$$\int_{-\infty}^{\infty} \phi(x)dx = 1; \tag{5}$$

(iii) The scaling function $\phi(x)$ is orthogonal to its translates, i.e.

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-n)dx = \delta_{n0}, \tag{6}$$

where the Kronecker δ_{n0} equals 1 when $n = 0$ and zero otherwise.

First, express the conditions in (i), (ii) and (iii) in terms of the filter taps of the low-pass filter H. Second, solve that system of five(*) equations for h_0, h_1, h_2 and h_3. How many solutions?

(*) Remarks:

(a) The quadratic condition coming from $n = 0$ in (iii) is actually redundant, but can and should be used to verify the results. So, one has to solve four (three linear, one quadratic) equations.

(b) The filter taps can be expressed in closed (irrational) form. Compare the closed forms with the numerical values. Do they match?

Task 3: Construction of the DAUB4 Scaling and Wavelets Functions

This task involves the construction of the scaling and wavelets functions, $\phi(x)$ and $\psi(x)$, for the Daubechies wavelets with two vanishing moments (DAUB4), where the low-pass and high-pass filters, H and G, have 4 filter taps. The computations can be done by hand or (easier) with a symbolic manipulation program such as Mathematica or Maple.

One of the solutions obtained in Task 2 reads

```
h0 = 0.4829629131445343
h1 = 0.836516303737808
h2 = 0.2241438680420134
h3 = -0.1294095225512604
```

(i) Use the *exact, closed form irrational* representation corresponding to the above numerical values of the filter $H = \{h_0, h_1, h_2, h_3\}$, of length $L = 4$, and the two-scale difference (refinement or dilation) equations for $\phi(x)$ and $\psi(x)$:

$$\phi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k\phi(2x-k) = \sum_{k=0}^{L-1} c_k\phi(2x-k) \quad \text{with} \quad c_k = \sqrt{2}h_k, \tag{7}$$

$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k\phi(2x-k) = \sum_{k=0}^{L-1} d_k\phi(2x-k) \quad \text{with} \quad d_k = \sqrt{2}g_k, \tag{8}$$

to determine the values of the scaling function $\phi(x)$ at the integers. To do this, go through the following steps:

(a) Prove that the scaling function $\phi(x)$ is compactly supported on the interval [0,L-1] = [0,3].

(b) Prove that $\phi(0) = \phi(3) = 0$.

(c) Set up the eigenvalue-eigenvector problem $\mathbf{A}\vec{v} = \lambda\vec{v}$ to determine $\phi(1)$ and $\phi(2)$.

What is the matrix $\mathbf{A}$? What is the vector $\vec{v}$? What are the eigenvalues of $\mathbf{A}$? Do you notice anything special about these eigenvalues? What are the eigenvectors of $\mathbf{A}$? Which eigenvalue and eigenvector will allow you to determine $\phi(1)$ and $\phi(2)$?

(d) Use the *resolution of the identity* (also called *partition of unity*) property of the scaling function,

$$\sum_{k\in\mathbf{Z}} \phi(x-k) = 1 \tag{9}$$

to normalize $\phi(1)$ and $\phi(2)$. Compute the closed forms of the irrational numbers $\phi(1)$ and $\phi(2)$?

(ii) What is the compact support of the wavelet function $\psi(x)$? Why?

(iii) Again, use the refinement equations for $\phi(x)$ and $\psi(x)$ to compute the values of these functions at the so-called diadic (dyadic) rational ($x = k\,2^{-j}$), k and j integer). Once the values of $\phi(x)$ and $\psi(x)$ at integers are computed, (7) and (8) determine the values at the halves, quarters, eights, etc. In general, by repeated applications of the refinement equations, you can obtain the values of $\phi(x)$ and $\psi(x)$ at diadic points $x = k\,2^{-j}$ for $k = ..., -2, -1, 0, 1, 2, ...; j = 1, 2, 3, ...$ For decent resolution of the pictures, it suffices to run this iteration process till $j = 4$ or 5. Determine explicitly the (closed-form irrational or numerical) values of $\phi(x)$ and $\psi(x)$ at $x = 0, \frac{1}{32}, \frac{3}{16}, \frac{1}{2}, \frac{5}{8}, \frac{3}{2}, \frac{11}{4}$, and 3.

(iv) Graph the various iterations of $\phi(x)$ and $\psi(x)$. Make sure that the horizontal and vertical axes are properly labelled.

Task 4: Scaling and Wavelet Functions via a Cascade Algorithm

Consider the following cascade (iterative) scheme based on the refinement equation for $\phi(x)$:

$$\phi^{(i+1)}(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k\, \phi^{(i)}(2x-k), \quad i \geq 0, \tag{10}$$

where the initial $\phi^{(0)}(x)$ must be given. We use the characteristic function on the unit interval

$$\phi^{(0)}(x) = \begin{cases} 1, & 0 \leq x < 1, \\ \\ 0, & \text{elsewhere.} \end{cases} \tag{11}$$

(i) For the Haar case ($L = 2$), take $h_0 = h_1 = \frac{1}{\sqrt{2}}$, and implement the cascade algorithm to compute the values of $\phi^{(1)}(x), \phi^{(4)}(x)$ and $\phi^{12)}(x)$.

(ii) Make pictures of these three scaling functions over their intervals of support. For a decent resolution of the graphs, use steps of $\frac{1}{64}$. Do not go beyond the boundaries of the supporting intervals.

Note: If you use Mathematica in order to efficiently compute the values of $\phi^{(i)}(x)$, use a construction like

```
phi[0,x_] := 0 /;  x<0.0
phi[0,x_] := 0 /;  x>=1.0
phi[0,x_] := 1 /;  (x<1.0 && x>=0.0)

h0=1/Sqrt[2];
h1=1/Sqrt[2];

phi[i_,x_] := phi[i,x] =

N[Sqrt[2]*(h0*phi[i-1,2*x]+h1*phi[i-1,2*x-1]),8] /; i >= 1
```

(iii) Still for the Haar case, use the relation (8) to compute the values of $\psi^{(12)}(x)$.

(iv) Make a picture of this wavelet function.

(v) What is the function corresponding to $\phi^{(\infty)}(x)$? Does it have an analytical form? What is it?

(vi) What is the function corresponding to $\psi^{(\infty)}(x)$? Is there an analytical form? What is it?

(vii) Repeat steps (i)-(iv) for the case of Daubechies wavelets (DAUB4) with two vanishing moments, where

```
h0 = 0.4829629131445343,
h1 = 0.836516303737808,
h2 = 0.2241438680420134,
h3 = -0.1294095225512604.
```

(viii) Answer the corresponding questions for the DAUB4 case: Can you still use (11) as the starting point? [Hint: Experiment with different initial conditions]. What could one use instead?

(ix) Assuming that (10) has a 'fixed point' for any choice of a low-pass filter, prove that the cascade algorithm always converges to the true scaling function corresponding to that filter.

(x) Prove or disprove that the 'fixed point' $\phi(x)$ depends on the initial $\phi^{(0)}(x)$.

[Hint: for (ix) and (x) work with the equivalent of (10) in the Fourier domain].

Question 4: Wavelets: solving the Quadrature Mirror Filter condition

Based on the Quadrature Mirror Filter (QMF) condition it is possible to compute Finite Impulse Response (FIR) low-pass and high-pass filters, $(h_0, h_1, h_2, ..., h_{L-1})$ and $(g_0, g_1, g_2, ..., g_{L-1})$, for the Daubechies wavelets. The QMF condition reads

$$H(z)H(\frac{1}{z}) + H(-z)H(-\frac{1}{z}) = 2, \qquad \forall z, \tag{12}$$

for

$$H(z) = \sum_{k=0}^{L-1} h_k z^k, \tag{13}$$

where L is the length of the filter and h_k are the low-pass filter taps.
The condition

$$H(1) = \sqrt{2} \tag{14}$$

normalizes the filter (and the scaling function).
For a wavelet $\psi(x)$ with M vanishing moments, i.e.

$$\int_{-\infty}^{\infty} x^m \psi(x) dx = 0, \quad m = 0, 1, 2, ..., M-1, \tag{15}$$

the polynomial $H(z)$ and its derivatives up to order $M-1$ must be zero at $z = -1$. Thus,

$$H(-1) = H'(-1) = H''(-1) = ... = H^{(M-1)}(-1) = 0. \tag{16}$$

(i) Use the normalization condition $H(1) = \sqrt{2}$ and QMF (12) to show that $H(-1) = 0$.
What does the latter condition imply in terms of the filters $(h_0, h_1, h_2,..., h_{L-1})$ and $(g_0, g_1, g_2,..., g_{L-1})$? What is relevance in signal processing?
(ii) Show that (14) and (16) imply that $H(z)$ is of the form

$$H(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^M Q(z), \tag{17}$$

where $Q(z)$ is polynomial and $Q(1) = 1$.

(iii) Show that for a FIR filter with M vanishing moments $Q(z)$ has degree $M-1$.

(iv) Consider the Haar case ($L = 2, M = 1$). What is $Q(z)$? Determine $H(z)$ and the filter coefficients from (17) by comparing with (13).

(v) If the QMF condition is rewritten in terms of $Q(z)$, one can show (see Lecture Notes, no need to prove this) that $Q(z)$ must satisfy

$$Q(z)Q(\frac{1}{z}) = \sum_{k=0}^{M-1} (-1)^k \frac{(M-1+k)!}{k!(M-1)!}(1-z)^{2k}(4z)^{-k}. \tag{18}$$

What does (18) reduce to for the Haar case?

(vi) Substitute (13) directly into (12) to compute the conditions for h_0, h_1 for the Haar case. How many conditions do you get? How many are trivial? How many are nontrivial? What do these conditions express? Compute h_0 and h_1 explicitly, using (14) to normalize.

(vii) Consider DAUB4, where $L = 4, M = 2$, corresponding to compactly supported wavelets with two vanishing moments. What is the degree of $Q(z)$? Use (18) and $Q(1) = 1$ to determine $Q(z)$ explicitly. Then, continuing with one of the (two possible) solutions, use (17) to compute $H(z)$. Finally, use (13) to find closed-form irrational expressions for the filter coefficients h_0, h_1, h_2, and h_3.

(viii) Substitute (13) directly into (12) to compute the conditions for the h_k for the DAUB4 low-pass filter ($L = 4$). How many conditions do you get? How many are trivial? How many are nontrivial? What do these conditions actually express? *Verify* that low-pass filter (h_0, h_1, h_2, h_3) computed in (vii) satisfies all conditions, including (14).

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Compiled by Willy Hereman——Updated: February 9, 2001

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