

Symbolic computation of solitary wave solutions and solitons through homogenization of degree

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Goals and Outline

- **Background & Motivation**
- **Homogenization of nonlinear PDEs**
 - ▶ Linearization of the Burgers equation
 - ▶ Homogenization of the KdV equation
- **A simplified Hirota method**
 - ▶ Hirota's transformation
 - ▶ Hirota operators and bilinear representation
 - ▶ Ignoring the bilinear form
 - ▶ Algorithm to find solitons

- **Application to 5th-order KdV equations**
 - ▶ Lax equation
 - ▶ Sawada-Kotera equation
 - ▶ Kaup-Kupershmidt equation
- **Application to non-solitonic PDEs**
 - ▶ Fisher equation (with convection term)
 - ▶ FitzHugh-Nagumo equation (with convection)
 - ▶ Burgers equation with cubic source term
 - ▶ Wave equation with cubic source term
 - ▶ Combined KdV-Burgers equation
 - ▶ An equation with two but not three solitons

- **Mathematica demonstration**
- **Concluding remarks**

Background & Motivation

Example 1: Bateman-Burgers equation

$$\frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{or} \quad u_t + 2uu_x - \nu u_{xx} = 0$$



Harry Bateman
(1882-1946)

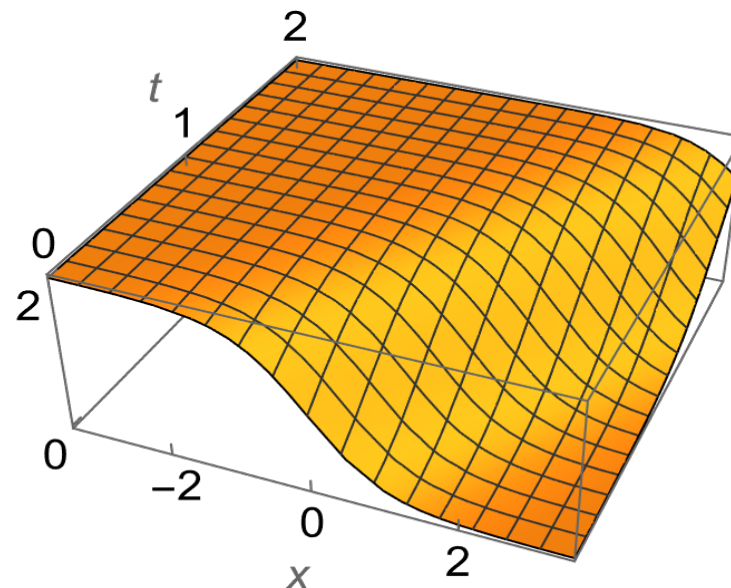
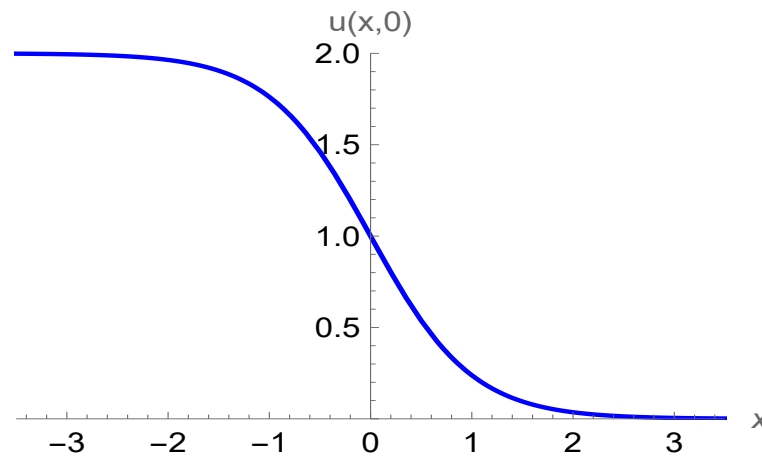


Johannes Burgers
(1895-1981)

Shock wave

$$u(x, t) = k\nu \left(1 - \tanh(kx - 2\nu k^2 t + \delta) \right)$$

Graphs of the kink solution for $k = \nu = 1$, $\delta = 0$.



Example 2: The Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad u_t + 6uu_x + u_{xxx} = 0$$



Diederik Korteweg
(1848-1941)



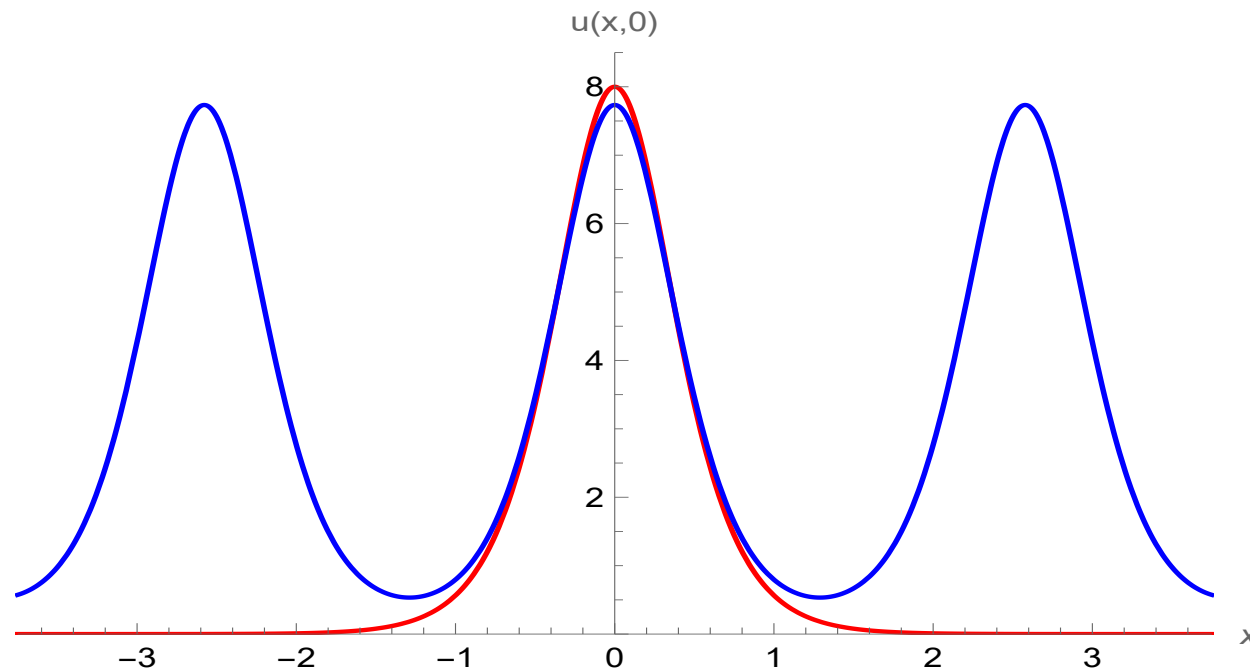
Gustav de Vries
(1866-1934)

Solitary wave and periodic solutions

$$u(x, t) = 2k^2 \operatorname{sech}^2(kx - 4k^3t + \delta) \quad \text{and}$$

$$u(x, t) = \frac{4}{3}k^2(1 - m) + 2k^2m \operatorname{cn}^2(kx - 4k^3t + \delta; m)$$

Graphs of the solitary wave (red) and cnoidal (blue) wave solutions for $k = 2$, $m = \frac{9}{10}$, $\delta = 0$.



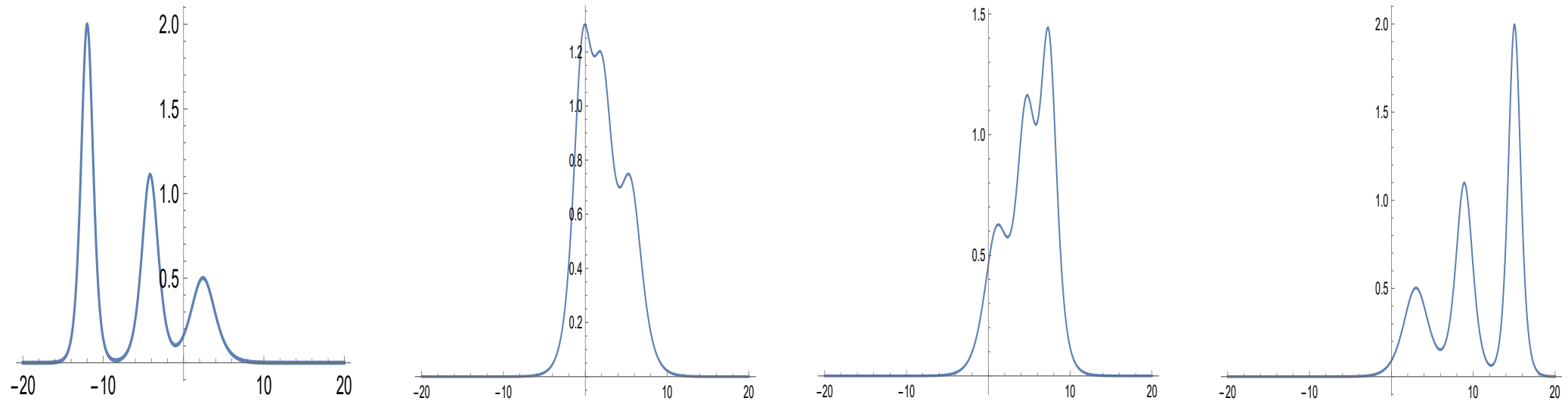


John Scott Russell (1808-1882)

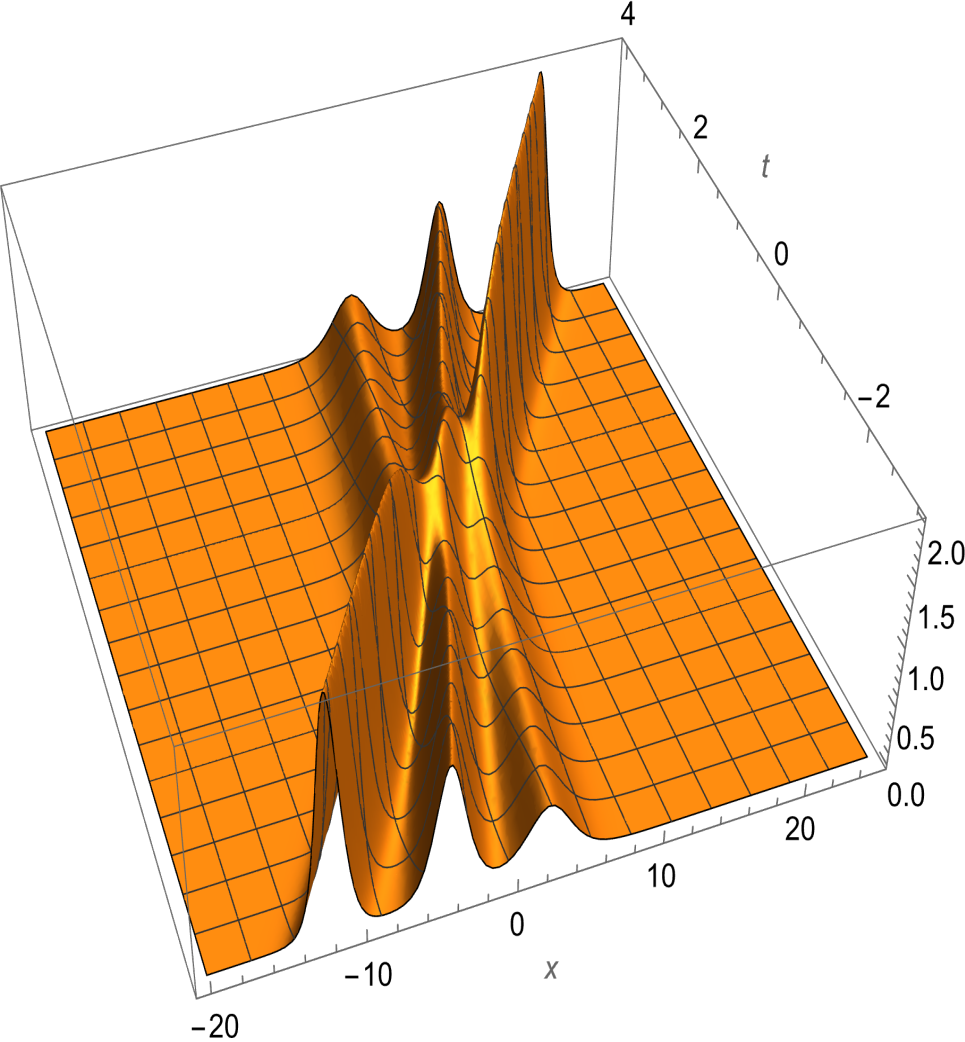
Soliton solutions

Norman Zabusky and Martin Kruskal (1965)

Collision of three-solitons for the KdV equation



Bird's eye view of a 3-soliton collision for the KdV equation. Notice the phase shift.



Computation of solitary wave & soliton solutions

- Direct integration (solitary wave solutions)
- Inverse Scattering Transform
- Hirota's bilinear method
- Wronskian determinant method
- Riemann-Hilbert approach
- $\bar{\partial}$ -dressing method
- Darboux & Bäcklund transformations
- Simplified Hirota method (homogenization of degree)

Homogenization of nonlinear PDEs

Example 1: The Burgers equation

The Cole-Hopf transformation

The Burgers equation

$$u_t + 2uu_x - \nu u_{xx} = 0$$

can be **linearized** with a logarithmic derivative transformation due to Cole (1951) and Hopf (1950).

Integrate the Burgers equation with respect to x :

$$\partial_t \left(\int^x u \, dx \right) + u^2 - \nu u_x = 0.$$

Substitute

$$u = c (\ln f)_x = c \left(\frac{f_x}{f} \right)$$

where c is a constant, to get

$$f(f_t - \nu f_{xx}) + (c + \nu) f_x^2 = 0.$$

Setting $c = -\nu$ yields the heat equation $f_t - \nu f_{xx} = 0$.

Then,

$$u(x, t) = -\nu (\ln f)_x = -\nu \left(\frac{f_x}{f} \right)$$

is the well-known Cole-Hopf transformation.

How to find the Cole-Hopf transformation?

As in the Painlevé test, substitute a Laurent series

$$u(x, t) = f^\alpha(x, t) \sum_{k=0}^{\infty} u_k(x, t) f^k(x, t)$$

$f(x, t)$ is manifold for the poles; α is negative integer.

Terms $f^{2\alpha-1}$ and $f^{\alpha-2}$ balance when $\alpha = -1$ and vanish for $u_0(x, t) = -\nu f_x$. Truncating the Laurent series at the constant level in f

$$u(x, t) = -\nu \left(\frac{f_x}{f} \right) + u_1(x, t) = -\nu (\ln f)_x + u_1(x, t)$$

where $u_1(x, t)$ satisfies the Burgers equation.

Setting $u_1 = 0$ yields the Cole-Hopf transformation.

Find a kink solution of the Burgers equation

Substitute $f(x, t) = 1 + e^\theta = 1 + e^{kx - \omega t + \delta}$ into $f_t - \nu f_{xx} = 0$. Then $\omega = -\nu k^2$.

$$\begin{aligned}u(x, t) &= -\nu \left(\frac{f_x}{f} \right) = -k\nu \left(\frac{e^\theta}{1 + e^\theta} \right) = -k\nu \left(\frac{e^\theta e^{-\frac{\theta}{2}}}{(1 + e^\theta) e^{-\frac{\theta}{2}}} \right) \\&= -k\nu \left(\frac{e^{\frac{\theta}{2}}}{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}} \right) = -\frac{1}{2}k\nu \left(\frac{2e^{\frac{\theta}{2}}}{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}} \right) \\&= -\frac{1}{2}k\nu \left(\frac{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}} + e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}}{e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}} \right) = -\frac{1}{2}k\nu \left(1 + \tanh \frac{\theta}{2} \right)\end{aligned}$$

with $\theta = kx + \nu k^2 t + \delta$.

Equivalently,

$$u(x, t) = \frac{1}{2} \tilde{k} \nu \left(1 - \tanh \frac{\tilde{\theta}}{2} \right)$$

$$\text{with } \tilde{k} = -k, \quad \tilde{\theta} = \tilde{k}x - \nu \tilde{k}^2 t - \delta$$

$$u(x, t) = \nu K (1 - \tanh \Theta)$$

$$\text{with } K = \frac{\tilde{k}}{2}, \quad \Theta = Kx - 2\nu K^2 t + \Delta, \quad \text{and } \Delta = -\frac{\delta}{2}.$$

Example 2: The Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Using a Laurent series, $f^{2\alpha-1}$ and $f^{\alpha-3}$ balance when $\alpha = -2$. The terms in f^{-5} and f^{-4} vanish when

$$u_0(x, t) = -f_x, \quad u_1(x, t) = 2f_{xx}.$$

Hence

$$u(x, t) = -\frac{2f_x^2}{f^2} + \frac{2f_{xx}}{f} + u_2(x, t) = 2(\ln f)_{xx} + u_2(x, t)$$

where $u_2(x, t)$ solves the KdV equation (auto-Bäcklund transformation). Setting $u_2 = 0$ yields the Hirota transformation that “bilinearizes” the KdV equation.

Hirota's method

Example 2 continued: The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Integrate with respect to x

$$\partial_t \left(\int^x u dx \right) + 3u^2 + u_{2x} = 0.$$

Substitute $u = c (\ln f)_{xx} = c \left(\frac{ff_{xx} - f_x^2}{f^2} \right)$ where c is a constant, to get

$$\begin{aligned} f^3(f_{xt} + f_{4x}) - f^2(f_x f_t - 3(c-1)f_{xx}^2 + 4f_x f_{3x}) \\ + 3(c-2)f_x^2(f_x^2 - 2ff_{xx}) = 0. \end{aligned}$$

Set $c = 2$ to get a PDE that is **homogenous** of **second** degree in f

$$f(f_{xt} + f_{4x}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} = 0$$

which can be written in bilinear form

$$B(f \cdot f) \equiv \left(D_x D_t + D_x^4 \right) (f \cdot f) = 0$$

using Hirota's bilinear operators

$$\begin{aligned} D_x^n (f \cdot g) &= (\partial_x - \partial_{x'})^n f(x, t) g(x', t) \Big|_{x'=x} \\ &= \sum_{j=0}^n \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \left(\frac{\partial^j f}{\partial x^j} \right) \left(\frac{\partial^{n-j} g}{\partial x^{n-j}} \right). \end{aligned}$$

Explicitly for $n = 4$

$$D_x^4(f \cdot g) = f_{4x}g - 4f_{3x}g_x + 6f_{xx}g_{xx} - 4f_xg_{3x} + fg_{4x}.$$

Leibniz rule for derivatives of products with every other sign flipped. Likewise

$$\begin{aligned} D_x^m D_t^n (f \cdot g) &= (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t) g(x', t') \Big|_{x'=x, t'=t} \\ &= \sum_{j=0}^m \sum_{i=0}^n \frac{(-1)^{(n+m-i-j)} m! n!}{j! (m-j)! i! (n-i)!} \left(\frac{\partial^{i+j} f}{\partial t^i \partial x^j} \right) \left(\frac{\partial^{n+m-i-j} g}{\partial t^{n-i} \partial x^{m-j}} \right). \end{aligned}$$

For example

$$D_x D_t (f \cdot g) = f_{xt}g - f_t g_x - f_x g_t + f g_{xt}.$$

Seek a solution of the form

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t) = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots$$

ϵ is a bookkeeping parameter (not a small quantity).

$$O(\epsilon^0) : B(1 \cdot 1) = 0$$

$$O(\epsilon^1) : B(1 \cdot f^{(1)} + f^{(1)} \cdot 1) = 0$$

$$O(\epsilon^2) : B(1 \cdot f^{(2)} + f^{(1)} \cdot f^{(1)} + f^{(2)} \cdot 1) = 0$$

$$O(\epsilon^3) : B(1 \cdot f^{(3)} + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)} + f^{(3)} \cdot 1) = 0$$

$$O(\epsilon^4) : B(1 \cdot f^{(4)} + f^{(1)} \cdot f^{(3)} + f^{(2)} \cdot f^{(2)} + f^{(3)} \cdot f^{(1)} + f^{(4)} \cdot 1) = 0$$

$$O(\epsilon^n) : B \left(\sum_{j=0}^n f^{(j)} \cdot f^{(n-j)} \right) = 0 \quad \text{with } f^{(0)} = 1.$$

How to find exact solutions if the bilinear form is not known? Use a simplified Hirota method!

The quadratic equation

$$f(f_{xt} + f_{4x}) - f_x f_t + 3f_{xx}^2 - 4f_x f_{3x} = 0$$

is of the form $f \mathcal{L}f + \mathcal{N}(f, f) = 0$.

\mathcal{L} is a linear differential operator, \mathcal{N} is a quadratic differential operator. For the KdV equation

$$\mathcal{L}f = f_{xt} + f_{4x}$$

$$\mathcal{N}(f, g) = -f_x g_t + 3f_{xx} g_{xx} - 4f_x g_{3x}$$

for auxiliary functions $f(x, t)$ and $g(x, t)$.

Substitute $f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t)$. Use Cauchy's product formula to group powers of ϵ :

$$\left(\sum_{r=1}^{\infty} \epsilon^r f^{(r)} \right) \left(\sum_{s=1}^{\infty} \epsilon^s f^{(s)} \right) = \sum_{n=2}^{\infty} \epsilon^n \sum_{j=1}^{n-1} f^{(j)} f^{(n-j)}$$

Set the coefficients of powers of ϵ to zero:

$$O(\epsilon^1) : \mathcal{L} f^{(1)} = 0$$

$$O(\epsilon^2) : \mathcal{L} f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)})$$

$$O(\epsilon^3) : \mathcal{L} f^{(3)} = - \left(f^{(1)} \mathcal{L} f^{(2)} + \mathcal{N}(f^{(1)}, f^{(2)}) + \mathcal{N}(f^{(2)}, f^{(1)}) \right)$$

$$\vdots \quad \quad \quad \vdots$$

$$O(\epsilon^n) : \mathcal{L} f^{(n)} = - \sum_{j=1}^{n-1} \left[f^{(j)} \mathcal{L} f^{(n-j)} + \mathcal{N}(f^{(j)}, f^{(n-j)}) \right], \quad n \geq 3.$$

The N-soliton solution of the KdV is generated from

$$f^{(1)} = \sum_{i=1}^N e^{\theta_i} \equiv \sum_{i=1}^N e^{k_i x - \omega_i t + \delta_i}.$$

Then $\mathcal{L}f^{(1)} = 0$ yields $\omega_i = k_i^3$.

Using $f^{(1)}$

$$\begin{aligned} -\mathcal{N}(f^{(1)}, f^{(1)}) &= - \sum_{i,j=1}^N 3k_i k_j^2 (k_i - k_j) e^{\theta_i + \theta_j} \\ &= \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i - k_j)^2 e^{\theta_i + \theta_j}. \end{aligned}$$

No terms in $e^{2\theta_i}$.

This determines the form of $f^{(2)}$

$$f^{(2)} = \sum_{1 \leq i < j \leq N} a_{ij} e^{\theta_i + \theta_j}$$

Next, compute

$$\mathcal{L}f^{(2)} = \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i + k_j)^2 a_{ij} e^{\theta_i + \theta_j}$$

and solve $\mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)})$ for

$$a_{ij} = \left(\frac{k_1 - k_j}{k_i + k_j} \right)^2, \quad 1 \leq i < j \leq N.$$

Similarly, compute $f^{(3)}$. For example, for $N = 3$

$$f^{(3)} = b_{123} e^{\theta_1 + \theta_2 + \theta_3}$$

with

$$b_{123} = a_{12} a_{13} a_{23} = \left(\frac{(k_1 - k_2)(k_1 - k_3)(k_2 - k_3)}{(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)} \right)^2.$$

For $N = 3$, one finds $f^{(n)} = 0$ for $n > 3$. Thus,

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} \\ + b_{123} e^{\theta_1 + \theta_2 + \theta_3}$$

setting $\epsilon = 1$. No terms in $e^{2\theta_1}$, $e^{2\theta_2}$, $e^{2\theta_1 + \theta_2}$, $e^{2\theta_2 + \theta_1}$,

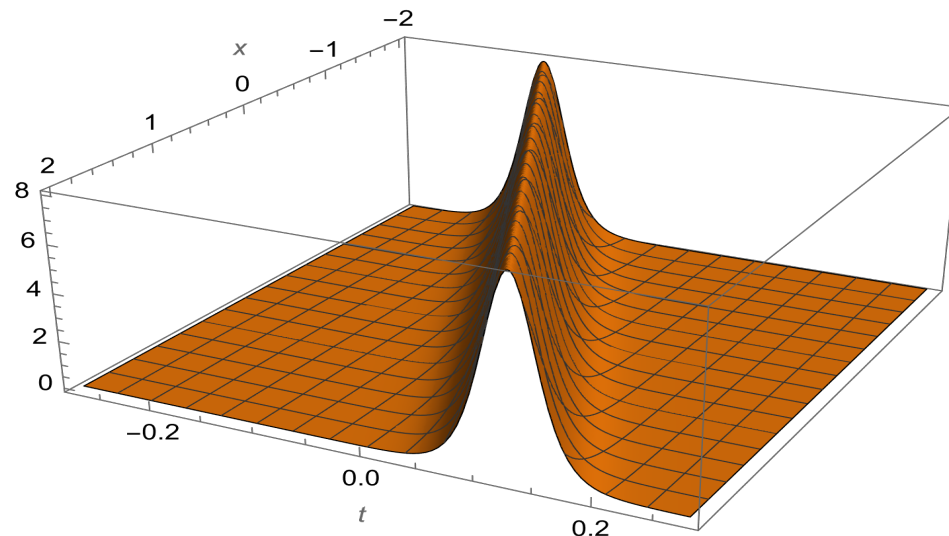
Finally, compute $u(x, t) = 2(\ln f)_{xx} = 2 \left(\frac{f f_{xx} - f_x^2}{f^2} \right)$.

One-soliton solution of the KdV equation

With $f = 1 + e^\theta = 1 + e^{kx - k^3 t + \delta}$

$$\begin{aligned} u(x, t) &= 2 \left(\frac{f f_{xx} - f_x^2}{f^2} \right) = \frac{2k^2 e^\theta}{(1 + e^\theta)^2} = \frac{2k^2 e^\theta e^{-\theta}}{\left(e^{-\frac{\theta}{2}} (1 + e^\theta) \right)^2} \\ &= \frac{1}{2} k^2 \operatorname{sech}^2 \left[\frac{1}{2} (kx - k^3 t + \delta) \right] = 2K^2 \operatorname{sech}^2 (Kx - 4K^3 t + \Delta) \end{aligned}$$

Graph of a hump solution for $K = 2$, $\Delta = 0$.



Two-soliton solution of the KdV equation

Using $f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$,

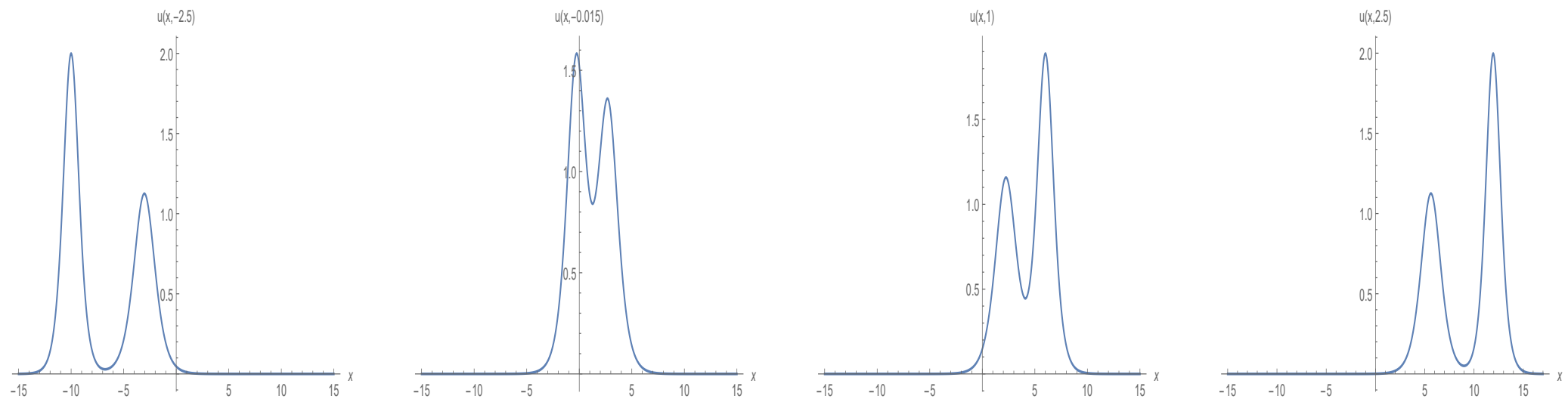
$$u(x, t) = \frac{2 \left[k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + 2(k_1 - k_2)^2 e^{\theta_1 + \theta_2} + a_{12} (k_2^2 e^{\theta_1} + k_1^2 e^{\theta_2}) e^{\theta_1 + \theta_2} \right]}{\left(1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \right)^2}$$

Setting $k_i = 2K_i$, $\delta_i = 2\Delta_i + \ln \left(\frac{K_2 + K_1}{K_2 - K_1} \right)$

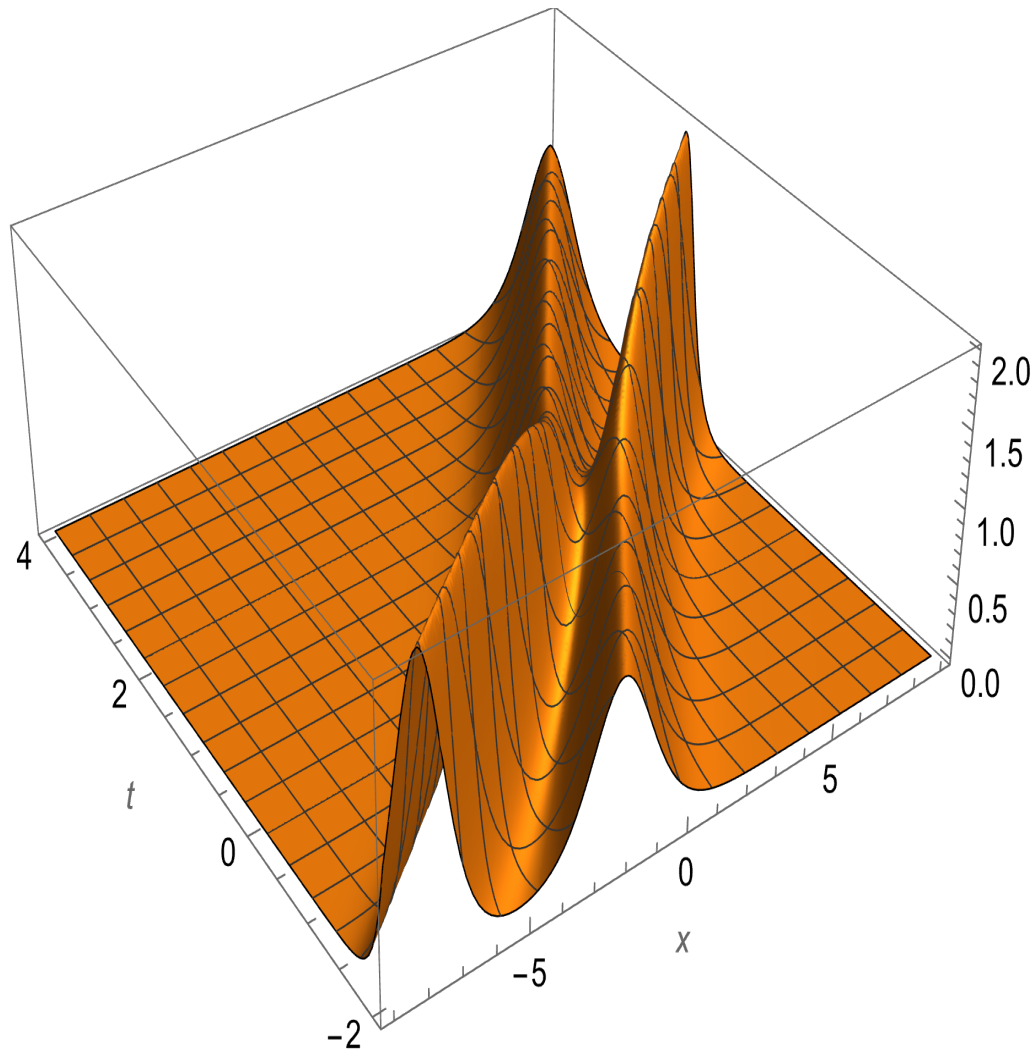
$$\begin{aligned} u(x, t) &= \frac{4(K_2^2 - K_1^2) \left[(K_2^2 - K_1^2) + K_1^2 \cosh(2\Theta_2) + K_2^2 \cosh(2\Theta_1) \right]}{\left[(K_2 - K_1) \cosh(\Theta_2 + \Theta_1) + (K_2 + K_1) \cosh(\Theta_2 - \Theta_1) \right]^2} \\ &= 2 \left(K_2^2 - K_1^2 \right) \left(\frac{K_1^2 \operatorname{sech}^2(\Theta_1) + K_2^2 \operatorname{csch}^2(\Theta_2)}{\left[K_1 \tanh(\Theta_1) - K_2 \coth(\Theta_2) \right]^2} \right) \end{aligned}$$

where $\Theta_i = K_i x - 4K_i^3 t + \Delta_i$.

Collision of two-solitons for the KdV equation
at different times with $k_1 = 2$, $k_2 = \frac{3}{2}$, $\delta_1 = \delta_2 = 0$.



Bird's eye view of a 2-soliton collision for the KdV equation; $k_1 = 2$, $k_2 = \frac{3}{2}$, $\delta_1 = \delta_2 = 0$.



N -soliton solution of the KdV equation

$$f = \sum_{\mu=0,1} e^{\left[\sum_{i<j}^{(N)} A_{ij} \mu_i \mu_j + \sum_{i=1}^N \mu_i \theta_i \right]}$$

$\sum_{\mu=0,1}$ is over all combinations of

$$\mu_1 = 0, 1, \quad \mu_2 = 0, 1, \quad \dots, \quad \mu_N = 0, 1.$$

$\sum_{i<j}^{(N)}$ is such that $0 < i < j \leq N$ and $a_{ij} = e^{A_{ij}}$.

Nice form of N -soliton solution (IST):

$$u(x, t) = 2 \left(\ln \det(I + M) \right)_{xx}$$

where I is the $N \times N$ identity matrix and

$$M_{\ell m} = \frac{e^{\Theta_\ell + \Theta_m}}{K_\ell + K_m} \quad \text{with} \quad \Theta_\ell = K_\ell x - 4K_\ell^3 t + \Delta_\ell.$$

Application: A family of fifth-order KdV equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

Scaling $u = \frac{1}{\gamma} \tilde{u}$ yields

$$\tilde{u}_t + \frac{\alpha}{\gamma^2} \tilde{u}^2 \tilde{u}_x + \frac{\beta}{\gamma} \tilde{u}_x \tilde{u}_{xx} + \tilde{u} \tilde{u}_{3x} + \tilde{u}_{5x} = 0.$$

Completely integrable cases:

$\frac{\alpha}{\gamma^2}$	$\frac{\beta}{\gamma}$	(α, β, γ)	Name
$\frac{3}{10}$	2	(30, 20, 10)	Lax
		(5, 5, 5)	
$\frac{1}{5}$	1	(45, 15, 15)	Sawada-Kotera
		(180, 30, 30)	Caudrey-Dodd-Gibbon
$\frac{1}{5}$	$\frac{5}{2}$	(20, 25, 10)	Kaup-Kupershmidt

Integrate

$$\partial_t \left(\int^x u \, dx \right) + \frac{1}{3} \alpha u^3 + \frac{1}{2} (\beta - \gamma) u_x^2 + \gamma u u_{xx} + u_{4x} = 0.$$

Substitute

$$u = c (\ln f)_{xx} = c \left(\frac{f f_{xx} - f_x^2}{f^2} \right)$$

where c is a constant, to get

$$\begin{aligned} & 6f^5 (f_{xt} + f_{6x}) - 3f^4 (2f_x f_t + \dots + 12f_x f_{5x}) \\ & + 2f^3 \left((\dots) f_{2x}^3 + \dots + (\dots) f_x^2 f_{4x} \right) \\ & + 3f^2 f_x^2 \left((\dots) f_{xx}^2 + (\dots) f_x f_{3x} \right) \\ & + 2f_x^4 (360 - 6\beta c + \alpha c^2 - 12\gamma c) (3f f_{xx} - f_x^2) = 0. \end{aligned}$$

Example 3: The Lax equation

Using $\alpha = \frac{3}{10}\gamma^2$, $\beta = 2\gamma$, and $c = \frac{20}{\gamma}$ one gets a **cubic** (homogenous) equation

$$f^2(f_{xt} + f_{6x}) - f(f_x f_t - 5f_{2x} f_{4x} + 6f_x f_{5x}) + 10(f_{2x}^3 - 2f_x f_{2x} f_{3x} + f_x^2 f_{4x}) = 0.$$

Bilinear form consists of two coupled equations

$$\left(D_x D_s + D_x^4 \right) (f \cdot f) = 0$$

$$\left(D_x D_t + D_x^6 \right) (f \cdot f) - \frac{5}{3} \left(D_s^2 + D_s D_x^3 \right) (f \cdot f) = 0$$

for only one function f but with an auxiliary time variable s . Ignore the bilinear form!

Write the cubic equation as

$$f^2 \mathcal{L}f + f \mathcal{N}_1(f, f) + \mathcal{N}_2(f, f, f) = 0$$

with

$$\mathcal{L}f = f_{xt} + f_{6x}$$

$$\mathcal{N}_1(f, g) = -(f_t g_x - 5f_{2x} g_{4x} + 6f_x g_{5x})$$

$$\mathcal{N}_2(f, g, h) = 10(f_{2x} g_{2x} h_{2x} - 2f_x g_{2x} h_{3x} + f_x g_x h_{4x}).$$

Substitute

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t).$$

Perturbation scheme becomes

$$O(\epsilon^1) : \mathcal{L}f^{(1)} = 0$$

$$O(\epsilon^2) : \mathcal{L}f^{(2)} = -\mathcal{N}_1(f^{(1)}, f^{(1)})$$

$$O(\epsilon^3) : \mathcal{L}f^{(3)} = - \left(2f^{(1)} \mathcal{L}f^{(2)} + \mathcal{N}_1(f^{(1)}, f^{(2)}) + \mathcal{N}_1(f^{(2)}, f^{(1)}) \right. \\ \left. + f^{(1)} \mathcal{N}_1(f^{(1)}, f^{(1)}) + \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) \right)$$

\vdots \vdots

Start from $f^{(1)} = \sum_{i=1}^N e^{\theta_i} = \sum_{i=1}^N e^{k_i x - \omega_i t + \delta_i}$ and proceed as in KdV case.

One-soliton solution of the Lax equation

$$\begin{aligned}u(x, t) &= \frac{5}{\gamma} k^2 \operatorname{sech}^2 \left[\frac{1}{2} (kx - k^5 t + \delta) \right] \\ &= \frac{20}{\gamma} K^2 \operatorname{sech}^2 \left(Kx - 16K^5 t + \Delta \right)\end{aligned}$$

where $k = 2K$ and $\delta = 2\Delta$ solves

$$u_t + \frac{3}{10} \gamma^2 u^2 u_x + 2\gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

Here, $\omega_i = k_i^5$, $a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2$ and $b_{123} = a_{12} a_{13} a_{23}$

(for 2- and 3-soliton solutions).

Example 4: The Sawada-Kotera equation

Using $\alpha = \frac{1}{5}\gamma^2$, $\beta = \gamma$, and $c = \frac{30}{\gamma}$ one gets a **quadratic** equation

$$f(f_{xt} + f_{6x}) - f_x f_t - 10f_{3x}^2 + 15f_{xx}f_{4x} - 6f_x f_{5x} = 0.$$

Bilinear representation

$$\left(D_x D_t + D_x^6 \right) (f \cdot f) = 0$$

is similar to one for the KdV equation. Ignore it!

Define

$$\mathcal{L}f = f_{xt} + f_{6x}$$

$$\mathcal{N}(f, g) = -f_x g_t - 10f_{3x} g_{3x} + 15f_{xx} g_{4x} - 6f_x g_{5x}.$$

Proceed as in the KdV case.

One-soliton solution of the Sawada-Kotera equation

$$\begin{aligned}u(x, t) &= \frac{15}{2\gamma} k^2 \operatorname{sech}^2 \left[\frac{1}{2} (kx - k^5 t + \delta) \right] \\ &= \frac{30}{\gamma} K^2 \operatorname{sech}^2 \left(Kx - 16K^5 t + \Delta \right).\end{aligned}$$

where $k = 2K$ and $\delta = 2\Delta$ solves

$$u_t + \frac{1}{5}\gamma^2 u^2 u_x + \gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

Here, $\omega_i = k_i^5$. For the 3-soliton solution:

$$\begin{aligned}a_{ij} &= \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)} = \frac{(k_i - k_j)^3 (k_i^3 + k_j^3)}{(k_i + k_j)^3 (k_i^3 - k_j^3)} \\ b_{123} &= a_{12} a_{13} a_{23}.\end{aligned}$$

Example 5: The Kaup-Kupershmidt equation

Using $\alpha = \frac{1}{5}\gamma^2$, $\beta = \frac{5}{2}\gamma$, and $c = \frac{15}{\gamma}$ yields a **quartic** equation

$$4f^3(f_{xt} + f_{6x}) - f^2(4f_t f_x - 5f_{3x}^2 + 24f_x f_{5x}) \\ - 30f f_x(f_{2x} f_{3x} - 2f_x f_{4x}) + 15f_x^2(3f_{2x}^2 - 4f_x f_{3x}) = 0.$$

Bilinear form consists of two coupled equations

$$\left(D_x D_t + \frac{1}{16} D_x^6\right)(f \cdot f) + \frac{15}{4} D_x^2(f \cdot g) = 0 \\ D_x^4(f \cdot f) - 4fg = 0$$

for two unknown functions f and g . Ignore bilinear form.

Continue with equation for f :

$$f^3 \mathcal{L}f + f^2 \mathcal{N}_1(f, f) + f \mathcal{N}_2(f, f, f) + \mathcal{N}_3(f, f, f, f) = 0$$

where

$$\mathcal{L}f = 4(f_{xt} + f_{6x})$$

$$\mathcal{N}_1(f, g) = -(4f_t g_x - 5f_{3x} g_{3x} + 24f_x g_{5x})$$

$$\mathcal{N}_2(f, g, h) = -30f_x (g_{2x} h_{3x} - 2g_x h_{4x})$$

$$\mathcal{N}_3(f, g, h, j) = 15f_x g_x (3h_{2x} j_{2x} - 4h_x j_{3x}).$$

First three equations of the 'perturbation' scheme:

$$O(\epsilon^1) : \mathcal{L}f^{(1)} = 0$$

$$O(\epsilon^2) : \mathcal{L}f^{(2)} = -\mathcal{N}_1(f^{(1)}, f^{(1)})$$

$$O(\epsilon^3) : \mathcal{L}f^{(3)} = - \left(3f^{(1)} \mathcal{L}f^{(2)} + 2f^{(1)} \mathcal{N}_1(f^{(1)}, f^{(1)}) \right) \\ + \mathcal{N}_1(f^{(2)}, f^{(1)}) + \mathcal{N}_1(f^{(1)}, f^{(2)}) \\ + \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)})$$

The one-soliton solution of the KK equation

With $f^{(1)} = e^\theta = e^{kx - \omega t + \delta}$, $\mathcal{L}f^{(1)} = 0$ yields $\omega = k^5$.

Compute

$$-\mathcal{N}(f^{(1)}, f^{(1)}) = 15k^6 e^{2\theta}.$$

Thus $f^{(2)}$ is of the form

$$f^{(2)} = ae^{2\theta}.$$

Since

$$\mathcal{L}f^{(2)} = 240ak^6e^{2\theta}$$

one gets $a = \frac{1}{16}$.

Next, verify that $f^{(n)} = 0$ for $n \geq 3$.

Setting $\epsilon = 1$, for the one-soliton solution

$$f = 1 + e^\theta + \frac{1}{16}e^{2\theta}.$$

Then $u = \frac{15}{\gamma} (\ln f)_{xx}$ yields

$$u = \frac{240}{\gamma} k^2 \left[\frac{e^\theta (16 + 4e^\theta + e^{2\theta})}{(16 + 16e^\theta + e^{2\theta})^2} \right]$$

which solves

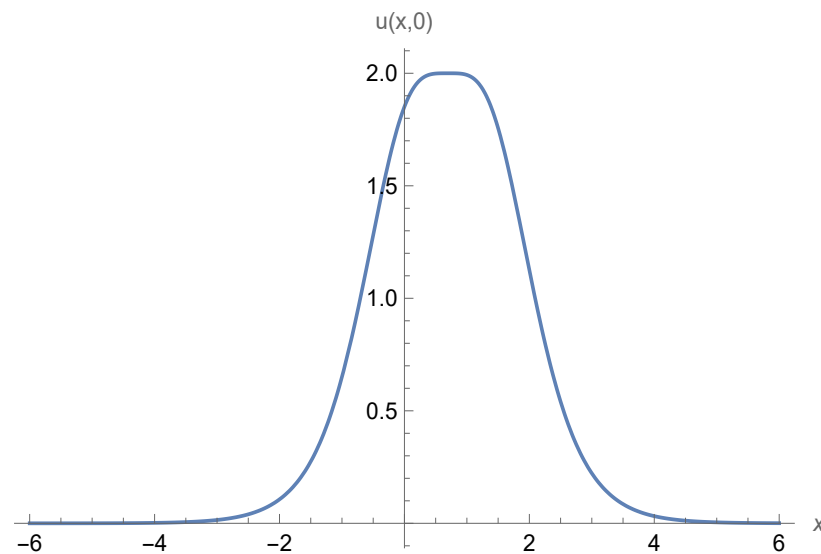
$$u_t + \frac{1}{5} \gamma^2 u^2 u_x + \frac{5}{2} \gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

One-soliton solution can also be written as

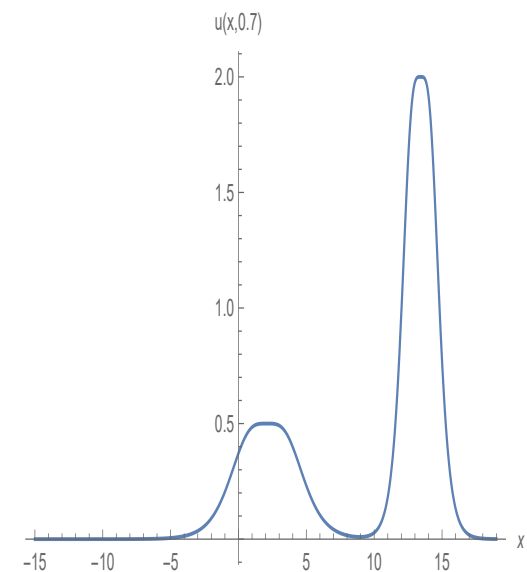
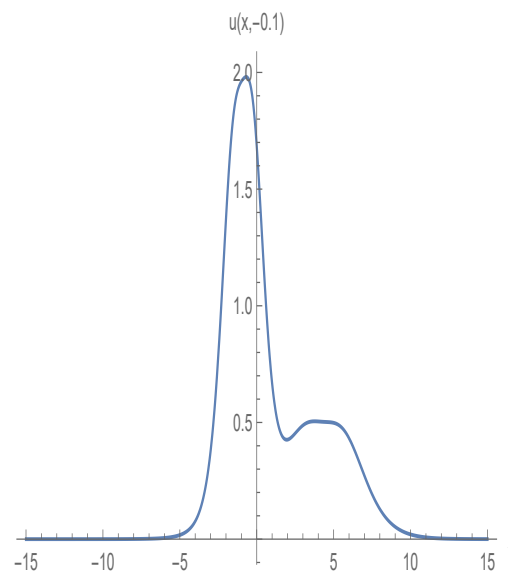
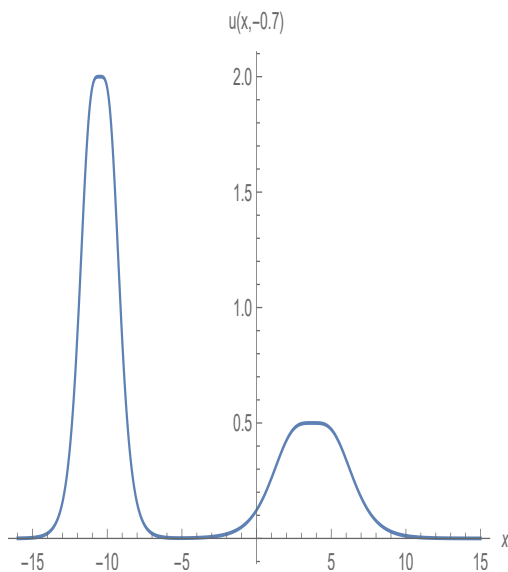
$$u = \frac{240}{\gamma} k^2 \left(\frac{\left[1 - \tanh^2\left(\frac{\theta}{2}\right) \right] \left[21 - 30 \tanh \frac{\theta}{2} + 13 \tanh^2\left(\frac{\theta}{2}\right) \right]}{\left[33 - 30 \tanh \frac{\theta}{2} + \tanh^2\left(\frac{\theta}{2}\right) \right]^2} \right)$$

where $\theta = kx - k^5 t + \delta$.

Graph of one-soliton solution ($\gamma = 10, k = 2, \delta = 0$).



Collision of two-solitons ($k_1 = 2, k_2 = 1, \delta_1 = \delta_2 = 0$).



The two-soliton solution of the KK equation

Start from

$$f^{(1)} = e^{\theta_1} + e^{\theta_2}$$

where $\theta_i = k_i x - \omega_i t + \delta_i$ with $\omega_i = k_i^5$ ($i = 1, 2$).

Compute

$$\begin{aligned} -\mathcal{N}_1(f^{(1)}, f^{(1)}) &= 15k_1^6 e^{2\theta_1} + 15k_2^6 e^{2\theta_2} \\ &\quad + 10k_1 k_2 (2k_1^4 - k_1^2 k_2^2 + 2k_2^4) e^{\theta_1 + \theta_2}. \end{aligned}$$

Thus $f^{(2)}$ must be of the form

$$f^{(2)} = a e^{2\theta_1} + b e^{2\theta_2} + a_{12} e^{\theta_1 + \theta_2}.$$

Proceed with

$$\begin{aligned}\mathcal{L}f^{(2)} &= 240ak_1^6e^{2\theta_1} + 240bk_2^6e^{2\theta_2} \\ &\quad + 20a_{12}k_1k_2(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)e^{\theta_1+\theta_2}.\end{aligned}$$

Hence, $a = b = \frac{1}{16}$, and

$$a_{12} = \frac{2k_1^4 - k_1^2k_2^2 + 2k_2^4}{2(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)}.$$

Therefore,

$$f^{(2)} = \frac{1}{16}e^{2\theta_1} + \frac{1}{16}e^{2\theta_2} + \frac{(2k_1^4 - k_1^2k_2^2 + 2k_2^4)e^{\theta_1+\theta_2}}{2(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)}.$$

At the next level

$$f^{(3)} = b_{12}\left(e^{\theta_1+2\theta_2} + e^{2\theta_1+\theta_2}\right)$$

with

$$b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{16(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)}.$$

Next,

$$f^{(4)} = b_{12}^2 e^{2(\theta_1 + \theta_2)} = \frac{(k_1 - k_2)^4(k_1^2 - k_1k_2 + k_2^2)^2 e^{2(\theta_1 + \theta_2)}}{256(k_1 + k_2)^4(k_1^2 + k_1k_2 + k_2^2)^2}.$$

After verification that all $f^{(n)}$ are zero, for $n \geq 5$

$$\begin{aligned} f &= 1 + e^{\theta_1} + e^{\theta_2} + \frac{1}{16}e^{2\theta_1} + \frac{1}{16}e^{2\theta_2} + a_{12}e^{\theta_1 + \theta_2} \\ &\quad + b_{12}\left(e^{2\theta_1 + \theta_2} + e^{\theta_1 + 2\theta_2}\right) + b_{12}^2 e^{2(\theta_1 + \theta_2)} \end{aligned}$$

and $u = \frac{15}{\gamma}(\ln f)_{xx}$.

The three-soliton solution of the KK equation

First six equations in the perturbation scheme must be solved. Start with $f^{(1)} = \sum_{i=1}^3 e^{\theta_i}$. Then

$$f^{(2)} = \frac{1}{16} \sum_{i=1}^3 e^{2\theta_i} + \sum_{1 \leq i < j \leq 3} a_{ij} e^{\theta_i + \theta_j}$$

with

$$a_{ij} = \frac{2k_i^4 - k_i^2 k_j^2 + 2k_j^4}{2(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, \quad 1 \leq i < j \leq 3.$$

Compute $f^{(3)}$ through $f^{(6)}$; at that term the expansion of f truncates. Verify that $f^{(n)} = 0$ for $n \geq 7$.

Summary of the results:

$$f^{(3)} = \sum_{1 \leq i < j \leq 3} b_{ij} \left(e^{2\theta_i + \theta_j} + e^{\theta_i + 2\theta_j} \right) + c_{123} e^{\theta_1 + \theta_2 + \theta_3}$$

where

$$b_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{16(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, \quad 1 \leq i < j \leq 3$$

and

$$\begin{aligned} c_{123} = & \frac{1}{D} \left[(2k_1^4 - k_1^2 k_2^2 + 2k_2^4)(k_3^8 + k_1^4 k_2^4) \right. \\ & + (2k_1^4 - k_1^2 k_3^2 + 2k_3^4)(k_2^8 + k_1^4 k_3^4) \\ & \left. + (2k_2^4 - k_2^2 k_3^2 + 2k_3^4)(k_1^8 + k_2^4 k_3^4) \right] \\ & - \frac{1}{2D} \left[(k_1^2 + k_2^2)(k_1^4 + k_2^4)(k_3^6 + k_1^2 k_2^2 k_3^2) \right. \\ & + (k_1^2 + k_3^2)(k_1^4 + k_3^4)(k_2^6 + k_1^2 k_2^2 k_3^2) \\ & + (k_2^2 + k_3^2)(k_2^4 + k_3^4)(k_1^6 + k_1^2 k_2^2 k_3^2) \\ & \left. + 12k_1^4 k_2^4 k_3^4 \right] \end{aligned}$$

with

$$D = 4 \prod_{1 \leq i < j \leq 3} (k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2).$$

Solve step by step to get

$$f^{(4)} = \sum_{1 \leq i < j \leq 3} b_{ij}^2 e^{2(\theta_i + \theta_j)} + 16 \left(a_{23} b_{12} b_{13} e^{2\theta_1 + \theta_2 + \theta_3} + a_{13} b_{12} b_{23} e^{\theta_1 + 2\theta_2 + \theta_3} + a_{12} b_{13} b_{23} e^{\theta_1 + \theta_2 + 2\theta_3} \right).$$

$$f^{(5)} = 16^2 b_{12} b_{13} b_{23} \left(b_{12} e^{2\theta_1 + 2\theta_2 + \theta_3} + b_{13} e^{2\theta_1 + \theta_2 + 2\theta_3} + b_{23} e^{\theta_1 + 2\theta_2 + 2\theta_3} \right).$$

$$f^{(6)} = 16 (16 b_{12} b_{13} b_{23})^2 e^{2(\theta_1 + \theta_2 + \theta_3)}.$$

Finally

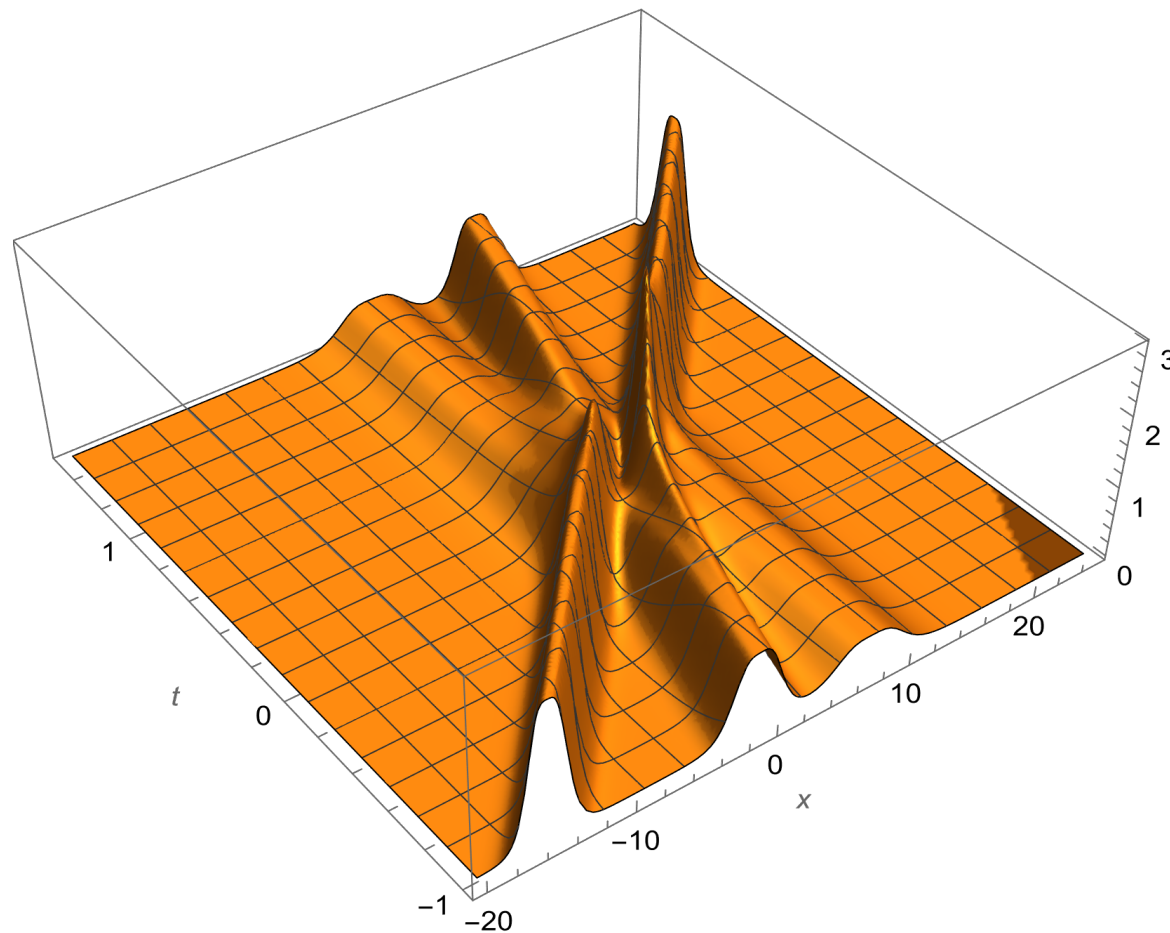
$$f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} + f^{(6)}.$$

Then $u = \frac{15}{\gamma} (\ln f)_{xx}$ solves

$$u_t + \frac{1}{5} \gamma^2 u^2 u_x + \frac{5}{2} \gamma u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0.$$

Mathematica demonstration

Bird's eye view of a 3-soliton collision for the KK equation for $k_1 = 2, k_2 = \frac{3}{2}, k_3 = 1, \delta_1 = \delta_2 = \delta_3 = 0$.



Application: Non-solitonic equations

Example 6: The Fisher equation with convection

$$u_t + \alpha u u_x - u_{xx} - u(1 - u) = 0.$$

Use $u(x, t) = -\frac{2}{\alpha}(\ln f)_x = -\frac{2}{\alpha} \left(\frac{f_x}{f} \right)$ to get a homogenous (quadratic) equation

$$f(f_{xxx} + f_x - f_{xt}) + f_x(f_t - f_{xx} + \frac{2}{\alpha} f_x) = f \mathcal{L}f + \mathcal{N}(f, f) = 0.$$

Seek $f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t)$.

$$\mathcal{L}f^{(1)} = \mathcal{L}(\sum_{i=1}^N e^{\theta_i}) \text{ determines } \omega_i = -(1 + k_i^2).$$

Consequently

$$\mathcal{L}f^{(2)} = - \sum_{i=1}^N k_i \left(1 + \frac{2}{\alpha} k_i\right) e^{2\theta_i} - \sum_{1 \leq i < j \leq N} \frac{4}{k} k_i k_j (k_i + k_j) e^{\theta_i + \theta_j}.$$

Including terms $e^{2\theta_i}$ in $f^{(2)}$ does not help (perturbation scheme does not terminate). Hence, $k_i = -\frac{\alpha}{2}$ and $N = 1$. Then

$$f(x, t) = 1 + e^\theta = 1 + e^{-\frac{\alpha}{2}x + \frac{1}{4}(4 + \alpha^2)t + \delta}.$$

With $u = -\frac{2}{\alpha} \left(\frac{f_x}{f}\right)$

$$u(x, t) = \frac{e^\theta}{1 + e^\theta} = \frac{1}{2} \left(1 - \tanh \left[\frac{1}{2} \left(\frac{\alpha}{2}x - \frac{1}{4}(4 + \alpha^2)t - \delta \right) \right] \right)$$

Example 7: The Fisher equation

$$u_t - u_{xx} - u(1 - u) = 0.$$

Truncated Laurent series reveals that

$$u(x, t) = -6(\ln f)_{xx} + \frac{6}{5}(\ln f)_t.$$

The equation for f is **quadratic**:

$$\begin{aligned} & f(f_{4x} + f_{xx} - \frac{6}{5}f_{xxt} + \frac{1}{5}f_{tt} - \frac{1}{5}f_t) \\ & - 4f_x f_{3x} + 3f_{xx}^2 - f_x^2 - \frac{6}{5}f_t f_{xx} + \frac{12}{5}f_x f_{xt} + \frac{1}{25}f_t^2 \\ & = f \mathcal{L}f + \mathcal{N}(f, f) = 0. \end{aligned}$$

$\mathcal{L}e^\theta = 0$ yields $\omega = -5k^2$ or $\omega = -(1 + k^2)$. Next, $\mathcal{N}(e^\theta, e^\theta) = 0$ determines $k = \pm \frac{1}{\sqrt{6}}$. Thus, $\omega = -\frac{5}{6}$.

With $f = 1 + e^\theta$

$$u(x, t) = \frac{e^{2\theta}}{(1 + e^\theta)^2} = \frac{1}{(1 + e^{-\theta})^2} = \frac{1}{4} \left(1 + \tanh \frac{\theta}{2} \right)^2.$$

For $k = -\frac{1}{\sqrt{6}}$

$$u(x, t) = \frac{1}{4} \left(1 - \tanh \left[\frac{1}{2} \left(\frac{1}{\sqrt{6}}x - \frac{5}{6}t - \delta \right) \right] \right)^2.$$

For $k = \frac{1}{\sqrt{6}}$

$$u(x, t) = \frac{1}{4} \left(1 + \tanh \left[\frac{1}{2} \left(\frac{1}{\sqrt{6}}x + \frac{5}{6}t + \delta \right) \right] \right)^2.$$

Example 8: The FitzHugh-Nagumo equation with convection term

$$u_t + \alpha u u_x - u_{xx} + u(1-u)(a-u) = 0.$$

Use $u = \sqrt{m} (\ln f)_x = \sqrt{m} \left(\frac{f_x}{f} \right)$ where $\alpha = \frac{m-2}{\sqrt{m}}$.

Then

$$\begin{aligned} & f(f_{3x} - a f_x - f_{xt}) + f_x \left(f_t - (m+1) f_{xx} + \sqrt{m}(1+a) f_x \right) \\ &= f \mathcal{L} f + \mathcal{N}(f, f) = 0. \end{aligned}$$

Seek $f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t)$.

$\mathcal{L} f^{(1)} = \mathcal{L}(\sum_{i=1}^N e^{\theta_i})$ determines $\omega_i = a - k_i^2$.

Solitary wave solutions:

$\mathcal{N}(e^\theta, e^\theta) = 0$ yields $k = \frac{1}{\sqrt{m}}$ or $k = \frac{a}{\sqrt{m}}$. Then, $\omega = \frac{am-1}{m}$
or $\omega = \frac{a(m-a)}{m}$. Thus,

$$u(x, t) = \frac{1}{2} \left(1 + \tanh \left[\frac{1}{2} \left(\frac{1}{\sqrt{m}} x - \frac{(am-1)}{m} t + \delta \right) \right] \right)$$

and

$$u(x, t) = \frac{1}{2} a \left(1 + \tanh \left[\frac{1}{2} \left(\frac{a}{\sqrt{m}} x - \frac{a(m-a)}{m} t + \delta \right) \right] \right).$$

Coalescent wave fronts:

Taking $f = 1 + e^{\theta_1} + e^{\theta_2}$ one gets

$$u(x, t) = \frac{e^{\theta_1} + a e^{\theta_2}}{1 + e^{\theta_1} + e^{\theta_2}}$$

where

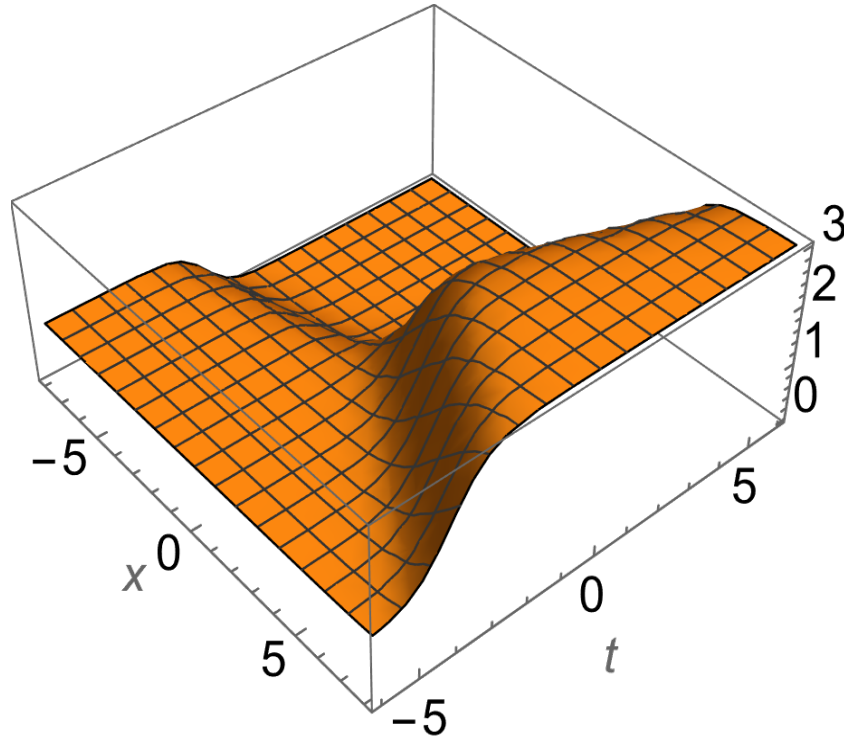
$$\theta_1 = \frac{1}{\sqrt{m}}x + \left(\frac{1-am}{m}\right)t + \delta_1 \quad \text{and} \quad \theta_2 = \frac{a}{\sqrt{m}}x + \left(\frac{a(a-m)}{m}\right)t + \delta_2.$$

The solution describes two coalescent wave fronts.

Since $\alpha = \frac{m-2}{\sqrt{m}}$ the values of m are

$$m = \frac{1}{2} \left(4 + \alpha^2 \pm \alpha \sqrt{8 + \alpha^2} \right).$$

Graph of $u(x, t)$ for $a = 3, \alpha = 1$ ($m = 4$), $\delta_1 = \delta_2 = 0$.



For $m = 2$ one gets the solution of FitzHugh-Nagumo equation without convection ($\alpha = 0$).

Example 9:

Burgers equation with cubic source term

$$u_t + \alpha u u_x - u_{xx} = 3u(2 - u)(u + 1).$$

Truncated Laurent series reveals that

$$u = \pm \sqrt{m} (\ln f)_x = \pm \sqrt{m} \left(\frac{f_x}{f} \right).$$

Homogeneous equation (for + sign):

$$\begin{aligned} f (6f_x - f_{xt} + f_{3x}) + f_x \left(f_t + 3\sqrt{m} f_x - (1 + 3m) f_{xx} \right) \\ \equiv f \mathcal{L}f + \mathcal{N}(f, f) = 0, \end{aligned}$$

provided $\alpha = \frac{3m-2}{\sqrt{m}}$.

Solitary wave solutions:

$\mathcal{L}e^\theta = 0$ yields $\omega = -(k^2 + 6)$ and $\mathcal{N}(e^\theta, e^\theta) = 0$ determines $k = -\frac{1}{\sqrt{m}}$ or $k = \frac{2}{\sqrt{m}}$. Thus, $\omega = -\frac{6m+1}{m}$ or $\omega = -\frac{2(3m+2)}{m}$, resp.

From $f = 1 + e^\theta$

$$u(x, t) = -\frac{1}{2} \left(1 - \tanh \left[\frac{1}{2\sqrt{m}}x - \frac{(6m+1)}{2m}t - \frac{\delta}{2} \right] \right)$$

and

$$u(x, t) = 1 + \tanh \left[\frac{1}{\sqrt{m}}x + \frac{(3m+2)}{m}t + \frac{\delta}{2} \right],$$

where in each case $m = \frac{1}{18} \left(12 + \alpha^2 \pm \alpha\sqrt{24 + \alpha^2} \right)$.

There is no two-soliton solution.

Coalescent wave fronts:

Taking $f = 1 + e^{\theta_1} + e^{\theta_2}$ where $\omega_i = -(k_i^2 + 6)$ yields

$$\begin{aligned} u(x, t) &= \frac{\sqrt{m}(k_1 e^{\theta_1} + k_2 e^{\theta_2})}{1 + e^{\theta_1} + e^{\theta_2}} \\ &= \frac{2 e^{\frac{2}{\sqrt{m}}x + \frac{2(3m+2)}{m}t + \delta_1} - e^{-\frac{1}{\sqrt{m}}x + \frac{(6m+1)}{m}t + \delta_2}}{1 + e^{\frac{2}{\sqrt{m}}x + \frac{2(3m+2)}{m}t + \delta_1} + e^{-\frac{1}{\sqrt{m}}x + \frac{(6m+1)}{m}t + \delta_2}}, \end{aligned}$$

where $k_1 = \frac{2}{\sqrt{m}}$ and $k_2 = -\frac{1}{\sqrt{m}}$ were used.

Since $\alpha = \frac{3m-2}{\sqrt{m}}$, $m = \frac{1}{18} \left(12 + \alpha^2 \pm \alpha \sqrt{24 + \alpha^2} \right)$.

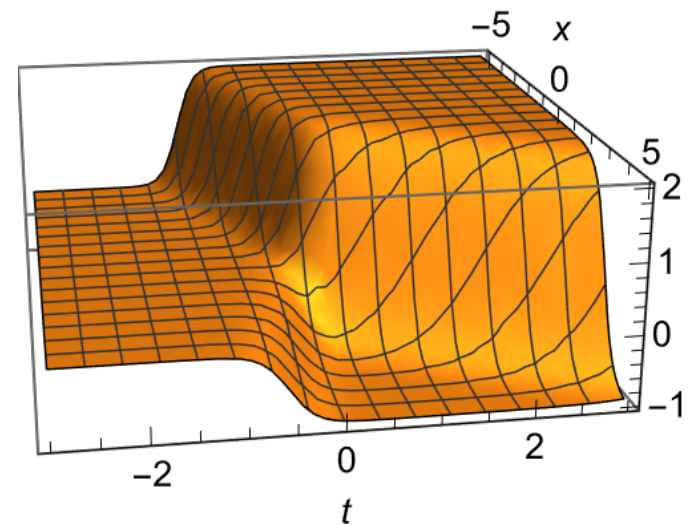
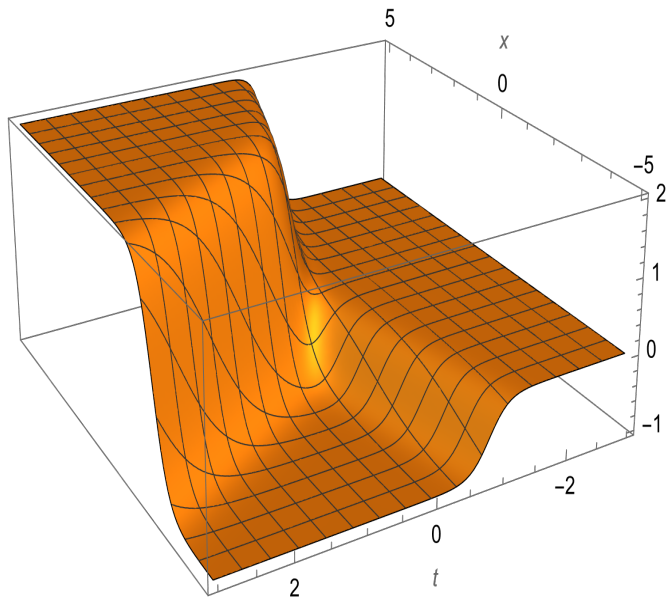
For $m = \frac{2}{3}$ (i.e., $\alpha = 0$), we have a solution of the equation without convection.

Example: For $m = 1$, i.e., $\alpha = 1$, one gets

$$u(x, t) = \frac{2e^{2x+10t+\delta_1} - e^{-x+7t+\delta_2}}{1 + e^{2x+10t+\delta_1} + e^{-x+7t+\delta_2}}.$$

Results for the $-$ sign: replace x by $-x$ and α by $-\alpha$.

3D graphs of the solutions for $m = 1, \alpha = 1$ (left) and $m = 1, \alpha = -1$ (right) with $\delta_1 = -\delta_2 = 1$.



Example 10:

A wave equation with cubic source term

$$\frac{1}{8}u_{tt} + u_t + uu_x - \frac{1}{8}u_{xx} = u(u-1)(u+2).$$

The Laurent series motivates the transformation

$$u = \frac{1}{2} [(\ln f)_t - (\ln f)_x] = \frac{1}{2} \left(\frac{f_t - f_x}{f} \right).$$

Homogeneous equation:

$$\begin{aligned} & f (16f_t + 8f_{tt} + f_{3t} - 16f_x - 8f_{xt} - f_{xtt} - f_{xxt} + f_{3x}) \\ & - (3f_t - f_x) (4f_t + f_{tt} - 4f_x - 2f_{xt} + f_{xx}) \\ & \equiv f \mathcal{L}f + \mathcal{N}(f, f) = 0. \end{aligned}$$

Solitary wave solutions:

$\mathcal{L}e^\theta = 0$ yields $\omega = -k$, $\omega = 4 - k$, or $\omega = 4 + k$.

From $f = 1 + e^\theta$, one readily obtains

$$u(x, t) = -\frac{1}{2}(\omega + k) \left(\frac{e^\theta}{1 + e^\theta} \right) = -\frac{1}{4}(\omega + k) \left(1 + \tanh \frac{\theta}{2} \right).$$

Case $\omega = -k$ must be rejected.

For $\omega = 4 - k$,

$$u(x, t) = - \left(1 + \tanh \left[\frac{1}{2} (kx - (4 - k)t + \delta) \right] \right).$$

For $\omega = 4 + k$, $\mathcal{N}(e^\theta, e^\theta) = 0$ determines $k = -2$ or $k = -3$ resulting in $\omega = 2$ or $\omega = 1$, resp.

Case $k = -2$ (i.e., $\omega = 2$) is rejected since $u(x, t) = 0$.

For $k = -3$ (i.e., $\omega = 1$) one gets

$$u(x, t) = \frac{1}{2} \left(1 - \tanh \left[\frac{1}{2} (3x + t - \delta) \right] \right).$$

There is no two-soliton solution.

Coalescent wave fronts:

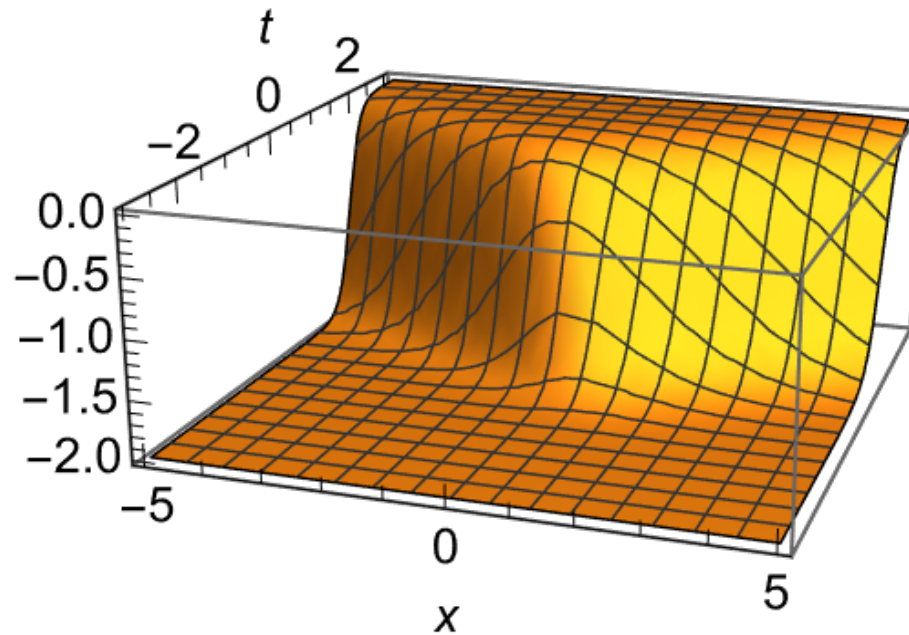
From $f = 1 + e^{\theta_1} + e^{\theta_2}$ with $\omega_i = 4 - k_i$, one gets

$$u(x, t) = -2 \left(\frac{e^{\theta_1} + e^{\theta_2}}{1 + e^{\theta_1} + e^{\theta_2}} \right),$$

where $\theta_1 = k_1 x - (4 - k_1)t + \delta_1$ and

$\theta_2 = k_2 x - (4 - k_2)t + \delta_2$.

3D graph of the solution for $k_1 = 1$, $k_2 = -2$, and $\delta_1 = \delta_2 = 0$.



Example 11: A combined KdV-Burgers equation

$$u_t + 6uu_x + u_{3x} - 5\beta u_{xx} = 0.$$

Truncated Laurent series reveals that

$$u(x, t) = 2(\ln f)_{xx} - 2\beta(\ln f)_x.$$

Substitute into the (integrated) KdV-Burgers equation

$$\partial_t \left(\int^x u dx \right) + 3u^2 + u_{xx} - 5\beta u_x = 0$$

to get

$$\begin{aligned} & f(f_{xt} - \beta f_t + 5\beta^2 f_{2x} - 6\beta f_{3x} + f_{4x}) \\ & - f_x f_t + \beta^2 f_x^2 + 6\beta f_x f_{2x} + 3f_{2x}^2 - 4f_x f_{3x} = 0. \end{aligned}$$

As before $f \mathcal{L}f + \mathcal{N}(f, f) = 0$. Then, $\mathcal{L}e^{kx - \omega t + \delta} = 0$ yields $(\beta - k)(\omega - k^3 + 5\beta k^2) = 0$.

Case 1: $\omega = k^2(k - 5\beta)$ and $\beta \neq k$.

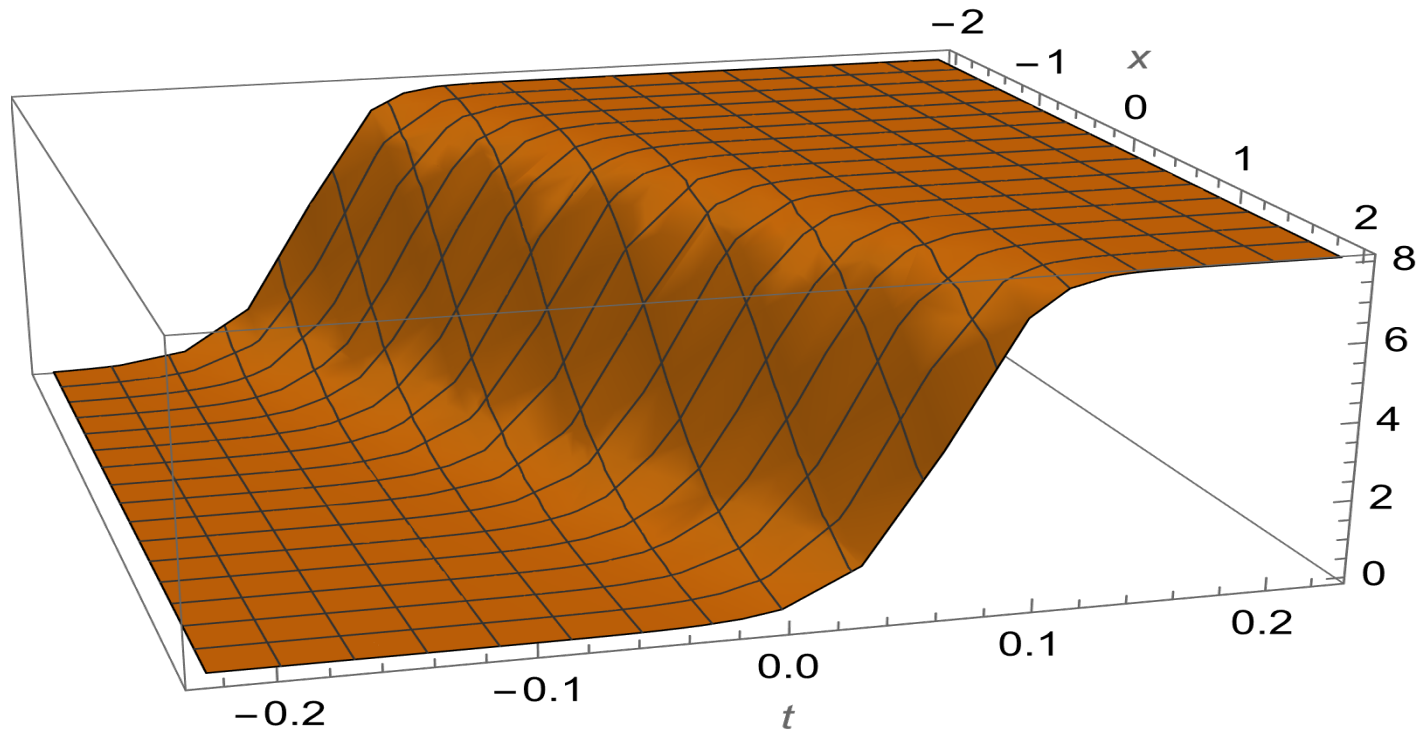
Then $\mathcal{N}(e^\theta, e^\theta) = 0$ determines $k = -\beta$. So, $\omega = -6\beta^3$.

First solution:

$$u(x, t) = 2\beta^2 \left(\frac{e^\theta (2 + e^\theta)}{(1 + e^\theta)^2} \right) = \frac{1}{2}\beta^2 \left(3 - \tanh \frac{\theta}{2} \right) \left(1 + \tanh \frac{\theta}{2} \right),$$

with $\theta = -\beta x + 6\beta^3 t + \delta$.

Graph of the solution for $\beta = 2, \delta = 0$.



Case 2: $\beta = k$.

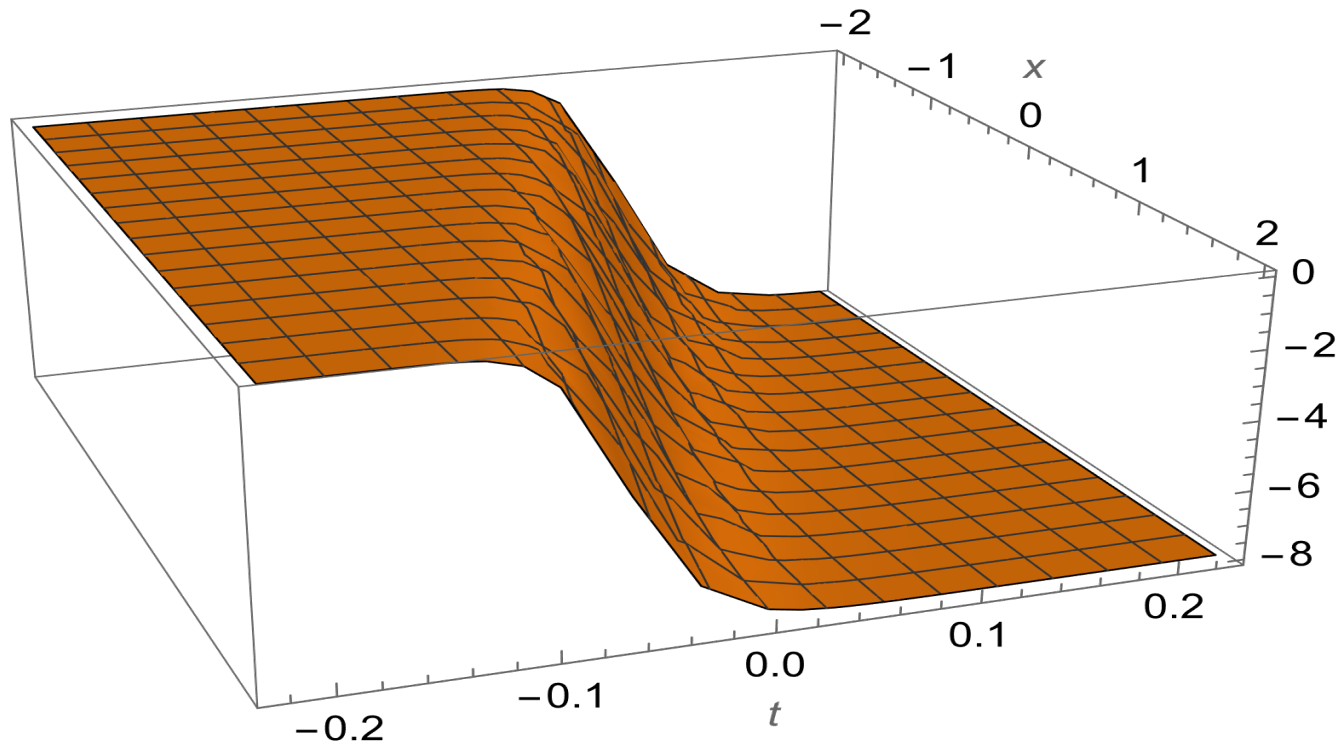
Then $\mathcal{N}(e^\theta, e^\theta) = 0$ determines $\omega = -6\beta^3$.

Second solution:

$$u(x, t) = -2\beta^2 \left(\frac{e^{2\theta}}{(1 + e^\theta)^2} \right) = -\frac{1}{2}\beta^2 \left(1 + \tanh \frac{\theta}{2} \right)^2$$

with $\theta = \beta x + 6\beta^3 t + \delta$.

Graph of the solution for $\beta = 2, \tilde{\delta} = 0$.



Example 12:

An equation with two but not three solitons

$$u_t + \frac{15}{784}u^3u_x + \frac{15}{28}uu_xu_{xx} + \frac{15}{56}u^2u_{3x} \\ + \frac{5}{2}u_{xx}u_{3x} + u_xu_{4x} + uu_{5x} + u_{7x} = 0.$$

Substituting

$$u = 56 (\ln f)_{xx} = 56 \left(\frac{f f_{xx} - f_x^2}{f^2} \right)$$

into the integrated form

$$\partial_t \left(\int^x u dx \right) + \frac{15}{3136}u^4 + \frac{15}{56}u^2u_{2x} + \frac{5}{4}u_{xx}^2 + uu_{4x} + u_{6x} = 0$$

yields

$$f (f_{xt} + f_{8x}) - f_x f_t + 35 f_{4x}^2 - 56 f_{3x} f_{5x} + 28 f_{2x} f_{6x} - 8 f_x f_{7x} \\ = f \mathcal{L}f + \mathcal{N}(f, f) = 0.$$

Solitary wave solution:

$$\mathcal{L}e^\theta = 0 \text{ yields } \omega = k^7.$$

$$\text{From } f = 1 + e^\theta$$

$$u(x, t) = 14k^2 \text{sech}^2 \left[\frac{1}{2} \left(kx - k^7 t + \delta \right) \right].$$

Two-soliton solution:

Take $f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$ with

$$a_{12} = \left(\frac{(k_1 - k_2) (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2) (k_1^2 + k_1 k_2 + k_2^2)} \right)^2$$

then

$$u(x, t) = 56 \left(\frac{k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + a_{12} e^{\theta_1+\theta_2} (k_1 + k_2)^2}{1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}} \right. \\ \left. - \frac{\left(k_1 e^{\theta_1} + k_2 e^{\theta_2} + a_{12} e^{\theta_1+\theta_2} (k_1 + k_2) \right)^2}{(1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2})^2} \right)$$

with $\theta_i = k_i x - k_i^7 t + \delta_i$.

Existence of a two-soliton solution is no surprise.

The PDE has a bilinear representation

$$\left(D_x D_t + D_x^8 \right) (f \cdot f) = 0.$$

But N -soliton solution does **not** exist for $N \geq 3$.

Reason:

$$S[P, n] =$$

$$\sum_{\sigma=\pm 1} P \left(\sum_{i=1}^n \sigma_i k_i, -\sum_{i=1}^n \sigma_i \omega_i \right) \prod_{i < j}^{(n)} P(\sigma_i k_i - \sigma_j k_j, -\sigma_i \omega_i + \sigma_j \omega_j) \sigma_i \sigma_j$$

must be **zero** for $n = 2, 3, \dots, N$.

In the formula $\sum_{\sigma=\pm 1}$ indicates summation over all possible combinations of $\sigma_1 = \pm 1, \sigma_2 = \pm 1, \dots,$

$\sigma_n = \pm 1$ and $\prod_{i < j}^{(n)}$ is the product over all possible combinations of n elements with $i < j$, and all k_i, ω_i subject to the dispersion law $\omega_i(k_i)$.

Here, $P(D_x, D_t) = D_x D_t + D_x^8$ and

$$S[P, 3] = -76832(k_1 k_2 k_3)^4 \left[(k_1^2 - k_2^2)(k_1^2 - k_3^2)(k_2^2 - k_3^2) \right]^2 \\ (k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2)(k_1^4 + k_2^4 + k_3^4 + k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2)$$

is only **zero** if the wave numbers are either equal, each other's opposites, or zero.

Concluding remarks

- Shed some light on how Hirota's method works.
- Hirota transformation and homogenization (of degree) are crucial.
- One can still proceed without knowing the bilinear forms.
- Bilinear form is useful, e.g., to proof existence of solitons solutions, etc.
- Simplified approach applies to nonlinear PDEs that are not soliton equation.
- Implementation in *Mathematica* in collaboration with Ünal Göktaş.

Thank You

Extra example: The mKdV equation

$$u_t + 24u^2u_x + u_{xxx} = 0.$$

Laurent series (two branches combined):

$$u = \frac{1}{2}i \left(\frac{F_x}{F} - \frac{G_x}{G} \right) = \frac{1}{2}i \left(\ln \left(\frac{F}{G} \right) \right)_x$$

Let $F = f + ig$, $G = f - ig$

$$u = \frac{1}{2}i \left(\ln \left(\frac{f + ig}{f - ig} \right) \right)_x = \left(\arctan \left(\frac{f}{g} \right) \right)_x = \frac{f_x g - f g_x}{f^2 + g^2}.$$

This is Hirota's transformation for the mKdV equation!

Integrate

$$u_t + 24u^2u_x + u_{xxx} = 0$$

with respect to x :

$$\partial_t \left(\int^x u dx \right) + 8u^3 + u_{xx} = 0.$$

Applying Hirota's transformation

$$u = \left(\arctan \left(\frac{f}{g} \right) \right)_x = \frac{f_x g - f g_x}{f^2 + g^2}$$

yields

$$\begin{aligned} & f^3(g_t + g_{3x}) - g^3(f_t + f_{3x}) - f^2(f_t g + 3f_x g_{xx} + 3f_{xx} g_x + f_{3x} g) \\ & + g^2(f g_t + 3f_{xx} g_x + 3f_x g_{xx} + f g_{3x}) + 6f g_x (f_x^2 + g_x^2) \\ & - 6f_x g (f_x^2 + g_x^2) + 6f g (f_x f_{xx} - g_x g_{2x}) = 0 \end{aligned}$$

which can be regrouped as

$$(f^2 + g^2)(f_t g - f g_t - f g_{3x} + 3f_x g_{xx} - 3f_{xx} g_x + f_{3x} g) \\ - 6(f_x g - f g_x)(f f_{xx} - f_x^2 + g g_{xx} - g_x^2) = 0,$$

and recast into bilinear form:

$$(D_t + D_x^3)(f \cdot g) = 0 \\ D_x^2(f \cdot f + g \cdot g) = 0.$$

Seek

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots \\ g = g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + \dots$$

Then $f^{(0)} = g^{(1)} = 0$ (or, equivalently, $g^{(0)} = f^{(1)} = 0$).

One-soliton solution of the mKdV equation

With $f = e^\theta$ and $g = 1$

$$\begin{aligned}u &= \frac{f_x}{1 + f^2} = k \frac{e^\theta}{1 + e^{2\theta}} = \frac{1}{2}k \operatorname{sech} \theta \\ &= \frac{1}{2}k \operatorname{sech} (kx - k^3t + \delta) = K \operatorname{sech} (2Kx - 8K^3t + \delta)\end{aligned}$$

with $k = 2K$.

Two-soliton solution of the mKdV equation

$$f = e^{\theta_1} + e^{\theta_2}$$

$$g = 1 - a_{12}e^{\theta_1+\theta_2}$$

with $\theta_i = k_i x - k_i^3 t + \delta_i$, $a_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$.

Eventually,

$$u = \frac{k_1 e^{\theta_1} + k_2 e^{\theta_2} + a_{12} (k_1 e^{\theta_2} + k_2 e^{\theta_1}) e^{\theta_1 + \theta_2}}{1 + e^{2\theta_1} + e^{2\theta_2} + \frac{8k_1 k_2}{(k_1 + k_2)^2} e^{\theta_1 + \theta_2} + a_{12}^2 e^{2\theta_1 + 2\theta_2}}.$$

Three-soliton solution of the mKdV equation

$$f = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - b_{123}e^{\theta_1+\theta_2+\theta_3}$$

$$g = 1 - a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3}$$

with $\theta_i = k_i x - k_i^3 t + \delta_i$, $a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2$, $b_{123} = a_{12}a_{13}a_{23}$.

N -soliton solution of the mKdV equation

$$u(x, t) = \frac{1}{2i} \left(\ln \frac{\det(I + iM)}{\det(I - iM)} \right)_x$$

where I is the $N \times N$ identity matrix and

$$M_{\ell m} = \frac{e^{\Theta_\ell + \Theta_m}}{K_\ell + K_m} \quad \text{with } \Theta_\ell = K_\ell x - 4K_\ell^3 t + \Delta_\ell.$$

Other Equations and Their Bilinear Forms

Type I

The KdV equation : $u_t + 6uu_x + u_{xxx} = 0$

$$u = 2(\ln f)_{xx}$$

$$(D_x D_t + D_x^4)(f \cdot f) = 0$$

Type II

The mKdV equation : $u_t + 6u^2u_x + u_{xxx} = 0$

$$u = 2 \left(\arctan \left(\frac{f}{g} \right) \right)_x$$

$$(D_t + D_x^3)(f \cdot g) = 0$$

$$D_x^2(f \cdot f + g \cdot g) = 0.$$

Alternative 1 : $u = \frac{G}{F}$

$$(D_t + D_x^3)(G \cdot F) = 0$$

$$D_x^2(F \cdot F) - 2G^2 = 0$$

Alternative 2 : $v = i \left(\ln \left(\frac{f^*}{f} \right) \right)_x$

$$(D_t + D_x^3)(f^* \cdot f) = 0$$

$$D_x^2(f^* \cdot f) = 0$$

Type III

The sine – Gordon equation : $u_{xt} = \sin u$

$$u = 2i \ln \left(\frac{f^*}{f} \right), \quad \sin u = \frac{1}{2i} \left(\left(\frac{f}{f^*} \right)^2 - \left(\frac{f^*}{f} \right)^2 \right)$$

$$D_x D_t (f \cdot f) = -\frac{1}{2} (f^{*2} - f^2)$$

Alternative 1 : $u = 4 \arctan \left(\frac{G}{F} \right)$

$$(D_x D_t - I)(F \cdot G) = 0$$

$$D_x D_t (F \cdot F - G \cdot G) = 0$$

Alternative 2 : $u = 4 \arctan \left(\frac{\tilde{F} - \tilde{G}}{\tilde{F} + \tilde{G}} \right)$

$$(D_x D_t - I)(\tilde{F} \cdot \tilde{F} - \tilde{G} \cdot \tilde{G}) = 0$$

$$D_x D_t (\tilde{F} \cdot \tilde{G}) = 0$$

Type IV

The NLS equation : $iu_t + u_{xx} + |u|^2u = 0$

$u = \frac{G}{F}$, with F real, $|u|^2 = 2(\ln F)_{xx}$

$$(iD_t + D_x^2)(G \cdot F) = 0$$

$$D_x^2(F \cdot F) = GG^*$$

Type V

The Benjamin – Ono equation : $u_t + 2uu_x + \mathcal{H}u_{xx} = 0$

Hilbert transform $\mathcal{H}(w)(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{w(t')}{t - t'} dt'$

$$u = i \left(\ln \left(\frac{f^*}{f} \right) \right)_x$$

$$(iD_t + D_x^2)(f \cdot f^*) = 0,$$

Coupled systems

The Hirota–Satsuma system

$$u_t + 3u_x v + u_{xxx} = 0$$

$$v_t - a(6vv_x + v_{xxx}) - 2buu_x = 0$$

$$u = \frac{G}{F}, \quad v = 2(\ln F)_{xx}$$

$$(D_t + D_x^3)(F \cdot G) = 0$$

$$(D_x D_t - aD_x^4)(F \cdot F) = bG^2$$

N -soliton solution only exists if $a = \frac{1}{2}$.

Polynomial versions of sine-Gordon equation

$$u_{xt} = \sin u$$

$$f = e^{iu}$$

$$f^3 - f(f_{xt} + 1) + f_x f_t = 0$$

Alternative :

$$\phi = u_x, \quad \psi = \cos u - 1$$

$$\phi_{xt} - \phi - \phi\psi = 0$$

$$\phi_t^2 + 2\psi + \psi^2 = 0$$